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Research Paper

THE NOETHERIAN DIMENSION OF MODULES VERSUS THEIR α-SMALL SHORTNESS

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Abstract. In this article, we first consider concept of small Noetherian dimension of a module, which is dual to the small krull dimension, denoted by $sn\text{-}\dim A$, and defined to be the codeviation of the poset of the small submodules of A. We prove that if an R-module A with finite hollow dimension, has small Noetherian dimension, then A has Noetherian dimension and $sn\text{-}\dim A \leq n\text{-}\dim A \leq sn\text{-}\dim A+1$. Last we introduce the concept of α-small short modules, i.e., for each small submodule S of A, either $n\text{-}\dim S \leqslant \alpha$ or $sn\text{-}\dim \frac{A}{S} \leqslant \alpha$ and α is the least ordinal number with this property and by using this concept, we extend some of the basic results of short modules to α-small short modules. In particular, we prove that if A is an α-small short module, then it has small Noetherian dimension and $sn\text{-}\dim A = \alpha$ or $sn\text{-}\dim A = \alpha + 1$. Consequently, we show that if A is an α-small short module with finite hollow dimension, then $\alpha \leq n\text{-}\dim A \leq \alpha + 2$.

1. Introduction

In [19], Lemonnier introduced the concept of the deviation of an arbitrary poset $(E, \leq)$, similar to the concept of Krull dimension of modules, see also [21]. The Krull dimension

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of module A, denoted by k-dim A and measures its deviation from being Artinian, was first introduced by Gabriel and Rentschler (for finite ordinals) in 1967. Later this definition was extended to infinite ordinals by Krause in 1970, see [16, 12, 11]. Lemonnier also defined the concept of the dual Krull dimension of E which he named the codeviation of E, as being the Krull dimension (i.e., the deviation) of E^0, the opposite poset of E, see [12]. We remind the reader that the dual Krull dimension of modules measures the deviation of a module from being Noetherian. We should emphasize, for the sake of record and the reader, that the dual Krull dimension of a module was first named Noetherian dimension by Karamzadeh in his 1974, Ph.D. thesis at Exeter university, England and later it is studied in [2, 6, 7, 8, 9, 11, 14, 15, 17]. Let us denote the dual Krull dimension of a module M by n-dim A. These dimensions have been investigated by many authors, see for example [4, 12, 18, 19, 21]. The module A satisfies the ascending chain condition(ACC, for short) on small submodules if and only if $Rad(A)$ is Noetherian, see [4, Theorem 5]. Motivated by this fact, one is tempted to extend it to ascending chain of small submodules of A. To this end, we first introduce and study the concept of small Noetherian dimension of a module A, which is the dual small krull dimension, see [3]. This dimension, denoted by sn-dim, A, is defined to be the codeviation of the poset of the small submodules of A. In some sense, it measure of how far small Noethrian dimension is from Noethrian dimension. We briefly study this dimension and observe that if A has Noethrian dimension, then $sn\text{-dim}\, A \leq n\text{-dim}\, A$. In this article, we introduce and study the concept of α-small short modules. We shall call an R-module A is called α-small short, if for each small submodule S of A, either $n\text{-dim}\, S \leqslant \alpha$ or $sn\text{-dim}\, \frac{A}{S} \leqslant \alpha$ and α is the least ordinal number with this property. Using this concept we extend some of basic result of α-short modules (i.e., for each submodule B of A, either $n\text{-dim}\, B \leqslant \alpha$ or $n\text{-dim}\, \frac{A}{B} \leqslant \alpha$ and α is the least ordinal number with this property , see also [7] and [9]) to α-small short modules. Let us give a brief sketch of this article. In Section 3, we introduce and study the concept of small Noetherian dimension of an R-module A, and briefly study the small atomic modules. In Section 4, we first introduce and study the concept of α-small short modules and a brief study of α-almost small Noetherian modules. We shall call an R-module A to be α-small short, if for each small submodule S of A, either $n\text{-dim}\, S \leqslant \alpha$ or $sn\text{-dim}\, \frac{A}{S} \leqslant \alpha$ and α is the least ordinal number with this property. Using this concept, we show that if A is an α-small short module, then either $sn\text{-dim}\, A = \alpha$ or $sn\text{-dim}\, A = \alpha + 1$. We also observe that if A is an α-small short module with finite hollow dimension, then $n\text{-dim}\, A \geq \alpha$. Consequently, every submodule of an α-small short module A with finite hollow dimension, where α is countable, is countably generated, see [15, Corollary 1.2]. Finally, in the last section we first study the relationship between Noetherian dimension and small Noetherian dimension. By Theorem 3.19, if A is a module with finite hollow dimension and $sn\text{-dim}\, A = \alpha$, then A has Noetherian dimension and

$n\text{-dim}\, A \leq \alpha + 1$. It is convenient that, when we are dealing with the latter dimensions, to begin our list of ordinals with -1. If an R-module A has Noetherian dimension and α is an ordinal number, then A is called α-conotable if $n\text{-dim}\, A = \alpha$ and $n\text{-dim}\, B < \alpha$ for all proper submodules B of A. An R-module A is called conotable if A is α-conotable for some ordinal α (note, conotable modules are also called atomic, dual critical and N-critical in some other articles, see for example [14, 17, 1, 6]). For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [5, 12, 15].
Throughout this paper, all rings are associative with $1 \neq 0$, and all modules are unital right modules. $B \subseteq A$ (resp., $B \subset A$) will mean B is a submodule (resp., a proper submodule) of A.

2. **Preliminaries**

Let us briefly recall some basic definitions and results from the literature.

Definition 2.1. Let A be an R-module. A proper submodule S of A is small in A if $S+B \neq A$ for every proper submodule B of A. We will indicate that S is a small submodule of A by notation $S \ll A$.

Lemma 2.2. [3, Lemma 2.2] *Let A be a module and suppose $C \subseteq B \subseteq A$ and $D \subseteq A$. Then*

(1) *$B \ll A$ if and only if $C \ll A$ and $\frac{B}{C} \ll \frac{A}{C}$.*
(2) *$D + C \ll A$ if and only if $D \ll A$ and $C \ll A$.*
(3) *If $\varphi : A \to M$ is a homomorphism and $B \ll A$, $\varphi(B) \ll M$.*
(4) *If $C \ll B$, then $C \ll A$.*
(5) *If B is a direct summand of A, $C \ll A$ if and only if $C \ll B$.*
(6) *If $A = A_1 \oplus A_2$ and $S_1 \subseteq A_1$, $S_2 \subseteq A_2$, $S_1 \oplus S_2 \ll A_1 \oplus A_2$ if and only if $S_1 \ll A_1$ and $S_2 \ll A_2$.*
(7) *$Rad(A)$ is the sum of all the small submodules of A.*
(8) *If $S \ll A$, A is finitely generated if and only if $\frac{A}{S}$ is finitely generated.*
(9) *$Soc(Rad(A))$ is small submodule of A. More generally, if B is small in A and $Soc(\frac{Rad(A)}{B}) = \frac{C}{B}$, then C is small in A.*

Definition 2.3. An R-module A is called hollow if $A \neq 0$ and every proper submodule B of A is small in A. Thus a non-zero module A is hollow if and only if sum of its two proper submodules is also a proper submodule.

Example 2.4. Atomic modules and uniserial modules are hollow.

Proposition 2.5. [25, 41.4] *The following statements are equivalent:*

(1) *A is hollow.*

(2) *For some proper submodule B of A, $\frac{A}{B}$ is hollow and $B \ll A$.*

(3) *Every proper factor module of A is indecomposable.*

Definition 2.6. A non-empty family $\{E_i\}_{i\in\ I}$ of proper submodules of an R-module A is called coindependent if, for any $k \in\ I$ and any finite subset $F \subseteq I \setminus \{k\}$, $E_k + \bigcap_{j\in F} E_j = A$

Definition 2.7. Let A be an R-module and $C \subseteq B \subseteq A$. We say B lies above C if $\frac{B}{C} \ll \frac{A}{C}$.

Proposition 2.8. [22, Corollary 13] *For any non-zero module A the following are equivalent:*

(1) *A does not contain an infinite coindependent family of submodules.*

(2) *For some $n \in \mathbb{N}$, A contains a coindependent family of submodules $\{E_1, E_2, ..., E_n\}$ such that $\bigcap_{i=1}^{n} E_i$ is small in A and $\frac{A}{E_i}$ is a hollow module for every $1 \leq i \leq n$.*

(3) $\sup\{k :$ *where k is the cardinality of a coindependent family of submodules in A* $\} = n$, *for some $n \in \mathbb{N}$.*

(4) *For any descending chain $B_1 \supseteq B_2 \supseteq ...$ of submodules of A there exists j, such that B_j lies above B_k in A for all $k \geq j$.*

(5) *There exists a small epimorphism from A to a finite direct sum of n hollow factor modules.*

Next, we give the definition of dual Goldie dimension (i.e., Hollow dimension), see [24, 10].

Definition 2.9. An R-module A is said to have finite hollow dimension if it satisfies one of the conditions in Proposition 2.8. In particular, if A satisfies condition (2) or (3), then A is said to have hollow dimension n, written as $h\dim A = n$.
If $A = 0$, we define $h\dim A = 0$ and if M does not finite hollow dimension we write $h\dim A = \infty$

Lemma 2.10. [20, Lemma 1.4.3] *(The Chinese Remainder Theorem):*
Let A be an R-module. For any coindependent family of subset of submodules $\{E_1, E_2, ..., E_n\}$, $\frac{A}{\bigcap_{i=1}^{n} E_i} \simeq \bigoplus_{i=1}^{n} \frac{A}{E_i}$ holds.

We also cite the following fact from part $(2')$ of the comment which follows [22, Corollary 13].

Proposition 2.11. *If A is an R-module and $h\dim A = n$, then there exists coindependent family of submodules $\{E_1, E_2, ..., E_n\}$, such that $\bigcap_{i=1}^{n} E_i \ll A$ and $\frac{A}{\bigcap_{i=1}^{n} E_i} \simeq \bigoplus_{i=1}^{n} \frac{A}{E_i}$ such that $\frac{A}{E_i}$ is hollow for all $i = 1, 2, ..., n$.*

Definition 2.12. An R-module A is said to have property $AB5^*$(or is said to be an $AB5^*$ module) if for every submodule B and inverse system $\{A_i\}_{i\in I}$ (i.e., for any finite number $i_1, i_2, ..., i_k$ of I, there exists $i_0 \in I$ such that $A_{i_0} \subseteq A_{i_1} \cap ... \cap A_{i_k}$) of submodules of A the following holds:

$B + \bigcap_{i\in I} A_i = \bigcap_{i\in I}(B + A_i)$.

Examples of modules with $AB5^*$ are Artinian modules or linearly compact modules, see [25, 29.8].

3. Dual the small Krull dimension and its properties

In this section, we consider the concept of dual small Krull dimension of an R-module A, which is a Noetherian-like dimension extension of the concept of ACC over small submodules. In other word, it is the codeviation of the poset of small submodules of A.

Next, we give our definition of small Noetherian dimension .

Definition 3.1. Let A be an R-module. The small Noetherian dimension of A denoted by $sn\text{-}\dim A$ is defined by transfinite recursion as follows: If $A = 0$, $sn\text{-}\dim A = -1$. If α is an ordinal number and $sn\text{-}\dim A \not< \alpha$, then $sn\text{-}\dim A = \alpha$ provided there is no infinite ascending chain of small submodules of A such as $S_0 \subseteq S_1 \subseteq S_2 \subseteq ...$ such that for each $i = 1, 2, ...,$ $sn\text{-}\dim \frac{S_{i+1}}{S_i} \not< \alpha$. In otherwise $sn\text{-}\dim A = \alpha$, if $sn\text{-}\dim A \not< \alpha$ and for each chain of small submodules of A such as $S_0 \subseteq S_1 \subseteq S_2 \subseteq ...$ there exists an integer n, such that for each $i \geq n$, $sn\text{-}\dim \frac{S_{i+1}}{S_i} < \alpha$. A ring R has small Noetherian dimension, if as an R-module has small Noetherian dimension. It is possible that there is no ordinal α such that $sn\text{-}\dim A = \alpha$, in this case we say A has no small Noetherian dimension.

If $sn\text{-}\dim A > \alpha$, there exists an infinite ascending chain $S_0 \subseteq S_1 \subseteq S_2 \subseteq ...$ of small submodules of A such that $sk\text{-}\dim \frac{S_{i+1}}{S_i} \geq \alpha$ for all i.

Clearly, $sn\text{-}\dim A = 0$ if and only if A satisfies ACC over its small submodules. Thus, we have the following.

Remark 3.2. Recall that, by [4, Theorem 5], [23, Theorem 2] and above definition, $Rad(A)$ is Noetherian if and only if A satisfies ACC on small submodules if and only if every small submodule of A is Noetherian if and only if $sn\text{-}\dim A = 0$.

The proofs of the following results are just a minor variant of the familiar argument for the small Krull dimension, see [3].

Lemma 3.3. *Let A be an R-module with small Noetherian dimension. Then for each small submodule S of A, $\frac{A}{S}$ has small Noetherian dimension and* $sn\text{-}\dim \frac{A}{S} \leq sn\text{-}\dim A$.

Lemma 3.4. *Let A be an R-module with small Noetherian dimension. Then for each submodule B of A, B has small Noetherian dimension and* $sn\text{-}\dim B \leq sn\text{-}\dim A$.

Lemma 3.5. *Let A be an R-module with Noetherian dimension. Then A has small Noetherian dimension and* $sn\text{-}\dim A \leq n\text{-}\dim A$.

It is well-known that the existence Krull dimension is equivalent to that of Noetherian dimension. Thus it is easy to see that a module A has small Krull dimension if and only if it has small Noetherian dimension. The following example shows that Lemma 3.5 in not true in general, see also [3].

Example 3.6. If we consider $A = (\bigoplus_{i\in\mathbb{N}} \mathbb{Z}) \bigoplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module where p is a prime number, then $Rad(A) = \mathbb{Z}_{p^\infty}$ is an Artinian module. So by Remark 3.2, we have $sk\text{-}\dim A = 0$ and so A has small Noetherian dimension. But A does not have Krull dimension(note, Goldie dimension of A is infinite).

Lemma 3.7. *Let A be an R-module. If for every $S \ll A$, $\frac{A}{S}$ has small Noetherian dimension, then so does A and $sn\text{-}\dim A = \sup\{sn\text{-}\dim \frac{A}{S} | S \ll A\}$.*

Lemma 3.8. *Let A be an R-module. If for every small submodule S of A, either S has Noetherian dimension or $\frac{A}{S}$ has small Noetherian dimension, then A has small Noetherian dimension.*

Lemma 3.9. *If A is an R-module, then for any small submodule S of A, $sn\text{-}\dim A = \sup\{n\text{-}\dim S, sn\text{-}\dim \frac{A}{S}\}$ if either side exists.*

The following result is a dual of [3, Proposition, 3.7].

Proposition 3.10. *An R-module A has small Noetherian dimension if and only if for all $S \ll A$, $n\text{-}\dim S$ exists. In this case $n\text{-}\dim S = sn\text{-}\dim S$ and $sn\text{-}\dim A \leq \sup\{n\text{-}\dim S : S \ll A\} + 1$.*

Proof. If $n\text{-}\dim S$ exists, $sn\text{-}\dim S$ exists and $sn\text{-}\dim S \leq n\text{-}\dim S$, by Lemma 3.5. Now in order to prove the equality, it suffices to prove the converse and show that $n\text{-}\dim S \leq sn\text{-}\dim S$. For this purpose, we proceed by transfinite induction on $sn\text{-}\dim A = \alpha$. If $\alpha = 0$, by Remark 3.2, S is Noetherian, so $n\text{-}\dim S = 0 = sn\text{-}\dim S$. Assume that $\alpha \geq 1$ and the result is true for all ordinals $\beta < \alpha$ and $S_0 \subseteq S_1 \subseteq S_2 \subseteq ...$ is a chain of submodules of S. Hence this is a chain of small submodules of A by Lemma 2.2(1), so there is integer k such that for every $i \geq k$, $sn\text{-}\dim \frac{S_{i+1}}{S_i} = \gamma < \alpha$. Hence, by induction hypothesis, $n\text{-}\dim \frac{S_{i+1}}{S_i}$ exists and $n\text{-}\dim \frac{S_{i+1}}{S_i} = sn\text{-}\dim \frac{S_{i+1}}{S_i} = \gamma < \alpha$, thus S has Noetherian dimension and $n\text{-}\dim S \leq \alpha$. Conversely, we note that the small submodules of A form a set and hence $\sup\{n\text{-}\dim S : S \ll A\}$ exists. Call it α. Given any chain $S_0 \subseteq S_1 \subseteq S_2 \subseteq ...$ of small submodules of A, $sn\text{-}\dim \frac{S_{i+1}}{S_i} \leq n\text{-}\dim \frac{S_{i+1}}{S_i} \leq n\text{-}\dim S_{i+1} = sn\text{-}\dim S_{i+1} \leq \alpha < \alpha + 1$, for all i. Therefore A has s.Noetherian dimension the least than or equal to $\alpha + 1$. □

Next, we give our definition of small atomic modules, which is similar to the concept of atomic modules.

Definition 3.11. An R-module A is called α-small atomic if sn-$\dim A = \alpha$ and for every $S \ll A$, sn-$\dim S < \alpha$. A is called small atomic if it is an α-small atomic for some α.

In view of the Propositions 3.10, 3.9, if A is an α-small atomic, then for every $S \ll A$, sn-$\dim \frac{A}{S} = \alpha$.

We note that an R-module A is 0-small atomic if and only if A has no non-zero small submodule.

We have the following definition.

Definition 3.12. If S is a small submodule of A, then $\frac{A}{S}$ is called a small quotient module of M.

Lemma 3.13. *Let A be an α-small atomic. If $\frac{B}{C}$ is a non-zero small quotient module of a small quotient module $\frac{A}{C}$, then $\frac{A/C}{B/C}$ is α-small atomic.*

Proof. Suppose that A be an α-s.atomic module. For every small quotient module $\frac{A}{C}$, sn-$\dim \frac{A}{C} = sn$-$\dim A = \alpha$. Now for every small submodule $\frac{B}{C}$ of $\frac{A}{C}$, since $B \ll A$, by Lemma 2.2(1), we have sn-$\dim B < \alpha = sn$-$\dim A$ (for A is α-s.atomic). But sn-$\dim \frac{A/C}{B/C} = sn$-$\dim \frac{A}{B} = \alpha = sn$-$\dim \frac{A}{C}$. Moreover, sn-$\dim \frac{B}{C} \leq sn$-$\dim B < sn$-$\dim A = \alpha = sn$-$\dim \frac{A}{C}$. Thus $\frac{A}{C}$ is α-s.atomic. □

Proposition 3.14. *Let A be an R-module with small Noetherian dimension. Then A has a nonzero small quotient module which is small atomic.*

Proof. We assume that every nonzero small quotient module is non-small atomic and seek a contradiction. Let $\frac{A}{S}$ be a nonzero small quotient module of A with the least small Noetherian dimension. Clearly, if $\frac{A}{S}$ has a small quotient module, then so does A, by Lemma 2.2(1). Thus without loss of generality we may assume that A has the least small Noetherian dimension amongst its nonzero small quotient modules. Now let sn-$\dim A = \alpha$, then by our assumption there exists a small submodule, S_1 say of A with sn-$\dim S_1 = \alpha$. But $\frac{A}{S_1}$ is not small atomic and sn-$\dim \frac{A}{S_1} = \alpha$. Now similarly there is a small submodule $\frac{S_2}{S_1}$ of $\frac{A}{S_1}$ with sn-$\dim \frac{S_2}{S_1} = \alpha$. If we repeat this process, we obtain an infinite ascending chain of small submodules $S_1 \subseteq S_2 \subset S_3 \subset ...$ in A with sn-$\dim \frac{S_{i+1}}{S_i} = \alpha$ for each i, which is a contradiction to sn-$\dim A = \alpha$. □

The following lemma shows that a hollow module A with small Noetherian dimension has Noetherian dimension with n-$\dim A \leq sn$-$\dim A + 1$.

Lemma 3.15. *Let A be a hollow module and sn-$\dim A = \alpha$. Then A has Noetherian dimension and n-$\dim A \leq \alpha + 1$.*

Proof. The Proposition 4.8 implies that, A has Noetherian dimension. For every proper submodule B of A, since $B \ll A$, $n\text{-}\dim B = sn\text{-}\dim B \leq \alpha$, by Proposition 3.10 and so $n\text{-}\dim A \leq \sup\{n\text{-}\dim B : B \subseteq A\} + 1 \leq \alpha + 1$. □

We recall that A is called local if it has exactly one maximal submodule that contains all proper submodules. It is clear that every local module and atomic modules are hollow. Therefore by Remark 3.2 and Lemma 3.15, we have the following corollaries.

Corollary 3.16. *Let A be a hollow module with small Noetherian dimension, whose $Rad(A)$ be Noetherian. Then $n\text{-}\dim A \leq 1$ and if $n\text{-}\dim A = 1$, A is a 1-atomic module.*

Corollary 3.17. *Let A be an α-atomic module and $Rad(A)$ be Noetherian. Then $\alpha = 1$, i.e., A is 1-atomic.*

It is well-known that every module with Noetherian dimension has a factor module which is atomic. Thus, we have the following fact.

Corollary 3.18. *Let A be an R-module with Noetherian dimension and $Rad(A)$ be Noetherian. Then A has a factor module which is 1-atomic.*

We may now present the following theorem.

Theorem 3.19. *Let A be an R-module with finite hollow dimension such that $sn\text{-}\dim A = \alpha$. Then A has Noetherian dimension and $n\text{-}\dim A \leq \alpha + 1$.*

Proof. First by Proposition 4.8, A has Noetherian dimension. Let $h\dim A = n$. If $n = 1$ is evident, by Lemma 3.15. We suppose that $n > 1$, so there exists coindependent set $\{E_1, E_2, ..., E_n\}$ such that $E_i \subseteq A$ and $E = \bigcap_{i=1}^{n} E_i$ is small in A and $\frac{A}{E_i}$ is hollow for every $i = 1, 2, ..., n$, by Propositions 2.8, 2.11, we get $\frac{A}{E} \simeq \bigoplus_{i=1}^{n} \frac{A}{E_i}$. Since $\frac{A}{E_i}$ is hollow for all i with s.Noetherian dimension(note, since $\frac{A}{E}$ has small Noetherian dimension, thus $\frac{A}{E_i}$ has small Noetherian dimension) and by Lemma 3.15, $n\text{-}\dim \frac{A}{E_i} \leq sn\text{-}\dim \frac{A}{E_i} + 1 \leq sn\text{-}\dim A + 1 = \alpha + 1$ and $n\text{-}\dim E = sn\text{-}\dim E$. But $n\text{-}\dim A = \sup\{n\text{-}\dim E, n\text{-}\dim \frac{A}{E}\} = \sup\{n\text{-}\dim E, n\text{-}\dim \frac{A}{E_1}, ..., n\text{-}\dim \frac{A}{E_n}\} \leq \sup\{\alpha, \alpha + 1\} = \alpha + 1$. That is $n\text{-}\dim A \leq \alpha + 1$. □

By Lemma 3.5, the following result is evident.

Corollary 3.20. *Let A be an R-module with finite hollow dimension such that $sn\text{-}\dim A = \alpha$. Then $\alpha \leq n\text{-}\dim A \leq \alpha + 1$.*

Corollary 3.21. *Let A be an R-module whose Noetherian dimension is a limit ordinal. Then $n\text{-}\dim A = sn\text{-}\dim A$.*

Motivated by Corollary 3.21, it would be interesting to characterize modules A with $n\text{-}\dim A = sn\text{-}\dim A$ (resp., $n\text{-}\dim A = sn\text{-}\dim A + 1$).

In view of Theorem 3.19 and Proposition 2.8, the following is evident.

Corollary 3.22. *Let A be an Artinian R-module and $sn\text{-}\dim A = \alpha$. Then $\alpha \leq n\text{-}\dim A \leq \alpha + 1$.*

4. α-small short modules

In this section, we introduce and study the concept of small short modules. Using this concept we extend some of basic result of α-short modules, see [7], to α-small short modules. We begin with the following definition.

Definition 4.1. An R-module A is called α-small short if for each small submodule S of A, either $n\text{-}\dim S \leqslant \alpha$ or $sn\text{-}\dim \frac{A}{S} \leqslant \alpha$ and α is the least ordinal number with this property.

Clearly, if A is a -1-small short module, then A has not nonzero small submodule.

In view of Lemmas 3.3, 3.4, we have the following results.

Remark 4.2. If A is an R-module with $sn\text{-}\dim A = \alpha$, then A is a β-small short module for some $\beta \leqslant \alpha$.

Remark 4.3. If A is an α-small short module, then each submodule and each small quotient module of A is β-small short for some $\beta \leqslant \alpha$.

In view of the Remark 4.2 and Lemma 4.18, we have the next result.

Lemma 4.4. *Let A be an α-small short module. Then A has small Noetherian dimension and $sn\text{-}\dim A \geq \alpha$.*

The following is now immediate.

Lemma 4.5. *An R-module A has small Noetherian dimension if and only if A is α-small short for some ordinal number α.*

Proposition 4.6. *If A is an α-small short module, then either $sn\text{-}\dim A = \alpha$ or $sn\text{-}\dim A = \alpha + 1$.*

Proof. In view of Lemma 4.4, we have $sn\text{-}\dim A \geqslant \alpha$. If $n\text{-}\dim A \neq \alpha$, then $sn\text{-}\dim A \geqslant \alpha + 1$. Let $A_1 \subseteq A_2 \subseteq ...$ be any ascending chain of small submodules of A. If there exists some n such that $sn\text{-}\dim \frac{A}{A_n} \leqslant \alpha$, then by Lemmas 3.3, 3.4, 2.2(1), $sn\text{-}\dim \frac{A_{i+1}}{A_i} \leqslant sn\text{-}\dim \frac{A}{A_i} = sn\text{-}\dim \frac{A/A_n}{A_i/A_n} \leqslant sn\text{-}\dim \frac{A}{A_n} \leqslant \alpha$ for each $i \geqslant n$. Otherwise $sn\text{-}\dim A_i \leqslant \alpha$ for each i, hence $sn\text{-}\dim \frac{A_{i+1}}{A_i} \leqslant n\text{-}\dim \frac{A_{i+1}}{A_i} \leqslant n\text{-}\dim A_{i+1} = sn\text{-}\dim A_{i+1} \leq \alpha$ for each i, by Proposition 3.10.

Thus in any case there exists an integer k such that for each $i \geq k$, $sn\text{-}\dim \frac{A_{i+1}}{A_i} \leqslant \alpha$. This shows that $sn\text{-}\dim A \leqslant \alpha + 1$, i.e., $sn\text{-}\dim A = \alpha + 1$. □

By Proposition 4.6 and Remark 3.2, we have the following result.

Corollary 4.7. *If A is a 0-small short module, then either $sn\text{-}\dim A = 1$ or $Rad(A)$ is Noetherian. Also, if A is -1-small short, then either $A = 0$ or $Rad(A)$ is Noetherian.*

The proof of the following proposition is similar to the proof of its dual in [3, Theorem 4.10].

Proposition 4.8. *Let A be an R-module with finite hollow dimension. Then A has Noetherin dimension if and only if it has small Noetherin dimension.*

In view of Lemma 4.4 and Proposition 4.8, the following result is evident.

Corollary 4.9. *If A is an α-small short module with finite hollow dimension, then $n\text{-}\dim A \geq \alpha$.*

We also cite the following fact from [15, Corollary 1.2].

Corollary 4.10. *Every submodule of an α-small short module A with finite hollow dimension, where α is countable, is countably generated.*

It is well-known that every module with Noetherian dimension has finite Goldie dimension, see [21, Lemma 6.2.6]. Thus we have the following result.

Corollary 4.11. *Every α-small short module with finite hollow dimension has finite Goldie dimension.*

Proposition 4.12. *Let A be an R-module, with $sn\text{-}\dim A = \alpha$, where α is a limit ordinal. Then A is α-small short.*

Proof. We know that A is β-small short for some $\beta \leqslant \alpha$. If $\beta < \alpha$, then by Proposition 4.6, $sn\text{-}\dim A \leqslant \beta + 1 < \alpha$, which is impossible. Thus A is α-small short. □

Proposition 4.13. *Let A be an R-module and $sn\text{-}\dim A = \alpha$, where $\alpha = \beta + 1$. Then A is either α-small short or β-small short.*

Proof. By Remark 4.2, A is γ-small short for some $\gamma \leqslant \alpha$. If $\gamma < \beta$, then Proposition 4.6 implies that $sn\text{-}\dim A \leqslant \gamma + 1 < \beta + 1$, which is a contradiction. Hence we are done. □

For the small atomic modules, we have the following proposition.

Proposition 4.14. *Let A be an α-small atomic R-module, where $\alpha = \beta + 1$, then A is a β-small short module.*

Proof. If $S \ll A$, by definition $sn\text{-}\dim S < \alpha$. Therefore $sn\text{-}\dim S \leqslant \beta$. It follows that A is δ-s.short for some $\delta \leqslant \beta$. If $\delta < \beta$, then $\delta + 1 \leqslant \beta < \alpha$. But $sn\text{-}\dim A \leqslant \delta + 1 \leqslant \beta < \alpha$, by Proposition 4.6, which is a contradiction. Thus $\delta = \beta$ and we are through. □

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 4.12, is not true in general.

Remark 4.15. Let A be an $\alpha + 1$-small atomic R-module, where α is a limit ordinal. Then A is an α-small short module but $sn\text{-}\dim A \neq \alpha$.

Proposition 4.16. *Let A be an R-module such that $sn\text{-}\dim A = \alpha + 1$. Then A is either an α-small short module or there exists a small submodule S of A such that $sn\text{-}\dim S = sn\text{-}\dim \frac{A}{S} = \alpha + 1$.*

Proof. We know that A is α-small short or an $\alpha+1$-small short R-module, by Proposition 4.13. Let us assume that A is not an α-small short module, hence there exists a small submodule S of A such that $sn\text{-}\dim S \geqslant \alpha + 1$ and $sn\text{-}\dim \frac{A}{S} \geqslant \alpha + 1$. This shows that $sn\text{-}\dim S = \alpha + 1$ and $sn\text{-}\dim \frac{A}{S} = \alpha + 1$ and we are through. □

Definition 4.17. An R-module A is called α-almost small Noetherian, if for each small submodule S of A, $n\text{-}\dim S < \alpha$ and α is the least ordinal number with this property.

In view of Lemma 3.10, we have the next results.

Lemma 4.18. *If A is an α-almost small Noetherian module, then A has small Noetherian dimension and $sn\text{-}\dim A \leqslant \alpha$. In particular, $sn\text{-}\dim A = \alpha$ if and only if A is α-small atomic.*

It is easy to see that if A is an α-almost small Noetherian, then each submodule of A is β-almost small Noetherian for some $\beta \leq \alpha$, by Lemmas 3.7, 3.9.

Lemma 4.19. *If A is a module with $sn\text{-}\dim A = \alpha$, then either A is α-small atomic, in which case it is α-almost small Noetherian, or it is $\alpha + 1$-almost small Noetherian.*

Lemma 4.20. *If A is an α-almost small Noetherian module, then either A is α-atomic or $\alpha = sn\text{-}\dim A + 1$. In particular, if A is an α-almost small Noetherian module, where α is a limit ordinal, then A is α-atomic.*

The following is now immediate.

Proposition 4.21. *An R-module A has small Noetherian dimension if and only if A is α-almost small Noetherian for some ordinal α.*

5. Properties of α-small short modules

In this section some properties of α- small short modules, α-almost small Noetherian modules over an arbitrary ring R are investigated.

The following result is a connection between α-short modules and α-small short modules.

Proposition 5.1. *Let A be an α-short module. Then A is a γ-small short module such that $\alpha \in \{\gamma, \gamma+1, \gamma+2\}$.*

Proof. Since every α-short module has Noetherian dimension, hence A has small Noetherian dimension. Let B be a small submodule of A, hence either $sn\text{-}\dim B = n\text{-}\dim B \leq \alpha$ or $sn\text{-}\dim \frac{A}{B} \leq n\text{-}\dim \frac{A}{B} \leq \alpha$, by Lemmas 3.4, 3.5. It follows that A is γ-small short for some $\gamma \leq \alpha$. But if A is γ-small short, then either $sn\text{-}\dim A = \gamma$ or $sn\text{-}\dim A = \gamma+1$. By Corollary 3.20, $\gamma \leq n\text{-}\dim A \leq \gamma+2$. From [7, Proposition 1.12], we obtain $\alpha \leq n\text{-}\dim A \leq \alpha+1$. Thus $\alpha \in \{\gamma, \gamma+1, \gamma+2\}$. □

Proposition 5.2. *Let A be an γ-small short module with hollow dimension. Then A is a α-short module such that $\alpha \in \{\gamma, \gamma+1, \gamma+2\}$.*

Proof. By Proposition 4.6, either $sn\text{-}\dim A = \gamma$ or $sn\text{-}\dim A = \gamma+1$. Corollary 3.20, gives that $\gamma \leq n\text{-}\dim A \leq \gamma+2$. Hence A is α-short for some ordinal number α, [7, Remark 1.2]. Therefore $\alpha \in \{\gamma, \gamma+1, \gamma+2\}$ and we are done. □

From Proposition 5.2 and Proposition 4.8, we get the following result.

Corollary 5.3. *Let A be an R-module with finite hollow dimension. If A is α-small short, then A has Noetherian dimension and $\alpha \leq n\text{-}\dim A \leq \alpha+2$.*

By Propositions 5.1 , 5.2, the following is evident.

Corollary 5.4. *Let A be an R-module, which is with finite hollow dimension and α be a limit ordinal. Then A is α-short if and only if is α-small short.*

Proposition 5.5. *Let S be a submodule of an R-module A such that S is α-small short and $\frac{A}{S}$ is β-small short. Let $\mu = \sup\{\alpha, \beta\}$, then A is γ-small short such that $\mu \leqslant \gamma \leqslant \mu+1$.*

Proof. Since B is α-small short, thus by Proposition 4.6, $sn\text{-}\dim S = \alpha$ or $sn\text{-}\dim S = \alpha+1$. Similarly since $\frac{A}{S}$ is β-small short, $sn\text{-}\dim \frac{A}{S} = \beta$ or $sn\text{-}\dim \frac{A}{S} = \beta+1$. We infer that A has small Noetherian dimension and $sn\text{-}\dim A = \sup\{sn\text{-}\dim S, sn\text{-}\dim \frac{A}{S}\}$, by Lemma 3.9. Therefore $\mu \leqslant sn\text{-}\dim A \leqslant \mu+1$. But by Remark 4.2, A is γ-small short for some ordinal number γ and by Proposition 4.6, $\gamma \leqslant sn\text{-}\dim A \leqslant \gamma + 1$. It follows that $\gamma = \mu$, or $\gamma = \mu + 1$ (note, we always have $\mu \leqslant \gamma$) and we are done. □

By Lemma 4.18, we have the following result which is the counterpart of Proposition 5.5, for α-almost small Noetherian modules.

Proposition 5.6. *Let S be a small submodule of an R-module A such that S is α-almost small Noetherian and $\frac{A}{S}$ is β-almost small Noetherian. Let $\mu = \sup\{\alpha, \beta\}$, then A is γ-almost small Noetherian such that $\mu \leqslant \gamma \leqslant \mu + 1$.*

Corollary 5.7. *Let A_1 is an α_1-small short(resp., α_1-almost small Noetherian) module and A_2 is an α_2-small short(resp., α_2-almost small Noetherian) module and $\alpha = \sup\{\alpha_1, \alpha_2\}$. Then $A_1 \oplus A_2$ is γ-small short(resp., γ-almost small Noetherian), for some ordinal number γ such that $\alpha \leq \gamma \leq \alpha + 1$.*

Proposition 5.8. *Let A be a nonzero α-small short module, which is not a small atomic module. Then A contains a small submodule S such that $sn\text{-}\dim \frac{A}{S} \leq \alpha$*

Proof. Since A is not small atomic, we conclude that there exists a small submodule $S \subset A$, such that $sn\text{-}\dim S = sn\text{-}\dim A$. The Proposition 4.6, implies that either $sn\text{-}\dim A = \alpha$ or $sn\text{-}\dim A = \alpha + 1$. If $sn\text{-}\dim A = \alpha$, then $sn\text{-}\dim \frac{A}{S} \leq \alpha$. Therefore we may assume that $sn\text{-}\dim A = \alpha + 1$. If $sn\text{-}\dim \frac{A}{S} = \alpha + 1$, then A is β-small short for some $\beta \geq \alpha + 1$, which is a contradiction. Thus $sn\text{-}\dim \frac{A}{S} \leq \alpha$. □

Proposition 5.9. *If for each small submodule S of A, $\frac{A}{S}$ is γ-small short for some ordinal number $\gamma \leq \alpha$. Then $sn\text{-}\dim A \leq \alpha + 1$, in particular, if A has finite hollow dimension, then A is μ-short for some ordinal $\mu \leq \alpha + 1$.*

Proof. Let S be any small submodule of A. Thus $\frac{A}{S}$ is γ-small short for some ordinal number $\gamma \leq \alpha$. By proposition 4.6, we infer that $sn\text{-}\dim \frac{A}{S} \leq \gamma + 1 \leq \alpha + 1$. Hence $sn\text{-}\dim A = \sup\{sn\text{-}\dim \frac{A}{S} | S \ll A\} \leq \alpha + 1$, by Lemma 3.7 and we are done. The final part is now evident.
□

The following result is the counterpart of Propositions 5.8, for α-almost small Noetherian modules.

Proposition 5.10. *If for each small submodule S of A, S (resp., $\frac{A}{S}$) is γ-almost small Noetherian for some ordinal number $\gamma \leq \alpha$. Then $sn\text{-}\dim A \leq \alpha + 1$. In particular, A is μ-almost small Noetherian for some ordinal $\mu \leq \alpha + 1$ (resp., $sn\text{-}\dim A \leq \alpha + 1$ and A is a μ-almost small Noetherian for some ordinal $\mu \leq \alpha + 1$).*

Before concluding this section with our last observation, let us cite the next result which is in [14, Theorem 2.9], see also [13, Theorem 3.2].

Theorem 5.11. *For a commutative ring R the following statements are equivalent.*

(1) *Every R-module with finite Noetherian dimension is Noetherian.*
(2) *Every Artinian R-module is Noetherian.*
(3) *Every R-module with Noetherian dimension is both Artinian and Noetherian.*

In view of Proposition 4.6 and Corollary 5.3, we have the following result.

Proposition 5.12. *Let R be a commutative ring. The following statements are equivalent.*

(1) *Every Artinian R-module is Noetherian.*
(2) *Every m-small short module with finite hollow dimension, is both Artinian and Noetherian for all integers $m \geq -1$.*
(3) *Every α-small short module with finite hollow dimension, is both Artinian and Noetherian for all ordinals α.*
(4) *Every m-almost small short module with finite hollow dimension, is both Artinian and Noetherian for all integers $m \geq -1$.*
(5) *Every α-almost small short module with finite hollow dimension, is both Artinian and Noetherian for all ordinals α.*

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