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R-NOTION OF CONJUGACY IN PARTIAL AND FULL INJECTIVE TRANSFORMATIONS

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Abstract. In this paper, we define a new notion of conjugacy in semigroups that reduces to the n-notion of conjugacy in an inverse semigroup. We compare our new notion with the existing notions. We characterize the notion in partial injective and in full injective transformations, and determine the conjugacy classes in these semigroups.

1. INTRODUCTION

The concept of conjugacy is essential as far as group theory is concerned. More importantly most of the famous results on finite groups involve the use of conjugacy in their proofs. Semigroups are a generalizations of groups, and the theory of semigroups has evolved as a result of generalizing the results of groups to semigroups. Like other notions of groups, it becomes natural to try to generalize the notion of conjugacy from groups to semigroups. Since the definition of conjugacy in a group involves the existence of inverses, the apparent choice for

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elements $a, b \in S$, where S is a semigroup, to be conjugate of each other is the existence of an element $g \in S^1$ (a semigroup obtained by adjoining identity 1) such that $ag = gb$. However, unlike groups, this relation is not necessarily transitive in an arbitrary semigroup. This prompted semigroup theorists to search for the best suitable notions of conjugacy, and as a result, various notions of conjugacy have been studied so far.

Before introducing the various notions of conjugacy, we recall some of the semigroup theoritical notions that will require in subsequent sections. We refer the reader to Howie [4] for any unexplained terminology in semigroups.

A semigroup S is called an *inverse semigroup* if for every $a \in S$, there is a unique $a^{-1} \in S$ (called the inverse of a) such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

For a non-empty set X, $\mathcal{P}(X)$ denotes the set of all *partial tranformations* on X and $\mathcal{T}(X)$ denotes the set of all *full tranformations* on X. We denote by $\mathcal{I}(X)$ the *symmetric inverse semigroup* on X, which is the subsemigroup of $\mathcal{P}(X)$ consisting of all partial injective transformations on X. We denote by $\mathcal{F}(X)$ the subsemigroup of $\mathcal{I}(X)$ consisting of all full injective transformations on X and by $\mathrm{sym}(X)$ the subsemigroup of $\mathcal{F}(X)$ consisting of all bijections on X. Next, we will introduce various notions of conjugacy.

Let G be a group. For $x, y \in G$, we say x is conjugate to y if there exists $p \in G$ such that $y = p^{-1}xp$, which is equivalent to $xp = py$. Due to this fact $\sim_l$ notion was introduced in a semigroup S defined as

$$x \sim_l y \Leftrightarrow \exists\ p \in S^1 \text{ such that } xp = py$$

where S^1 is S with an identity adjoined. If $x \sim_l y$, we say x is left conjugate to y [10, 12, 13]. The relation $\sim_l$ is always reflexive and transitive in any semigroup but not symmetric in general. Lallement [9] has defined the conjugate elements of a free semigroup S as those related by $\sim_l$ and showed that $\sim_l$ is equal to the following equivalence on the free semigroup S:

$$x \sim_p y \Leftrightarrow \exists\ u, v \in S^1 \text{ such that } x = uv \text{ and } y = vu$$

The relation $\sim_p$ is always reflexive and symmetric but not transitive in general.

The relation $\sim_l$ has been restricted to $\sim_o$ [10], and $\sim_p$ has been extended to $\sim_p^*$ [7, 8], in such a way that the modified relations are equivalences on an arbitrary semigroup S:

$$x \sim_o y \Leftrightarrow \exists\ p, q \in S^1 \text{ such that } xp = py \text{ and } yq = qx.$$

$\sim_p^*$ is the transitive closure of $\sim_p$. The relation $\sim_o$ is not useful for semigroups S with zero since for every such S, we have $\sim_o = S \times S$. This deficiency has been remedied in [3], where the following relation has been defined on an arbitrary semigroup S,

$$x \sim_c y \Leftrightarrow \exists p \in \mathbb{P}^1(x), q \in \mathbb{P}^1(y) \text{ such that } xp = py \text{ and } yq = qx,$$

where for $x \neq 0$, $\mathbb{P}(x) = \{p \in S : (mx)p \neq 0 \text{ for all } mx \in S^1x \setminus \{0\}\}$, $S^1x \setminus \{0\}$ denotes the left principal ideal generated by x and $\mathbb{P}(0) = \{1\}$. The relation $\sim_c$ is an equivalence relation and it does not reduce to $S \times S$ if S has a zero, and it is equal to $\sim_o$ if S does not have a zero.

Furthermore, J. Konieczny in [6] introduced the $\sim_n$ notion of conjugacy in semigroups. If S is a semigroup and let $x, y \in S$. Then,

$$x \sim_n y \Leftrightarrow \exists\ p, q \in S^1 \text{ such that } xp = py, yq = qx, x = pyq \text{ and } y = qxp.$$

This relation is an equivalence relation in any semigroup and does not get reduced to a universal relation in a semigroup with zero.

The aim of this paper is to introduce a new definition of conjugacy in an arbitrary semigroup. The new notion $\sim_r$ is an equivalence relation in any semigroup and does not get reduced to a universal relation in a semigroup with zero. The beauty of r-notion is due to the following properties.

(1) It contains $\sim_n$ notion of conjugacy. i.e, $\sim_n \subseteq \sim_r$.
(2) It coincides with $\sim_n$ and $\sim_i$ notion of conjugacy in an inverse semigroup.
(3) Unlike the n-notion of conjugacy, where there are only two conjugators, we have more freedom in r-notion; the number of conjugators is four, here.

2. The notion $\sim_r$ of conjugacy

Let S be a semigroup and let $a, b \in S$. Then,

$$x \sim_r y \Leftrightarrow \exists\ p, q, u, v \in S^1 \text{ such that } xp = py, yq = qx, x = pyu \text{ and } y = qxv.$$

In the following result we show that r-notion is an equivalence relation in any semigroup and it does not get reduced to a universal relation in a semigroup with zero.

Theorem 2.1. *If S is a semigroup, then*

(1) *$\sim_r$ is an equivalence relation in any semigroup.*
(2) $[0]_r = \{0\}$.
(3) *If S is a group, then $\sim_r$ reduces to the usual notion of conjugacy.*

Proof. (1) Let $x \sim_r y$ then there exist $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$.

(i) **Reflexivity:** We take $p = q = u = v = 1$, and we get the required result.
(ii) **Symmetry:** This follows by definition.
(iii) **Transitivity:** Let $x \sim_r y$ and $y \sim_r z$. Then there exist p_1, q_1, u_1, v_1 and p_2, q_2, u_2, v_2 such that $xp_1 = p_1y, yq_1 = q_1x, x = p_1yu_1$ and $y = q_1xv_1$ and $yp_2 = p_2z, zq_2 = q_2y, y = p_2zu_2$ and $z = q_2yv_2$. Now $ap_1p_2 = p_1yp_2 = p_1p_2z$, $zq_2q_1 = q_2yq_1 = q_2q_1x$, $x = p_1yu_1 = p_1p_2zu_2u_1$ and $z = q_2yv_2 = q_2q_1xv_1v_2$. Hence $x \sim_r z$.

(2) Let $x \neq 0$ and let $x \sim_r 0$. Then there exist $p, q, u, v \in S^1$ such that $xp = p0, 0q = qx, x = p0u$ and $0 = qxv$. This means $x = 0$. So we get $[0]_r = \{0\}$.

(3) Let $x \sim_r y$. Then there exist $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$. From $xp = py$, we can pre-multiply by p^{-1} on both sides to get $y = g^{-1}xg$, which is the usual notion of conjugacy.□

In the next result, we compare the r-notion with the notions $\sim_n, \sim_c$ and $\sim_o$.

Theorem 2.2. *Let S be semigroup. Then* $\sim_n \subseteq \sim_r \subseteq \sim_c \subseteq \sim_o$.

Proof. Let $x \sim_n y$. Then there exist $p, q \in S^1$ such that $xp = py, yq = qx, x = pyq$ and $y = qxp$. we can take $u = q$ and $v = p$ so that we get $x \sim_r y$. Thus $\sim_n \subseteq \sim_r$. Next we prove $\sim_r \subseteq \sim_c$. If $x = 0$ then $y = 0$ since $[0]_r = 0$. Suppose $x \neq 0$ and let $x \sim_r y$. Then there exist $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$. Now let $m \in S^1$ be such that $mx \neq 0$. Then $(mx)p \neq 0$ since if $(mx)p = 0$ then $mpy = 0$ which implies $mpyu = 0$. This implies $mx = 0$, which is a contradiction. Hence $(mx)p \neq 0$. Similarly, if $m \in S^1$ is such that $my \neq 0$ then $(my)q \neq 0$. So, $p \in \mathbb{P}^1(x)$ and $q \in \mathbb{P}^1(y)$. Hence $x \sim_c y$. Since $\sim_c \subseteq \sim_o$ is obvious. Hence we get the required result.□

Let S be an inverse semigroup and let $x, y \in S$. Then $x \sim_i y$ if there exists $p \in S^1$ such that $x = pyp^{-1}$ and $y = p^{-1}xp$.

Theorem 2.3. [6, Theorem 2.6] *Let S be an inverse semigroup and let $a, b \in S$. Then $a \sim_n b$ if and only if there exists $g \in S^1$ such that $g^1ag = b$ and $gbg^1 = a$.*

The semigroup $\mathcal{I}(X)$ is universal for the class of inverse semigroups because of the Vagner-Preston theorem, which states that every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [4, Theorem 5.1.7]. This is analogous to the Cayley theorem for groups, which states that every group can be embedded in some symmetric group $\mathrm{Sym}(X)$.

We now prove that $\sim_n$ reduces to $\sim_r$ in inverse semigroups.

Theorem 2.4. *Let S be an inverse semigroup and let $a, b \in S$. Then $a \sim_r b$ if and only if $a \sim_n b$.*

Proof. By Theorem 2.2, $\sim_n \subseteq \sim_r$. So $a \sim_n b$ implies $a \sim_r b$.

For the converse, we may assume by the Vagner-Preston theorem that S is a subsemigroup of some symmetric inverse semigroup $\mathcal{I}(X)$. Let $a \sim_r b$. Then there exists $g, h, u, v \in S^1$ such that

$$ag = gb, bh = ha, a = gbu \text{ and } b = hav.$$

We claim $agg^{-1} = a$. Clearly $\mathrm{dom}(agg^{-1}) \subseteq \mathrm{dom}(a)$. Let $x \in \mathrm{dom}(a)$ implies $xa \in \mathrm{im}(a) \subseteq \mathrm{dom}(g)$ implies $(xa) \in \mathrm{dom}(g)$, which implies $(xa)g \in \mathrm{dom}(g^{-1})$. Hence $x \in \mathrm{dom}(agg^{-1})$, which implies $\mathrm{dom}(a) \subseteq \mathrm{dom}(agg^{-1})$. Thus $\mathrm{dom}(a) = \mathrm{dom}(agg^{-1})$. Next for every $x \in \mathrm{dom}(a)$, $x(agg^{-1}) = (xa)gg^{-1} = xa$. So $agg^{-1} = a$. Since $ag = gb$ implies $agg^{-1} = gbg^{-1}$ and so $a = gbg^{-1}$.

Next we claim that $g^{-1}gb = b$. We have

$$\begin{aligned}
&g^{-1}gb \neq b \\
\Rightarrow\ & g^{-1}ag \neq b \\
\Rightarrow\ & g^{-1}agg^{-1} \neq bg^{-1} \\
\Rightarrow\ & g^{-1}a \neq bg^{-1} \\
\Rightarrow\ & g^{-1}gbu \neq bg^{-1} \\
\Rightarrow\ & gg^{-1}gbu \neq gbg^{-1} \\
\Rightarrow\ & gbu \neq gbg^{-1} \\
\Rightarrow\ & a \neq gbg^{-1}
\end{aligned}$$

which is a contradiction. Hence $g^{-1}gb = b$. Since $ag = gb$, we have $g^{-1}ag = g^{-1}gb$ we have $g^{-1}ag = b$. Thus $a \sim_i b$ and so by Theorem 2.3, $a \sim_n b$.□

By Theorem 2.3 and Theorem 2.4, we have $\sim_n = \sim_r = \sim_i$ in $\mathcal{I}(X)$.

3. $\sim_r$ NOTION OF CONJUGACY IN PARTIAL INJECTIVE TRANSFORMATIONS $\mathcal{I}(X)$

Let X be any set and let R be a binary relation on X. Then $\Gamma = (X, R)$ is called a *directed graph* (or a *digraph*). We call any $x \in X$ a vertex and any $(x, y) \in R$ an arc of Γ. For example, Let $X = \{a, b, c, d, e, f\}$ and $R = \{(a, e), (b, f)\}$. Then the digraph Γ is as follows,

$$\overset{a}{\bullet} \longrightarrow \overset{e}{\bullet}\ \overset{b}{\bullet} \longrightarrow \overset{f}{\bullet}\ \overset{c}{\bullet}\ \overset{d}{\bullet}.$$

For any $\sigma \in \mathcal{I}(X)$, $\Gamma(\sigma) = (X, R_\sigma)$ represents a digraph, where for all $x, y \in X$, $(x, y) \in R_\sigma$ if and only if $x \in \mathrm{dom}(\sigma)$ and $x\sigma = y$. For example, If $X = \{1, 2, 3\}$ and $R_\sigma = \{(1, 2), (2, 1)\}$. Then the digraph $\Gamma(\sigma)$ is represented as

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}\ .$$

A vertex $x \in X$ for which there is no y in X such that $(x, y) \in R$ is called a *terminal vertex* of Γ. A vertex $x \in X$ is said to be an initial vertex if there is no $y \in X$ for which $(y, x) \in R$. A vertex $x \in X$ is said to be a non-initial vertex if $(y, x) \in R$ for some $y \in X$.

Let $\Gamma_1 = (X_1, R_1)$ and $\Gamma_2 = (X_2, R_2)$ be digraphs. A mapping φ from X_1 to X_2 is called a *homomorphism* from Γ_1 to Γ_2 if for all $x, y \in X_1, (x, y) \in R_1$ implies $(\mathrm{x}\varphi,\ \mathrm{y}\varphi) \in R_2$.

A partial mapping φ from X_1 to X_2 is called a *partial homomorphism* from Γ_1 to Γ_2 if for all $x, y \in \mathrm{dom}(\varphi)$, $(x, y) \in R_1$ implies $(x\varphi, y\varphi) \in R_2$.

Definition 3.1. A partial homomorphism φ from X_1 to X_2 is said to be a *restrictive partial homomorphism* from Γ_1 to Γ_2 if the following hold:

(a) If $(x, y) \in R_1$, then $x, y \in \mathrm{dom}(\varphi)$ and $(x\varphi, y\varphi) \in R_2$.

(b) If x is a terminal vertex in Γ_1 and $x \in \mathrm{dom}(\varphi)$, then $x\varphi$ is a terminal vertex in Γ_2.

We say that Γ_1 is *rp-homomorphic* to Γ_2 if there is an rp-homomorphism from Γ_1 to Γ_2.

For any $\sigma \in \mathcal{P}(X)$ span(σ) represents $\mathrm{dom}(\sigma) \cup \mathrm{im}(\sigma)$. By $\sigma \neq 0$ we mean $\mathrm{dom}(\sigma) \neq \emptyset$.

For any semigroups S and T, by $S \leq T$ we mean S is a subsemigroup of T.

Theorem 3.2. [11, Theorem 3.5] *Let $S \leq \mathcal{P}(X)$ and $\sigma, \tau \in S$. Then $\sigma \sim_r \tau$ if and only if there are $\alpha, \beta, \varphi, \psi \in S^1$ for which α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

If $\sigma, \tau \in \mathcal{T}(X)$, then every homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ is an rp-homomorphism. So we have the following corollary.

Corollary 3.3. [11, Corollary 3.6] *Let $S \leq \mathcal{T}(X)$ and $\sigma, \tau \in S$. Then $\sigma \sim_r \tau$ if and only if there are $\alpha, \beta, \varphi, \psi \in S^1$ such that α is a homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is a homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Definition 3.4. Let $\cdots, p_{-2}, p_{-1}, p_0, p_1, p_2, \cdots$ be pairwise distinct elements of T. Then

(1) A $\delta \in \mathcal{P}(X)$ is called a *cycle* of length k if $\delta = (p_0 p_1 p_2 \cdots p_{k-1})$ where $(k \geq 1)$. i.e., $p_j = p_{j-1}\delta$, $j = 1, 2, \cdots, k$ and $p_0 = p_{k-1}\sigma$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots \to p_{k-1} \to p_0.$$

(2) A $\nu \in \mathcal{P}(X)$ is called a *right ray* if $\nu = [p_0\ p_1\ p_2\ \cdots >$. i.e., $p_j = p_{j-1}\nu$, $j \geq 1$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots .$$

(3) A $\omega \in \mathcal{P}(X)$ is called a *double ray* if $\omega =< \cdots p_{-1}\ p_0\ p_1\ \cdots >$. i.e., $p_j = p_{j-1}\omega$, $j \in \mathbb{Z}$ and we write it as

$$\cdots \to p_{-1} \to p_0 \to p_1 \to p_2 \to \cdots .$$

(4) A $\lambda \in \mathcal{P}(X)$ is called a *left ray*, if $\lambda =< \cdots\ p_2\ p_1\ p_0]$. i.e., $p_j\lambda = p_{j-1}$, $j \geq 1$ and we write it as

$$\cdots \to p_2 \to p_1 \to p_0.$$

(5) A $\theta \in \mathcal{P}(X)$ is called a *chain* of length k if $\theta = [p_0\ p_1\ p_2\ \cdots\ p_k]$. i.e., $p_j = p_{j-1}\theta$, $j = 1, 2, \cdots, k$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots \to p_k.$$

These are called basic partial maps.

Let $\sigma \in \mathcal{P}(X)$ and let α be a basic partial map with $\alpha \subset \sigma$. Then α is *maximal* in σ if $x \notin \text{dom}(\alpha)$ implies $x \notin \text{dom}(\sigma)$ and $x \notin \text{im}(\alpha)$ implies $x \notin \text{im}(\sigma)$ for every $x \in \text{span}(\alpha)$.

For example, let $\sigma = [p\ q\ r\ s \cdots > \cup\ [a\ b\ c\ p] \in \mathcal{P}(\mathbb{Z})$. Then σ contains infinitely many right rays. For example, $[c\ p\ q\ r \cdots >$ but only two of them, namely $[p\ q\ r\ s \cdots >$ and $[a\ b\ c\ p\ q\ r\ s \cdots >$ are maximal in σ.

For any $\eta \in \{\delta, \theta, \omega, \nu, \lambda\}$ and any $\varphi \in \mathcal{I}(X)$ such that $\text{span}(\eta) \subseteq \text{dom}(\varphi)$, we define $\eta\varphi^*$ to be η in which p_i has been replaced with $p_i\varphi$. For example,

$$\delta\varphi^* = (p_0\varphi p_1\varphi \cdots p_{k-1}\varphi) \text{ and } \lambda\varphi^* =< \cdots p_2\varphi p_1\varphi p_0\varphi]$$

Consider $\theta = [p_0p_1 \cdots p_k]$, $\omega =< \cdots p_{-1}p_0p_1 \cdots >$, $\nu = [p_0p_1p_2 \cdots >$, and $\lambda =< \cdots p_2p_1p_0]$ in $\mathcal{I}(X)$. Then any $[p_ip_{i+1} \cdots p_k]$ $(0 \leq i < k)$ is a terminal segment of θ; any $[p_ip_{i+1}p_{i+2} \cdots >$ is a terminal segment of ω; any $[p_ip_{i+1}p_{i+2} \cdots >$ $(i \geq 0)$ is a terminal segment of ν; and any $[p_ip_{i-1} \cdots](i \geq 1)$ is a terminal segment of λ.

For $\sigma \neq 0$, Δ_σ denotes the set of cycles of σ and Θ_σ denotes the set of chains of σ. For $k \geq 1$, Δ^k_σ denotes the set of cycles of length k in σ and Θ^k_σ denotes the set of chains of length k in σ. Ω_σ denotes the set of double rays of σ. Υ_σ denotes the set of right rays of σ and Λ_σ denotes the set of left rays of σ.

Proposition 3.5. [2, Proposition 2.10] *Let $\sigma, \tau, \alpha \in \mathcal{I}(X)$. Then α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ if and only if for all $k \geq 1$, $\delta \in \Delta^k_\sigma$, $\theta \in \Theta^k_\sigma$, $\omega \in \Omega_\sigma$, $\nu \in \Upsilon_\sigma$ and $\lambda \in \Lambda_\sigma$*

(1) $\delta\alpha^* \in \Delta^k_\tau, \omega\alpha^* \in \Omega_\tau$ *and* $\lambda\alpha^* \in \Lambda_\tau$.

(2) *either there is a unique $\theta_1 \in \Theta^m_\tau$ with $m \geq k$ such that $\theta\alpha^*$ is a terminal segment of θ_1 or there is a unique $\lambda_1 \in \Lambda_\tau$ such that $\theta\alpha^*$ is a terminal segment of λ_1.*

(3) *either there is a unique $\nu_1 \in \Upsilon_\tau$ such that $\nu\alpha^*$ is a terminal segment of ν_1 or there is a unique $\omega_1 \in \Lambda_\tau$ such that $\omega\alpha^*$ is a terminal segment of ω_1.*

The following proposition follows easily from Theorem 3.2 and Proposition 3.5.

Proposition 3.6. *Let $\sigma, \tau \in \mathcal{I}(X)$. Then $\sigma \sim_r \tau$ if and only if there exist $\alpha, \beta, \varphi, \psi \in \mathcal{I}(X)$ such that the following conditions hold:*

1. *For all $k \geq 1$, $\delta \in \Delta^k_\sigma$, $\theta \in \Theta^k_\sigma$, $\omega \in \Omega_\sigma$, $\nu \in \Upsilon_\sigma$ and $\lambda \in \Lambda_\sigma$ such that*

   (i) $\delta\alpha^* \in \Delta^k_\tau, \omega\alpha^* \in \Omega_\tau$ and $\lambda\alpha^* \in \Lambda_\tau$.

   (ii) either there is a unique $\theta_1 \in \Theta^m_\tau$ with $m \geq k$ such that $\theta\alpha^*$ is a terminal segment of θ_1 or there is a unique $\lambda_1 \in \Lambda_\tau$ such that $\theta\alpha^*$ is a terminal segment of λ_1.

   (iii) either there is a unique $\nu_1 \in \Upsilon_\tau$ such that $\nu\alpha^*$ is a terminal segment of ν_1 or there is a unique $\omega_1 \in \Lambda_\tau$ such that $\omega\alpha^*$ is a terminal segment of ω_1
2. *For all $k \geq 1$ $\delta' \in \Delta^k_\tau$, $\theta' \in \Theta^k_\tau$, $\omega' \in \Omega_\tau$, $\nu' \in \Upsilon_\tau$ and $\lambda' \in \Lambda_\tau$ such that*

   (i) $\delta'\beta^* \in \Delta^k_\sigma, \omega'\beta^* \in \Omega_\sigma$ and $\lambda'\beta^* \in \Lambda_\sigma$.

   (ii) either there is a unique $\theta'_1 \in \Theta^m_\sigma$ with $m \geq k$ such that $\theta'\beta^*$ is a terminal segment of θ'_1 or there is a unique $\lambda'_1 \in \Lambda_\sigma$ such that $\theta'\beta^*$ is a terminal segment of λ'_1.

   (iii) either there is a unique $\nu'_1 \in \Upsilon_\sigma$ such that $\nu'\beta^*$ is a terminal segment of ν'_1 or there is a unique $\omega'_1 \in \Lambda_\sigma$ such that $\omega'\beta^*$ is a terminal segment of ω'_1.
3. *$q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Let $\sigma \sim_r \tau$. Then by Theorem 3.2, there are $\alpha, \beta, \varphi, \psi \in S^1$ for which α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$. Therefore by Proposition 3.5 we get the required result.

Conversely, let (1), (2) and (3) hold. Then by Proposition 3.5, α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$. So by (3) and Theorem 3.2 we get $\sigma \sim_r \tau$.□

For a countable set A, we define two cardinal numbers that will be crucial in our characterization of r-conjugacy in the semigroup $\mathcal{I}(X)$. We denote by $\mathbb{Z}_+$ the set of positive integers and by $\mathbb{N}$ the set $\mathbb{Z}_+ \cup \{0\}$.

Definition 3.7. Let A be countable and suppose that $\sigma \in \mathcal{I}(X)$. We define $k_\sigma \in \mathbb{N} \cup \{\aleph_0\}$ by

$$k_\sigma = \sup\{k \in \mathbb{Z}_+ : \Theta_\sigma \neq \emptyset\}.$$

If $\Theta_\sigma^k = \emptyset$ for every $k \in \mathbb{Z}_+$, we define k_σ to be 0.

Suppose $k_\sigma \in \mathbb{Z}_+$, that is, k_σ is the largest positive integer k such that $\Theta_\sigma^k \neq \emptyset$. We define $m_\sigma \in \mathbb{N}$ by

$$m_\sigma = \mathrm{maA}\{m \in \{1, 2, \cdots, k_\sigma\} : |\Theta_\sigma^m| = \aleph_0\}.$$

If Θ_σ^m is finite for every $m \in \{1, 2, \cdots, k_\sigma\}$, we define m_σ to be 0.

For any chain $\theta \in \mathcal{I}(X)$, we denote the length of θ by $\mathrm{l}(\theta)$. For example, if $\theta = [1234]$ then $\mathrm{l}(\theta) = 3$.

Lemma 3.8. [2, Lemma 2.13] *Let A be countably infinite and let $\sigma, \tau \in \mathcal{I}(X)$. Suppose that $k_\sigma = k_\tau = \aleph_0$. Then there exists an injective mapping $p : \Theta_\sigma \to \Theta_\tau$ such that for every $\theta \in \Theta_\sigma$, if $\theta \in \Theta_\sigma^k$ and $\theta p \in \Theta_\tau^m$, then $m \geq k$.*

Theorem 3.9. [2, Theorem 2.14] *Suppose that A is countable. Let $\sigma, \tau \in \mathcal{I}(X)$. Then $\sigma \sim_c \tau$ if and only if the following conditions are satisfied:*

1. *$|\Delta_\sigma^k| = |\Delta_\tau^k|$ for every $k \in \mathbb{Z}_+, |\Omega_\sigma| = |\Omega_\tau|$ and $|\Lambda_\sigma| = |\Lambda_\tau|$;*
2. *If Ω_σ is finite, then $|\Upsilon_\sigma| = |\Upsilon_\tau|$; and*
3. *If Λ_σ is finite, then*
   - *(i) $k_\sigma = k_\tau$; and*
   - *(ii) If $k_\sigma \in \mathbb{Z}_+$, then $m_\sigma = m_\tau$ and for every $k \in \{m_\sigma + 1, \cdots, k_\sigma\}, |\Theta_\sigma^k| = |\Theta_\tau^k|$.*

In the next result we characterize the r-notion in $\mathcal{I}(X)$.

Proposition 3.10. *Suppose that A is countable. Let $\sigma, \tau \in \mathcal{I}(X)$. Then $\sigma \sim_r \tau$ if and only if the following conditions are satisfied:*

1. *$|\Delta_\sigma^k| = |\Delta_\tau^k|$ for every $k \in \mathbb{Z}_+, |\Omega_\sigma| = |\Omega_\tau|$ and $|\Lambda_\sigma| = |\Lambda_\tau|$;*
2. *If Ω_σ is finite, then $|\Upsilon_\sigma| = |\Upsilon_\tau|$; and*
3. *If Λ_σ is finite, then*
   - *(i) $k_\sigma = k_\tau$; and*
   - *(ii) If $k_\sigma \in \mathbb{Z}_+$, then $m_\sigma = m_\tau$ and for every $k \in \{m_\sigma + 1, \cdots, k_\sigma\}, |\Theta_\sigma^k| = |\Theta_\tau^k|$.*
4. *There are $\alpha, \beta, \varphi, \psi \in S^1$ such that $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Suppose $\sigma \sim_r \tau$. Then by Theorem 3.2, there are $\alpha, \beta, \varphi, \psi \in S^1$ for which α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with

$q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$ As $\sigma \sim_r \tau$ implies $\sigma \sim_c \tau$, therefore by Proposition 3.9, (1), (2) and (3) hold.

Conversely, suppose condition (1), (2), (3) and (4) holds We will define an injective homomorphism φ from $\Gamma(\sigma)$ to $\Gamma(\tau)$. By (1), for every $k \in \mathbb{Z}^+$, there is an injective mapping $f_k : \Delta^k_\sigma \to \Delta^k_\tau$.

Suppose that both Ω_σ and Λ_σ are infinite. Then $|\Omega_\sigma \cup \Upsilon_\sigma| = |\Omega_\tau|$ and $|\Lambda_\sigma \cup \Theta_\sigma| = |\Lambda_\tau|$ and so there are injective mappings $g : \Omega_\sigma \cup \Upsilon_\sigma \to \Upsilon_\tau$ and $d : \Lambda_\sigma \cup \Theta_\sigma \to \Lambda_\tau$. For all $k \geq 1, \delta \in \Delta^k_\sigma, \omega \in \Omega_\sigma, \lambda \in \Lambda_\sigma, \nu \in \Upsilon_\sigma$ and $\theta \in \Theta_\sigma$, we define φ on span$(\delta)\cup$ span$(\omega)\cup$ span$(\lambda)\cup$ span$(\nu)\cup$ span(θ) in such a way that $\delta\varphi^* = \delta f_k, \omega\varphi^* = \omega g, \lambda\varphi^* = \lambda d$, $\nu\varphi^*$ is a terminal segment of νg, and $\theta\varphi^*$ is a terminal segment of θd. Note that this defines φ for every vertex x in $\Gamma(\sigma)$. By the definition of φ and Proposition 3.5, $\varphi \in \mathcal{I}(X)$ and φ is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$.

Suppose that Ω_σ is finite and Λ_σ is infinite. Then $|\Upsilon_\sigma| = |\Upsilon_\tau|$ by (2), and so there exists an injective mapping $j : \Upsilon_\sigma \to \Upsilon_\tau$. Let $f_k : \Delta^k_\sigma \to \Delta^k_\tau$ $(k \in \mathbb{Z}^+)$ and $d : \Lambda_\sigma \cup \Theta_\sigma \to \Lambda_\tau$ be the injective mappings defined in the previous paragraph. Since $|\Omega_\sigma| = |\Omega_\tau|$, there exists an injective mapping $g : \Omega_\sigma \to \Omega_\tau$. We define φ as in the previous paragraph, except that $\nu\varphi^* = \nu j$ for every $\nu \in \Upsilon_\sigma$. Again, $\varphi \in \mathcal{I}(X)$ and φ is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$.

Suppose that Ω_σ is infinite and Λ_σ is finite. Then $k\sigma = k\tau$ by (3)(i). Let $f_k : \Delta^k_\sigma \to \Delta^k_\tau (k \in \mathbb{Z}^+)$ and $g : \Omega_\sigma \cup \Upsilon_\sigma \to \Omega_\tau$ be the injective mappings defined in the case in which both Ω_σ and Λ_σ are infinite. Since $|\Lambda_\sigma| = |\Lambda_\tau|$, there exists an injective mapping $d : \Lambda_\sigma \to \Lambda_\tau$.

Suppose that $k_\sigma = \aleph_0$. Then by lemma 3.4, there is an injective mapping $p : \Theta_\sigma \to \Theta_\tau$ such that for every $\theta \in \Theta_\sigma$, if $\theta \in \Theta^k_\sigma$ and $\theta_p \in \theta^m_\tau$, then $m \geq k$. We define φ as in the case in which both Ω_σ and Λ_σ are infinite, except that $\theta\varphi^*$ is a terminal segment of θ_p for every $\theta \in \Theta_\sigma$. Again, $\varphi \in \mathcal{I}(X)$ and φ is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$.

Suppose that $k_\sigma < \aleph_0$. If $k_\sigma = 0$, then $\Theta_\sigma = \Theta_\tau = \emptyset$. Suppose that $k_\sigma \in \mathbb{Z}^+$. Then by (3)(ii), $m\sigma = m\tau$ and for every $k \in \{m_\sigma + 1, \cdot\cdot\cdot, k_\sigma\}$, $|\Theta^k_\sigma| = |\Theta^k_\tau|$. Let $m = m\sigma$. We have $|\Theta^1_\sigma \cup \cdot\cdot\cdot \cup \Theta^m_\sigma| = |\Theta^m_\tau| = \aleph_0$ and $|\Theta^k_\sigma| = |\Theta^k_\tau|$ for every $k > m$. Thus, there are injective mappings $s : \Theta_\sigma \cup \cdot\cdot\cdot \cup \Theta^m_\sigma \to \Theta^m_\sigma$ and $t_k : \Theta^k_\sigma \to \Theta^k_\tau$ for every $k > m$. We define φ (whether k_σ is 0 or not) as in the case when both Ω_σ and are infinite, except that for every $\theta \in \Theta_\sigma$, $\theta\varphi^*$ is a terminal segment of θ_s if $\theta \in \Theta^k_\sigma$ with $1 \leq k \leq m$, and $\theta\varphi^*$ is a terminal segment of θt_k if $\theta \in \Theta^k_\sigma$ with $k > m$. As in the previous cases, $\varphi \in \mathcal{I}(X)$ and φ is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$.

Finally, if both Ω_σ and Ω_σ are finite, we define an injective rp-homomorphism φ from $\Gamma(\sigma)$ to $\Gamma(\tau)$ as in the case in which Ω_σ is infinite and Λ_σ is finite, except that $\nu\varphi^* = \nu j$ for every

$\nu \in \Upsilon_\sigma$, where $j : \Upsilon_\sigma \to \Upsilon_\tau$ is an injective mapping from the case in which Ω_σ is finite and Λ_σ is infinite.

We have proved that there exists an injective rp-homomorphism φ from $\Gamma(\sigma)$ to $\Gamma(\tau)$. By symmetry, there exists an injective rp-homomorphism ψ from $\Gamma(\tau)$ to $\Gamma(\sigma)$. Hence, $\sigma \sim_r \tau$ by Theorem 3.2.□

4. $\sim_r$ NOTION OF CONJUGACY IN FULL INJECTIVE TRANSFORMATIONS $\mathcal{F}(X)$

For $\sigma \in \mathcal{F}(X)$ we denote by X_σ, Y_σ and Z_σ the set of maximal right rays contained in σ, the set of double rays contained in σ and the set of cycles contained in σ.

For $\mu = [a_0a_1a_2 \cdots >, \omega =< \cdots a_{-1}a_0a_1 \cdots >, \delta = (a_0a_1 \cdots a_{k-1})$ and any φ in $\mathcal{F}(X)$, we define:

$$\mu\varphi^* = [a_0\varphi a_1\varphi a_2\varphi \cdots >,$$

$$\omega\varphi^* =< \cdots a_{-1}\varphi a_0\varphi a_1\varphi \cdots >,$$

$$\delta\varphi^* = (a_0\varphi a_1\varphi \cdots a_{k-1}\varphi).$$

Proposition 4.1. [3, Proposition 7.3] *Let $\sigma, \tau, \alpha \in \mathcal{F}(X)$. Then α is a homomorphism from $\Gamma(\sigma)$to $\Gamma(\tau)$ if and only if for all $\mu \in X_\sigma, \omega \in Y_\sigma$, and $\delta \in Z_\sigma$:*

1. *either there is a unique $\mu_1 \in X_\tau$ such that $\mu\alpha^* \subseteq \mu_1$ or there is a unique $\omega_1 \in Y_\tau$ such that $\mu\alpha^* \subset \omega_1$.*
2. *$\omega\alpha^* \in Y_\tau$ and $\delta\alpha^* \in Z_\tau$.*

Proposition 4.2. *Let $\sigma, \tau \in \mathcal{F}(X)$. Then $\sigma \sim_r \tau$ if and only if there are $\alpha, \beta, \varphi, \psi \in S^1$ such that*

1. *For all $\mu \in X_\sigma, \omega \in Y_\sigma$, and $\delta \in Z_\sigma$:*
   (i) either there is a unique $\mu_1 \in X_\tau$ such that $\mu\alpha^* \subseteq \mu_1$ or there is a unique $\omega_1 \in Y_\tau$ such that $\mu\alpha^* \subset \omega_1$.
   (ii) $\omega\alpha^* \in Y_\tau$ and $\delta\alpha^* \in Z_\tau$.
2. *For all $\mu' \in X_\tau, \omega' \in Y_\tau$, and $\delta' \in Z_\tau$:*
   (i) either there is a unique $\mu'_1 \in X_\sigma$ such that $\mu'\beta^* \subseteq \mu'_1$ or there is a unique $\omega'_1 \in Y_\sigma$ such that $\mu'\beta^* \subset \omega'_1$.
   (ii) $\omega'\beta^* \in Y_\sigma$ and $\delta'\psi^* \in Z_\sigma$.
3. *$q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Let $\sigma \sim_r \tau$ then by Corollary 3.3, there are $\alpha, \beta, \varphi, \psi \in \mathcal{F}(X)$ such that α is a homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is a homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\varphi = q$

for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$. By Proposition 4.1, (1) and (2) hold.

Conversely, let (1), (2) and (3) holds. Then by Proposition 4.1, α is a homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Similarly β is a homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$. Then by Corollary 3.3, $\sigma \sim_r \tau$.□

Lemma 4.3. [3, Lemma 7.5] *Let A_1, B_1, A_2 and B_2 be sets such that $A_1 \cap B_1 = \emptyset$, $A_2 \cap B_2 = \emptyset, |A_1| + |B_1| \leq |A_2| + |B_2|$ and $|B_1| \leq |B_2|$. Then there is an injective mapping $f : A_1 \cup B_1 \to A_2 \cup B_2$ such that $xf \in B_2$ for every $x \in B_1$.*

Theorem 4.4. [3, Theorem 7.6] *Let $\sigma, \tau \in \mathcal{F}(X)$. Then $\sigma \sim_c \tau$ in $\mathcal{F}(X)$ if and only if $|X_\sigma| + |Y_\sigma| = |X_\tau| + |Y_\tau|, |Y_\sigma| = |Y_\tau|$ and $|Z^n_\sigma| = |Z^n_\tau|$ for every $n \geq 1$.*

In the next result we characterize the r-notion in $\mathcal{F}(X)$.

Theorem 4.5. *Let $\sigma, \tau \in \mathcal{F}(X)$. Then $\sigma \sim_r \tau$ in $\mathcal{F}(X)$ if and only if $|X_\sigma| + |Y_\sigma| = |X_\tau| + |Y_\tau|, |Y_\sigma| = |Y_\tau|$ and $|Z^n_\sigma| = |Z^n_\tau|$ for every $n \geq 1$ and there are $\alpha, \beta, \varphi, \psi \in \mathcal{F}(X)$ such that $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Suppose $\sigma \sim_r \tau$ in $\mathcal{F}(X)$. Then by Corollary 3.3, there are $\alpha, \beta, \varphi, \psi \in \mathcal{F}(X)$ such that α is a homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is a homomorphism from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$. Since $\sim_r \subseteq \sim_c$, we obtain the required result by Theorem 4.4, we get required.

Conversely, suppose $|X_\sigma| + |Y_\sigma| = |X_\tau| + |Y_\tau|, |Y_\sigma| = |Y_\tau|$ and $|Z^n_\sigma| = |Z^n_\tau|$ for every $n \geq 1$ and there are $\alpha, \beta, \varphi, \psi \in \mathcal{F}(X)$ such that $q\alpha\varphi = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non-initial vertex k of $\Gamma(\tau)$. By Lemma 4.3, the mapping $f : X_\sigma \cup Y_\sigma \to X_\tau \cup Y_\tau$ is injective such that $\omega f \in Y_\tau$ for every $\omega \in Y_\sigma$. For every $n \geq 1$, fix a bijection $g_n : Z^n_\sigma \to Z^n_\tau$. Let $n \geq 1$. For all $\mu \in X_\sigma, \omega \in Y_\sigma$ and $\delta \in Z_\sigma$, we define α on dom$(\mu)\cup$ dom$(\omega)\cup$ dom(δ) in such a way that $\mu\alpha^* \subset \mu f, \omega\alpha^* = \omega f$ and $\delta\alpha^* = \delta g_n$ if $\mu \in X_\sigma, \omega \in Y_\sigma$ and $\delta \in Z^n_\sigma$. Note that this defines α for every $x \in X$. By the definition of α and Proposition 4.1, $\alpha \in \mathcal{F}(X)$ and α is a homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$. By symmetry, there is an injective homomorphism β from $\Gamma(\tau)$ to $\Gamma(\sigma)$. Hence $\sigma \sim_r \tau$ by Corollary 3.3. □

5. NUMBER OF CONJUGACY CLASSES

J.Koneiczny in [6] proved that if X is a finite set with n elements, then the symmetric inverse semigroup $\mathcal{I}(X)$ has $\sum_{r=0}^{n} p(r)p(n-r)$, n-conjugacy classes and if X is infinite, then $\mathcal{I}(X)$ has $\kappa^{\aleph_0}$, n-conjugacy classes. Also he proved that if X is finite with $|X| = n$, then $\mathcal{F}(X)$ has $p(n)$,

n-conjugacy classes and if X is infinite then both $\mathrm{Sym}(X)$ and $\mathcal{F}(X)$ have $\kappa^{\aleph_0}$, n-conjugacy classes. By Theorem 2.4, as $\sim_n = \sim_r$ in $\mathcal{I}(X)$. Since $\mathrm{sym}(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{I}(X)$. Therefore $\sim_n = \sim_r$ in $\mathcal{F}(X)$ and $\mathrm{sym}(X)$. These facts enable us to have the following results.

Theorem 5.1. *Let X be a non-empty set. Then*

1. *If X is finite with $|X| = n$ then $\mathcal{I}(X)$ has $\sum_{r=0}^{n} p(r)p(n-r)$ r conjugacy classes;*
2. *If X is infinite with $|X| = \aleph_\varepsilon$ then $\mathcal{I}(X)$ has $\kappa^{\aleph_0}$ r conjugacy classes where $\kappa = \aleph_0 + |\varepsilon|$.*

Theorem 5.2. *Let X be a non-empty set. Then*

1. *If X is finite with $|X| = n$, then $\mathcal{F}(X)$ has $p(n)$ r-conjugacy classes.*
2. *If X is infinite with $|X| = \aleph_\varepsilon$, then Sym(X) and $\mathcal{F}(X)$ have $\kappa^{\aleph_0}$ r- conjugacy classes , where $\kappa = \aleph_0 + |\varepsilon|$.*

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