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Research Paper

MODULES WHOSE SURJECTIVE ENDOMORPHISMS HAVE A γ-SMALL KERNELS

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Abstract. In this paper, we introduce a proper generalization of that of Hopfian modules, called γ-Hopfian modules. A right R-module M is said to be γ-Hopfian, if any surjective endomorphism of M has a γ-small kernel. Some basic characterizations of γ-Hopfian modules are proved. We prove that a ring R is semisimple cosingular if and only if every R-module is γ-Hopfian. Further, we prove that the γ-Hopfian property is preserved under Morita equivalences. Some other properties of γ-Hopfian modules are also obtained with examples.

1. Introduction

Throughout this paper all rings have identity and all modules are unital right modules. We use the notations $\subseteq$, $\leq$ and $\leq^{\oplus}$ to denote inclusion, submodule and direct summand, respectively, and $E(M)$, $Z^*(M)$, $\mathrm{End}(M)$ will denote the injective hull, the cosingular submodule, and the ring of endomorphisms of a module M. Recall that a submodule K of M is

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said to be small in M ($K \ll M$), if for every submodule $L \leq M$ with $K + L = M$ implies $L = M$. For a right R-module M, Ozcan [11], defined the submodule $Z^*(M)$ as a dual of singular submodule to be the set of all elements $m \in M$ such that mR is a small module, that is, $Z^*(M) = \{m \in M : mR \ll E(M)\}$. A right R-module M is called cosingular (resp., noncosingular) if $Z^*(M) = M$ (resp., $Z^*(M) = 0$). A submodule K of an R-module M is said to be μ-small in M ($K \ll_\mu M$), if for every submodule $L \leq M$ such that $K + L = M$ with M/L cosingular implies $M = L$ ([13]). It is clear that if A is a small submodule of M, then A is a μ-small submodule of M, the converse is not true in general. A submodule K of an R-module M is said to be γ-small in M ($K \ll_\gamma M$), if for every submodule $L \leq M$ such that $K + L = M$ with M/L noncosingular implies $M = L$ (see [8]). It is clear that if K is a small submodule of M, then K is a γ-small submodule of M, but the converse is not true in general.

The study of modules by properties of their endomorphisms has long been of interest. In [7], Hiremath introduced the concepts of Hopfian modules and rings. Later, in [12], Varadarajan, introduced the notion of co-Hopfian modules. An R-module M is called co-Hopfian (resp. Hopfian) if any injective (resp. surjective) endomorphism of M is an automorphism. Note that any Artinian module is co-Hopfian, and any Noetherian module is Hopfian, but the converse is not true in general. The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian non-Artinian $\mathbb{Z}$-module, which is Hopfian and co-Hopfian. The notions Hopfian, co-Hopfian modules and their generalizations have been investigated by several authors, see, for instance, ([3], [4], [5], [6], [7], [12]).

In [5], Ghorbani and Haghany introduced the notion of generalized Hopfian modules. A right R-module M is called generalized Hopfian, if any surjective endomorphism of M has a small kernel.

In [4], we introduced and studied the concept of μ-Hopfian modules. A right R-module M is called μ-Hopfian, if any surjective endomorphism of M has a μ-small kernel.

By works mentioned we are motivated in this paper to introduce the notion of γ-Hopfian modules which is a proper generalization of that of Hopfian modules (Example 2.4), and in particular Noetherian modules. We call a module γ-Hopfian if any its surjective endomorphism has a γ-small kernel.

Recall that the module M is called Dedekind finite, if $fg = 1$ implies $gf = 1$ for each $f, g \in \mathrm{End}(M)$. Consequently, M is a Dedekind finite module if and only if M is not isomorphic to any proper direct summand of itself. In [5, Corollary 1.4], it is shown that the concepts of Dedekind finite, Hopfian and generalized Hopfian modules equivalent for every (quasi-)projective module. It is clear that every generalized Hopfian module is γ-Hopfian, but the converse is not true in general (Example 3.5). Also, this example shows that a γ-Hopfian module need not be Dedekind finite.

Therefore, we obtain the following diagram:

$$\begin{tikzcd}
 & & \boxed{\gamma\text{-Hopfian}} & \\
\boxed{\text{Noetherian}} \arrow[r] & \boxed{\text{Hopfian}} \arrow[ur] \arrow[dr] \arrow[rr] & & \boxed{\text{Dedekind finite}} \\
 & & \boxed{\text{generalized Hopfian}} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

At the end of the paper, some open problems are given.

We list some properties of cosingular modules that will be used in the paper.

Lemma 1.1. [11]. For any ring R, the class of cosingular R-modules is closed under submodules, homomorphic images and direct sums but not (in general) under essential extensions or extensions.

Lemma 1.2. [11].

Let R be a right cosingular ring. Then any (right) R-module is cosingular

We list some properties of γ-small submodules that will be used in the paper.

Lemma 1.3. [8]. Let M be an R-module. Then the following statements hold.

(1) Let $A \leq B \leq M$. Then $B \ll_\gamma M$ if and only if $A \ll_\gamma M$ and $B/A \ll_\gamma M/A$.
(2) Let A, B be submodules of M with $A \leq B$, if $A \ll_\gamma B$, then $A \ll_\gamma M$.
(3) Let $f : M \to M'$ be an epimorphism such that $A \ll_\gamma M$, then $f(A) \ll_\gamma M'$.
(4) Let $M = M_1 \oplus M_2$ be an R-module and let $A_1 \leq M_1$ and $A_2 \leq M_2$. Then $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$ if and only if $A_1 \ll_\gamma M_1$ and $A_2 \ll_\gamma M_2$.

Definition 1.4. [6]. A right R-module M is called weakly co-Hopfian if any injective endomorphism of M is essential.

Examples 1.5. The following facts are well known:

(1) Any Artinian R-module M (i.e., M has DCC on submodules), is co-Hopfian and it is weakly co-Hopfian [1].
(2) The additive group $\mathbb{Q}$ of rational numbers is a non-Artinian $\mathbb{Z}$-module, which is co-Hopfian and weakly co-Hopfian.

Definition 1.6. [9]. An R-module M is called quasi-projective if for any surjective homomorphism g of M onto N and any homomorphism, γ of M to N, there exists an endomorphism h of M such that: $\gamma = gh$ (i.e., there exists $h : M \to M$ such that the diagram

$$
\begin{tikzcd}
 & M \arrow[dl, dotted, "h"'] \arrow[d, "\gamma"] & \\
M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}
$$

commute). Clearly, every projective module is quasi-projective.

Definition 1.7. [3]. A module M is called semi Hopfian if any surjective endomorphism of M has a direct summand kernel, i.e. any surjective endomorphism of M splits.

Examples 1.8. [3].

(1) Any semisimple module is semi Hopfian.
(2) Any quasi-projective module is semi Hopfian.
(3) A vector space V over a field F is Hopfian if and only if it is finite dimensional, by [7, Theorem 16(ii)]. Thus an infinite-dimensional vector space over a field is semi Hopfian, but it is not Hopfian.
(4) Any module with D2 is semi Hopfian. (Recall that a module M has D2 if any submodule N such that M/N is isomorphic to a direct summand of M is a direct summand of M).

Definition 1.9. [2]. An R-module M is said to be Fitting if for any endomorphism f of M, there exists a positive integer $n \geq 1$ such that $M = Kerf^n \oplus Imf^n$.

Remarks 1.10. The following facts are well known:

(1) An R-module M is Fitting if and only if $\text{End}(M)$ is strongly π-regular. (i.e., for every $f \in \text{End}(M)$, there exists $g \in \text{End}(M)$ and an integer n such that $f^n = gf^{n+1} = f^{n+1}g$). [2]
(2) Every Artinian and Noetherian R-module is Fitting. [1]
(3) Every Fitting R-module is Hopfian and co-Hopfian. [1]

2. γ-Hopfian modules

Motivated by the notion of Hopfian modules and the concept of generalized Hopfian modules, we define a γ-Hopfian module as follows.

Definition 2.1. Let M be an R-module. We say that M is γ-Hopfian if any surjective endomorphism of M has a γ-small kernel.

The next result gives several equivalent conditions for a γ-Hopfian module.

Theorem 2.2. Let M be an R-module. The following are equivalent:

(1) M is γ-Hopfian.

(2) For every surjective endomorphism f of M, if $N \ll_\gamma M$, then $f^{-1}(N) \ll_\gamma M$.

(3) For any epimorphism $f : M/N \to M$, we have $N \ll_\gamma M$.

(4) If f is a surjective endomorphism of M and if M/N is nonzero and noncosingular for some $N \leq M$, then $f(N) \neq M$.

Proof. (1) $\Rightarrow$ (2) Let $f : M \to M$ is a surjective endomorphism and $N \ll_\gamma M$. Let $f^{-1}(N) + K = M$ for some $K \leq M$, where $Z^*(M/K) = 0$. Then $N + f(K) = M$. Since M/K is noncosingular and $M/f(K)$ is an image of M/K, $M/f(K)$ is noncosingular. Hence $N + f(K) = M$ and $N \ll_\gamma M$, giving $f(K) = M$. So $K + Ker(f) = M$. Since M is γ-Hopfian, $Ker(f) \ll_\gamma M$. Hence M/K is noncosingular implies that $K = M$. Thus $f^{-1}(N) \ll_\gamma M$.

(2) $\Rightarrow$ (3) Let $f : M/N \to M$ be an epimorphism and $\pi : M \to M/N$ be a canonical epimorphism. It is clear that $N \leq Ker(f\pi)$. By (2), $Ker(f\pi) = (f\pi)^{-1}(0) \ll_\gamma M$. Therefore $N \ll_\gamma M$ by Lemma 1.3.

(3) $\Rightarrow$ (4) Let N be a proper submodule of M such that M/N is noncosingular and $f : M \to M$ a surjective endomorphism with $f(N) = M$. Then $M = Ker(f) + N$, moreover $g : M/Ker(f) \to M$ is an epimorphism, then $Ker(f) \ll_\gamma M$ by (3). Hence $M = N$, contradiction.

(4) $\Rightarrow$ (1) Let $f : M \to M$ be a surjective endomorphism. If $M = N + Ker(f)$, with M/N is noncosingular, hence $M = f(M) = f(N)$. Then $N = M$ by (4). Therefore $Ker(f) \ll_\gamma M$. □

Corollary 2.3. Let M be a γ-Hopfian module, $g \in \mathrm{End}(M)$ an epimorphism and $K \leq M$. Then $K \ll_\gamma M$ if and only if $g(K) \ll_\gamma M$ if and only if $g^{-1}(K) \ll_\gamma M$.

The following example shows that Hopfian modules form a proper subclass of γ-Hopfian modules.

Example 2.4. Let $M = \mathbb{Z}_{p^\infty}$. As any submodule of M is γ-small in M, we see that M is a γ-Hopfian $\mathbb{Z}$-modules. However M is not Hopfian since the multiplication by p induces an epimorphism of M which is not an isomorphism.

Theorem 2.5. Let M be an R-module. The following are equivalent:

(1) M is γ-Hopfian.

(2) There exists a fully invariant γ-small submodule N of M such that M/N is γ-Hopfian.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) Assume that N is a fully invariant γ-small submodule of M with M/N is γ-Hopfian. Let $f : M \to M$ be a surjective endomorphism. Then $g : M/N \to M/N$ given by $g(m + N) = f(m) + N$ is a well-defined surjective endomorphism, since M/N is γ-Hopfian,

$Ker(g) \ll_\gamma M/N$. Suppose $Ker(g) = L/N$ for some appropriate submodule L of M, then $L/N \ll_\gamma M/N$. Since $N \ll_\gamma M$, $L \ll_\gamma M$ by Lemma 1.3. As $Ker(f)$ is a submodule of L, $Ker(f) \ll_\gamma M$. Therefore M is γ-Hopfian. □

Proposition 2.6. Let M be an R-module and let N be a γ-Hopfian fully invariant submodule of M such that M/N is Hopfian. Then M is γ-Hopfian.

Proof. Let $f : M \to M$ be a surjective endomorphism. Since the induced map $g : M/N \to M/N$ is surjective, it must be an isomorphism, thus $N = f^{-1}(N)$. Therefore $f|_N : N \to N$ is a surjective endomorphism. Now if N is γ-Hopfian, $Ker(f) \cap N \ll_\gamma N$. Since $Ker(f)$ is a submodule of N, then $Ker(f) \ll_\gamma N \leq M$. Hence by Lemma 1.3, $Ker(f) \ll_\gamma M$ and M is γ-Hopfian. □

Lemma 2.7. Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies ACC on non γ-small submodules N such that M/N has the property P, then M is γ-Hopfian.

Proof. Suppose M is not γ-Hopfian. Then there exists a submodule N_1 with N_1 not γ-small in M and $M/N_1 \simeq M$. Hence M/N_1 is not γ-Hopfian but satisfies P. Then there exists a submodule $N_2 \supseteq N_1$ with N_2/N_1 not γ-small in M/N_1 and $M/N_2 \simeq M/N_1$. So we get $N_1 \subseteq N_2$ and both non γ-small in M with $M/N_i \simeq M$ for $i = 1, 2$. Repeating the process yields a chain of submodules of the type that contradicts our hypothesis. Then M is γ-Hopfian. □

Corollary 2.8. Let M be a weakly co-Hopfian module with ACC on non γ-small submodules N of M. If M/N is weakly co-Hopfian, then M is γ-Hopfian.

Proof. We may assume M is a weakly co-Hopfian module with ACC on non γ-small submodules and that P is the property of being weakly co-Hopfian. This property is preserved under isomorphism. Then by Lemma 2.7, M is γ-Hopfian. □

Example 2.9. Let R be a semisimple cosingular ring. Hence by Theorem 3.4, $M = R^{(\mathbb{N})}$ is a γ-Hopfian R-module. As $M \oplus M \cong M$ and $M \neq 0$, then M is not weakly co-Hopfian by [6, Theorem 1.1].

Proposition 2.10. Let M be an R-module with ACC on non γ-small submodules. Then M is γ-Hopfian.

Proof. We may assume M is nonzero with ACC on non γ-small submodules and that P is the property of being nonzero. By Lemma 2.7, M is γ-Hopfian. □

Remarks 2.11. (1) Every Noetherian R-module is γ-Hopfian.

(2) By [5, Remarks 1.19(iii)], the module $M = \sum \oplus \overline{Z}_p$ is generalized Hopfian. Hence it is γ-Hopfian. But M fail ACC on non γ-small submodules. Thus the converse of Proposition 2.10 do not hold in general.

Proposition 2.12. Let M be an R-module. If M satisfies DCC on non γ-small submodules, then M is γ-Hopfian.

Proof. Assume that M satisfies DCC on non γ-small submodules and M is not γ-Hopfian. Hence there exists an epimorphism $f : M \to M$ such that $K = Ker(f)$ is not a γ-small submodule of M. Then each submodule L of M, which contains K, is not a γ-small submodule of M. As M is not γ-Hopfian, then it is not generalized Hopfian and it is not Artinian by [5, Remarks 1.19(i)]. Hence $M/K \cong M$ is not Artinian and there is a descending chain $L_1/K \supset L_2/K \supset L_3/K \supset ...$ of submodules of M/K. Thus $L_1 \supset L_2 \supset L_3 \supset ...$ is a descending chain of non γ-small submodule of M, a contradiction. □

Remarks 2.13. (1) Every Artinian R-module is γ-Hopfian.

(2) The module $M = \sum \oplus \overline{Z}_p$ is generalized Hopfian by [5, Remarks 1.19(iii)], then it is γ-Hopfian. But M fail DCC on non γ-small submodules. Thus the converse of Proposition 2.12 do not hold in general.

Proposition 2.14. Let M be an R-module with the property that for any endomorphism f of M there exists an integer $n \geq 1$ such that $Kerf^n \cap Imf^n \ll_\gamma M$. Then M is γ-Hopfian.

Proof. Let $f : M \to M$ be an homomorphism. Then there exists $n \geq 1$ such that $Kerf^n \cap Imf^n \ll_\gamma M$. If f is surjective then so is f^n, i.e., $Imf^n = M$, so we get that $Kerf^n \ll_\gamma M$. Since $Kerf \leq Kerf^n$, $Kerf \ll_\gamma M$ by Lemma 1.3. Therefore M is γ-Hopfian. □

Examples 2.15. .

(1) Every proper submodule of $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ is γ-small, then for any endomorphism f of M there exists an integer $n \geq 1$ such that $Kerf^n \cap Imf^n \ll_\gamma M$. Hence M is a γ-Hopfian $\mathbb{Z}$-module.

(2) If M is a Noetherian module, then for any endomorphism f of M there exists an integer $n \geq 1$ such that $Kerf^n \cap Imf^n = 0$. Hence M is γ-Hopfian.

Proposition 2.16. Any direct summand of a γ-Hopfian module M is γ-Hopfian.

Proof. Let K be a direct summand of M. Then there exists a submodule N of M such that $M = K \oplus N$. Let $f : K \to K$ be a surjective endomorphism of K, then f induces a surjective endomorphism of M, $f \oplus 1_N : M \to M$ with $(f \oplus 1_N)(k + n) = f(k) + n$, where $k \in K$ and $n \in N$. Since M is γ-Hopfian, then $Ker(f \oplus 1_N) \ll_\gamma M$. Hence $Kerf \ll_\gamma K$ by Lemma 1.3, and K is γ-Hopfian. □

Proposition 2.17. Let $M = M_1 \oplus M_2$ be an R-module. If for every $i \in \{1, 2\}$, M_i is a fully invariant submodule of M, then M is γ-Hopfian if and only if M_i is γ-Hopfian for each $i \in \{1, 2\}$.

Proof. $\Rightarrow$) Clear from Proposition 2.16.

$\Leftarrow$) Let $f = (f_{ij})$ be an epimorphism of M, where $f_{ij} \in Hom(M_i, M_j)$ and $i, j \in \{1, 2\}$. Since M_i is a fully invariant submodule of M, then $Hom(M_i, M_j) = 0$ for every $i, j \in \{1, 2\}$ with $i \neq j$. Since f is an epimorphism, f_{ii} is an epimorphism of M_i for each $i \in \{1, 2\}$. As M_i is γ-Hopfian for each $i \in \{1, 2\}$, $Ker(f_{ii}) \ll_\gamma M_i$. Then $Ker(f) = Ker(f_{11}) \oplus Ker(f_{22}) \ll_\gamma M_1 \oplus M_2 = M$ by Lemma 1.3. Hence M is γ-Hopfian. □

Definition 2.18. Let M and N be two R-modules. M is called γ-Hopfian relative to N, if for any epimorphism $f : M \to N$, $Ker(f) \ll_\gamma M$.

In view of the above definition, an R-module M is γ-Hopfian if and only if M is γ-Hopfian relative to M.

In the following Proposition, we characterize the γ-Hopfian modules in terms of their direct summands and factor modules.

Proposition 2.19. Let M and N be two R-modules. Then the following are equivalent:

(1) M is γ-Hopfian relative to N.

(2) For each $L \leq^\oplus M$, L is γ-Hopfian relative to N.

(3) For each $L \leq M$, M/L is γ-Hopfian relative to N.

Proof. (1) $\Rightarrow$ (2) Let $L \leq^\oplus M$ say $M = L \oplus K$, where $K \leq M$ and $f : L \to N$ an epimorphism. Let $\pi : M \to L$ be the natural projection. Then $f\pi : M \to N$ is an epimorphism and so $Ker(f\pi) \ll_\gamma M$ by (1). It is clear that $Ker(f\pi) = Ker(f) \oplus K$. Then $Ker(f\pi) = Ker(f) \oplus K \ll_\gamma M$. Hence by Lemma 1.3, $Ker(f) \ll_\gamma L$.

(2) $\Rightarrow$ (1) Clear, take $L = M$.

(1) $\Rightarrow$ (3) Let $L \leq M$ and $f : M/L \to N$ be an epimorphism. Then $f\pi : M \to N$ is an epimorphism, where $\pi : M \to M/L$ is the natural homomorphism. As $Ker(f\pi) =$

$\pi^{-1}(Ker(f))$ and $Ker(f\pi) \ll_\gamma M$, $\pi(Ker(f\pi)) = Ker(f) \ll_\gamma M/L$ by Lemma 1.3. Hence M/L is γ-Hopfian relative to N.

(3) $\Rightarrow$ (1) Clear, take $L = 0$. □

Proposition 2.20. Let M be a semi Hopfian R-module. If M is co-Hopfian, then it is γ-Hopfian.

Proof. Let $f : M \to M$ be a surjective endomorphism. Since M is a semi Hopfian R-module, f splits, and hence there exists an endomorphism $g : M \to M$, such that $fg = 1$. This implies that g is an injective endomorphism. Now since M is co-Hopfian, g is an automorphism. Therefore f is an automorphism and M becomes a γ-Hopfian R-module. □

Corollaries 2.21.

(1) Let M be an R-module with D2. If M is co-Hopfian, then it is γ-Hopfian.
(2) Every semisimple co-Hopfian R-module is γ-Hopfian.
(3) Every quasi-projective co-Hopfian R-module is γ-Hopfian.

3. Characterizations the class of rings R for which every R-module is γ-Hopfian

Lemma 3.1. Let M be an R-module and $N \leq M$. The following are equivalent.

(1) $N \ll_\gamma M$.
(2) If $X + N = M$, then $X \leq^\oplus M$ with M/X is a semisimple cosingular module.

Proof. (1) $\Rightarrow$ (2) Let $Y \leq M$ such that $M/(X \oplus Y)$ is semisimple and injective, hence by [10, Lemma 1(iii)] $Z^*(M/(X \oplus Y)) = 0$. Since $X + Y + N = M$ and $N \ll_\gamma M$, then $X \oplus Y = M$. To see that $M/X \cong Y$ is semisimple cosingular.

Let A be a submodule of Y . Then $X + A + N = M$. Arguing as above with $X + A$ replacing X, we have that $X + A = X \oplus A$ is a direct summand of M, thus A is a direct summand of Y, so M/X is semisimple.

Write $Y = Z^*(Y) \oplus C$, where C is noncosingular. Then $M/(X \oplus Z^*(Y)) = (X \oplus Y)/(X \oplus Z^*(Y)) \cong C$ is noncosingular. Since $M = (X + Z^*(Y)) + N$, by (1) $X \oplus Z^*(Y) = M$. This shows that $C = 0$, implies $Z^*(Y) = Y$, then Y is cosingular.

(2) $\Rightarrow$ (1) Let $K \leq M$ such that $K + N = M$ and $Z^*(M/K) = 0$. By (2) M/K is semisimple cosingular, (i.e., $Z^*(M/K) = M/K$). Hence $M/K = 0$. Then $M = K$ and $N \ll_\gamma M$. □

Theorem 3.2. Let M be an R-module. The following are equivalent:

(1) M is γ-Hopfian.

(2) For every right R-module Y. If there is an epimorphism $M \to M \oplus Y$, then Y is semisimple and cosingular.

Proof. (1) $\Rightarrow$ (2) Let $f : M \to M \oplus Y$ be an epimorphism, and $\pi : M \oplus Y \to M$ the natural projection. It is clear that $Ker(\pi f) = f^{-1}(0 \oplus Y)$. Since M is γ-Hopfian, $Ker(\pi f) \ll_\gamma M$. By Lemma 1.3, $0 \oplus Y = f[f^{-1}(0 \oplus Y)] = f(Ker(\pi f)) \ll_\gamma M \oplus Y$. Therefore $Y \ll_\gamma Y$ by Lemma 1.3. So, by Lemma 3.1, Y is semisimple and cosingular.

(2) $\Rightarrow$ (1) Let f be a surjective endomorphism of M and $Ker(f) + L = M$ for some $L \leq M$, where $Z^*(M/L) = 0$. Since $\frac{M}{Ker(f)} \oplus \frac{M}{L} \cong M \oplus \frac{M}{L}$, the epimorphism $M \to M \oplus \frac{M}{L}$ exists. By (6), M/L is semisimple and cosingular, then $Z^*(M/L) = M/L$. Then $M/L = 0$. Therefore $M = L$ and $Ker(f) \ll_\gamma M$. □

Theorem 3.3. Let M be a (quasi-)projective R-module. Then the following are equivalent:

(1) M is γ-Hopfian.

(2) If f is a surjective endomorphism of M, then $Ker(f)$ is semisimple cosingular.

Proof. (1) $\Rightarrow$ (2) Let $f \in \mathrm{End}(M)$ be a surjective endomorphism of M. Then by (1), $Ker(f) \ll_\gamma M$. Since M is (quasi-)projective, then there exists g in $\mathrm{End}(M)$ such that $fg = 1 \in \mathrm{End}(M)$. It is clear that $Ker(f) = (1 - gf)M$ and $M = Ker(f) \oplus (gf)M$. So, $Ker(f)$ is semisimple and cosingular by Lemma 3.1.

(2) $\Rightarrow$ (1) Let $f \in \mathrm{End}(M)$ be a surjective endomorphism of M. Then by (2), $Ker(f)$ is semisimple cosingular. We prove that $Ker(f) \ll_\gamma M$. Let $Ker(f) + L = M$ for some $L \leq M$. Since $Ker(f)$ is semisimple, $(Ker(f) \cap L) \oplus K = Ker(f)$ for some $K \leq Ker(f)$. Therefore $K \oplus L = M$. As K is semisimple cosingular by Lemma 1.1, hence $Ker(f) \ll_\gamma M$ by Lemma 3.1. □

In the following, we characterize the class of rings R for which every (free) R- module is γ-Hopfian.

Theorem 3.4. Let R be a ring. Then the following are equivalent:

(1) Any R-module is γ-Hopfian.

(2) Any projective R-module is γ-Hopfian.

(3) Any free R-module is γ-Hopfian.

(4) R is semisimple cosingular.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let $M = R^{(\mathbb{N})}$, by (3) M is a γ-Hopfian R-module. Since $M \cong M \oplus M$, hence by Theorem 3.2, M is semisimple cosingular. Therefore R is semisimple cosingular.

(4) $\Rightarrow$ (1) Let R be a semisimple cosingular ring and M be an R-module. Hence M is projective and for each surjective endomorphism f of M, $Ker(f)$ is semisimple cosingular by Lemma 1.1 and Lemma 1.2. Hence by Theorem 3.3, M is γ-Hopfian. □

It is clear that every generalized Hopfian module is γ-Hopfian. The following example shows that the converse is not true, in general. Also, it shows that a γ-Hopfian module need not be Dedekind finite.

Example 3.5. Let R be a semisimple cosingular ring. Hence by Theorem 3.4, $M = R^{(\mathbb{N})}$ is a γ-Hopfian R-module. As $M \cong M \oplus M$ and $M \neq 0$, then M is not generalized Hopfian and it is not Dedekind finite by [5, Corollary 1.4].

The following result shows γ-Hopfian property is preserved under Morita equivalences.

Theorem 3.6. γ-Hopfian is a Morita invariant property.

Proof. Let T and S be Morita equivalent rings with inverse category equivalences

$$\alpha : \text{Mod-}T \to \text{Mod-}S, \;\; \beta : \text{Mod-}S \to \text{Mod-}T.$$

Let $M \in \text{Mod-}T$ be a γ-Hopfian module. To prove that $\alpha(M)$ is γ-Hopfian in Mod-S. Assume that $f : \alpha(M) \to \alpha(M) \oplus Y$ be an S-module epimorphism where Y is a right S-module. Since any category equivalence preserves direct sums and epimorphisms, we have $\beta(f) : \beta\alpha(M) \to \beta\alpha(M) \oplus \beta(Y)$, as an epimorphism in Mod-T.

Since $\beta\alpha(M) \cong M$, we have an epimorphism $M \to M \oplus \beta(Y)$ in Mod-T. This implies that $\beta(Y)$ is semisimple cosingular as an T-module, by Theorem 3.2. As any category equivalence preserves semisimple and cosingularity properties, Y is semisimple cosingular as an S-module. Therefore $\alpha(M)$ is γ-Hopfian, by Theorem 3.2. □

Corollary 3.7. Let R be a ring. The following are equivalent for $n \geq 2$:

(1) Any n-generated R-module is γ-Hopfian.

(2) Any cyclic $M_n(R)$-module is γ-Hopfian.

Proof. Let $K = T^n$ and $S = \text{End}(K)$. Then, it is known that

$$Hom_T(K, .) : N_T \to Hom({}_SK_T, N_T)$$

defines a Morita equivalence between Mod-T and Mod-S with the inverse equivalence.

$$- \otimes_S K : M_S \to M \otimes K.$$

Moreover, for any cyclic S-module M, $M \otimes_S K$ is an n-generated T-module and if N is a n-generated T-module, then $Hom_T(K, N)$ is a cyclic S-module. By Theorem 3.6, a Morita equivalence preserves the γ-Hopfian property of modules. Therefore, every cyclic S-module is γ-Hopfian if and only if every n-generated T-module is γ-Hopfian. □

In the following Corollary, we characterize the rings R for which every finitely generated free R-module is γ-Hopfian.

Corollary 3.8. Let R be a ring. Then the following statements are equivalent:

(1) Every finitely generated free R-module is γ-Hopfian.
(2) Every finitely generated projective R-module is γ-Hopfian.
(3) $M_n(R)$ is γ-Hopfian $M_n(R)$-module for each $n \geq 1$.

Proof. (1) $\Rightarrow$ (2) Clear from Proposition 2.16.

(2) $\Rightarrow$ (1) Clear.

(1) $\Leftrightarrow$ (3) Let n be a positive integer and $S = M_n(R)$. By Theorem 3.6 and the proof of Corollary 3.7, if $M = R^n$ is γ-Hopfian, then $Hom_R(M, M)$ is γ-Hopfian as an S-module. Conversely, if S is γ-Hopfian as an S-module, then $S \otimes_S M$ is γ-Hopfian as an R-module. □

4. Conclusion

In this paper the notion of γ-Hopfian modules are present. The relation between the class of γ-Hopfian and other classes of Hopfian modules are given. Some basic characterizations of γ-Hopfian modules are proved. And some other properties of γ-Hopfian modules are also obtained with examples.

For further studies we shall be interested in the following problems:

- What is the structure of rings whose finitely generated right modules are γ-Hopfian?
- Let R be a ring with identity, and M be a γ-Hopfian module. Is $M[X, X^{-1}]$ γ-Hopfian in $R[X, X^{-1}]$-module?
- Let R be a γ-Hopfian ring and $n \geq 1$ an integer. Is the matrix ring $M_n(R)$ γ-Hopfian?

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