

Research Paper

ON THE ESSENTIAL CP -SPACES

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Dedicated to professor F. Azarpanah on the occasion of his 70th birthday.

ABSTRACT. Let $C_c(X)$ be the functionally countable subalgebra of $C(X)$. Essential CP -spaces are introduced and investigated algebraically and topologically. It is shown that if X is a proper essential CP -space, then $mC_c(X)$ is compact if and only if $\{\eta\}$ is a G_δ , where η is the only non CP -point of X and $mC_c(X)$ is the space of minimal prime ideals of $C_c(X)$ which are not maximal. Quasi F_c -spaces, c -basically disconnect spaces, almost CP -spaces and almost essential CP -spaces are introduced and studied via essential CP -spaces. Finally, $C_c(X)$ as a CSV -ring where X is an essential CP -space is investigated.

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1. INTRODUCTION

All topological spaces X are considered to be infinite Tychonoff, unless otherwise mentioned. Let $C(X)$ (resp., $C^*(X)$) be the ring of real-valued continuous functions (resp., bounded functions) on a space X . For each $f \in C(X)$, the set of zeros of f which is denoted by $Z(f)$, is called the zero-set of f and $X \setminus Z(f)$ is the cozero-set of f . The set of all zero-sets (resp., cozero-sets) in X is denoted by $Z(X)$ (resp., $Coz(X)$). βX denotes the Stone-Ćech compactification of X and vX is the Hewit real-compactification of X . For a zero dimensional space X the counterpart of vX is v_0X . The subalgebra $C^*(X)$ of $C(X)$ has an important role in study the relation between topological properties of X and algebraic properties of $C(X)$. But it is shown that, for any topological space X , $C^*(X) \cong C(\beta X)$. The subring of $C(X)$ consisting of those functions with countable image, which is denoted by $C_c(X)$ is introduced and studied by Karamzadeh et al. in [7, 8]. It is shown that for each topological space X does not necessarily exist a topological space Y where $C_c(X) \cong C(Y)$ despite $C_c(X)$ behaves like $C(X)$ and this fact motivated us enough to study more $C_c(X)$, see [4], [7], [8], and [15]. In this paper we aim to introduce essential CP -spaces and investigate the relations between topological properties of X and algebraic properties of $C_c(X)$ via essential CP -spaces and some related spaces. We remind the reader that the set of zeros of f for each $f \in C_c(X)$ denoted by $Z(f)$. An ideal I in $C_c(X)$ is called a z_c -ideal, if $Z(f) \subseteq Z(g)$ (where $f \in I$ and $g \in C_c(X)$) implies that $g \in I$. A subset S of X is called C_c^* -embedded in X if for each $f \in C_c^*(S)$ there exists $\bar{f} \in C_c^*(X)$ such that $\bar{f}|_S = f$. A space X is called F_c -space if every cozero-set is C_c^* -embedded, see [4]. The space X in which $\text{cl}(coz(f))$ is open for each $f \in C_c(X)$ is called c -basically disconnected. A topological space X with a base of clopen sets is called zero dimensional. Banaschewski has shown that every zero dimensional space X has a zero dimensional compactification, denoted by β_0X . In [4], it is shown that X is an F_c -space if and only if β_0X is an F_c -space. We recall that X is an F -space if and only if βX is an F -space. It is shown that X is strongly zero dimensional whenever βX is zero dimensional. We note that F -spaces and F_c -spaces coincide for a strongly zero dimensional space X , see [4]. A point p of X is called a CP -point if $f(p) = 0$ (where $f \in C_c(X)$) implies that $p \in \text{int}(Z(f))$. We recall that $p \in X$ is a P -point if for every $f \in C(X)$, $f(p) = 0$ implies that $p \in \text{int}(Z(f))$. The subspace of all CP -points of space X denotes by $CP(X)$. A topological space X is called a CP -space whenever each point of X is a CP -point, see [7]. We define the ideals O_c^p and M_c^p in $C_c(X)$ for $p \in \beta_0X$ such that, $O_c^p = \{f \in C_c(X) : p \in \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f)\}$ and $M_c^p = \{f \in C_c(X) : p \in \text{cl}_{\beta_0X} Z(f)\}$, see [4]. The space X is called quasi F_c -space if each dense cozero-set in $C_c(X)$ is C_c^* -embedded. We remind the reader that an element f of a commutative ring R with identity element is called a Von Neumann regular element if there is an element $g \in R$ such that $f^2g = f$. The ring $C_c(X)$ is Von Neumann regular (VNR) (X is called a CP -space), if and only if all elements in the

ring, are Von Neumann regular. It is shown that any P -space is a CP -space but the converse is not necessarily true, see [7]. For a zero dimensional space X , P -spaces and CP -spaces coincide. X is called an essential CP -space whenever all points except almost one point of X are CP -points. If X has exactly one non CP -point, it is called a proper essential CP -space and its non CP -point is denoted by η . It is evident that every essential P -space is an essential CP -space, but the converse is not necessarily true. The ring R is called a Von Neumann local ring (VNL) whenever for each $a \in R$ either a or $1 - a$ is a Von Neumann regular element. It is shown that the ring $C_c(X)$ is a Von Neumann local ring if and only if X is an essential CP -space. In [1], it is shown that if a has a Von Neumann inverse (i.e., there exists $b \in R$ such that $a^2b = a$), then there is a unit u of R such that au is an idempotent. Quasi F_c -spaces are introduced and investigated versus quasi F -spaces and essential CP -spaces. Also, c -basically disconnected spaces, almost CP -spaces and almost essential CP -spaces are defined and studied. It is shown that whenever X is an essential CP -space and $\{\eta\}$ is a G_δ , then F_c -spaces, quasi F_c -spaces and c -basically disconnected spaces coincide. A Tychonoff space X is called a CSV -space whenever $\frac{C_c(X)}{P}$ is a valuation domain for each prime ideal P . It is shown that for a proper essential CP -space X , X is a CSV -space if and only if $\frac{C_c(X)}{P}$ is a valuation domain, for each minimal prime ideal P contained in $M_{c\eta}$.

2. CHARACTERIZATION OF AN ESSENTIAL CP -SPACES

We introduced essential CP -spaces and investigate the relation between topological properties of X and algebraic properties of $C_c(X)$.

Definition 2.1. A topological space X is called an essential CP -space whenever all points except almost one point of X are CP -point. If X has exactly one non CP -point, it is called a proper essential CP -space and its non CP -point denotes by η .

It is evident that every essential P -space is an essential CP -space but the following example shows that the converse does not necessarily hold.

Example 2.2. Let $\Sigma = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$ and define a topology on Σ as follow, all points of \mathbb{N} are isolated and the neighborhood of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$, where \mathcal{U} be a free ultrafilter on \mathbb{N} . The ideal O_σ in Σ is prime but it is not maximal. So Σ is not a P -space but it is an proper essential P -space. Since \mathbb{N} is a discrete space, we infer that it is a P -space. σ is an only non P -point of Σ , therefore Σ is an essential CP -space too. Now, let $X = \Sigma \cup [-1, 0]$. Clearly, X is not an essential P -space but it is an essential CP -space.

We recall that $O_{cp} = \{f \in C_c(X) : p \in \text{int}_X(Z(f))\}$, see [7]. The next proposition is the counterpart of [2, Proposition 1.2]

Proposition 2.3. *Let X be a zero dimensional proper essential CP -space and P is a non-maximal prime ideal of $C_c(X)$, then $P \subset M_{c\eta}$, where $M_{c\eta} = \{f \in C_c(X) : \eta \in Z(f)\}$.*

Proof. Let P be a nonmaximal prime ideal of $C_c(X)$, so there exists $p \in \beta_0 X$ such that $O_c^p \subseteq P \subseteq M_c^p$. If $p \in \beta_0 X \setminus X$, then $P = O_c^p = M_c^p$ which is a contradiction by hypothesis. Let $p \in X$, if $p \neq \eta$, then $M_{cp} = O_{cp} = P$ that is impossible. So $p = \eta$ and it infers that $P \subset M_{c\eta}$. \square

In the next proposition which is the counterpart of [1, Theorem 5.2], the equivalent conditions with essential CP -spaces are characterized for $C_c(X)$.

Proposition 2.4. *Let X be a topological space, then the following statements are equivalent.*

- (1) $C_c(X)$ is a VNL -ring (Von Neumann Local ring).
- (2) For each $f \in C_c(X)$, either $Z(f)$ or $Z(1 - f)$ is open.
- (3) If $Z(f) \cap Z(g) = \emptyset$, then $Z(f)$ or $Z(g)$ is open.
- (4) X is an (proper) essential CP -space.
- (5) For each ideal I of $C_c(X)$, I is a z_c -ideal or $I \subset M_{c\eta}$.
- (6) For each ideal I of $C_c(X)$, $I = (f)$ or $I = (1 - f)$ is a z_c -ideal.
- (7) If $f \in C_c(X)$, then $\text{coz}(f)$ or $\text{coz}(1 - f)$ is C_c -embedded in X .
- (8) Each ideal I is an intersection of prime ideals or $I \subset M_{c\eta}$.
- (9) Each ideal I is an intersection of maximal ideals or $I \subset M_{c\eta}$.

Proof. (1) \iff (2) $C_c(X)$ is a VNL -ring if and only if for $f \in C_c(X)$ there is a $g \in C_c(X)$ such that $f = f^2g$. Therefore $f(1 - fg) = 0$, so $Z(f) \cup Z(1 - fg) = X$ and $Z(f) \cap Z(1 - fg) = \emptyset$. Hence $Z(f) = X \setminus Z(1 - fg)$, i.e., $Z(f)$ is open.

(2) \iff (3) Let $h = \frac{f^2}{f^2 + g^2}$. $h \in C_c(X)$, since $Z(f) \cap Z(g) = \emptyset$ and $Z(h) = Z(f)$, $Z(1 - h) = Z(g)$, therefore (2) and (3) are equivalent.

(3) \implies (4) Let p_1 and p_2 are two distinct non CP -points of X contained in disjoint neighborhoods U_1 and U_2 . So there are f_1 and f_2 in $C_c(X)$ such that $Z(f_1) \subseteq U_1$, $Z(f_2) \subseteq U_2$ and $p_1 \notin \text{int}(Z(f_1))$, $p_2 \notin \text{int}(Z(f_2))$. So $Z(f_1)$ and $Z(f_2)$ are two disjoint zero sets of X neither of which is open.

(4) \implies (1) Suppose that X is a proper essential CP -space with non CP -point η and $f \in C_c(X)$. If $\eta \notin Z(f)$, then $Z(f)$ is open. Let $g(x) = \frac{1}{f}(x)$ where $x \in \text{coz}(f)$ and otherwise $g(x) = 0$. $g \in C_c(X)$ and $f = f^2g$. If $\eta \in Z(f)$, then $\eta \notin Z(1 - f)$ and by the similar argument $1 - f$ has Von Neumann inverse. Hence $C_c(X)$ is a VNL -ring.

(4) \implies (5) Let I be an any ideal of $C_c(X)$. It is sufficient to show that \sqrt{I} is a z_c -ideal or $I \subseteq M_{c\eta}$. Suppose that $P \in \text{Min}(I)$. If P is a maximal ideal, then it is a z_c -ideal, so \sqrt{I} is a z_c -ideal, thus I is a z_c -ideal. Otherwise, if there is a $P_0 \in \text{Min}(I)$ such that P_0 is not maximal

ideal, then by Proposition 2.3, $P_0 \subset M_{c\eta}$, so $I \subset P_0 \subset M_{c\eta}$.

(5) \implies (4) Suppose that there is $x \in X$ such that $x \neq \eta$. We show that $M_{cx} = O_{cx}$. Let $f \in M_{cx} - O_{cx}$, then there exists a prime ideal P that is not z_c -ideal and $O_{cx} \subseteq P$. From That P is not z_c -ideal by assumption $P \subseteq M_{c\eta}$. So $O_{cx} \subset M_{c\eta}$ and it is a contradiction since M_{cx} is an only maximal ideal including O_{cx} .

(2) \iff (6) It follows immediately from that $I = (f)$ is a z_c -ideal if and only if $Z(f)$ is open.

(2) \implies (7) Let $f \in C_c(X)$, suppose that $Z(f)$ is open. Let $V = X \setminus Z(f) = \text{coz}(f)$. So V and $Z(f)$ are two disjoint open sets in X . If $g \in C_c(V)$ let $h(x) = g(x)$, for each $x \in V$ and $h(x) = 0$, otherwise. So $h \in C_c(X)$ and $h|_V = g$. Thus $\text{coz}(f)$ is C_c -embedded in X . If $Z(1 - f)$ is open similarity $\text{coz}(1 - f)$ is C_c -embedded in X .

(7) \implies (1) If $f \in C_c(X)$, put $Z_1 = Z(f)$ and $Z_2 = Z(1 - f)$. From (7), $V = X \setminus Z_1$ or $W = X \setminus Z_2$ are C_c -embedded in X . If V is C_c -embedded in X . Let $f_0 = \frac{1}{(f|_V)}$, so $f_0 \in C_c(V)$ and there is a $g \in C_c(X)$ such that $g|_V = f_0$ and $f^2g = f$. Hence f has a Von Neumann inverse.

(5) \implies (8) If I is a z_c -ideal, then it is a semiprime. So I is an intersection of prime ideals.

(8) \implies (1) Let $f \in C_c(X)$, then ideal (f^2) is an intersection of prime ideals or $(f^2) \subseteq M_{c\eta}$. Suppose that (f^2) is an intersection of prime ideals. So $(f^2) = \bigcap \{P : P \text{ is prime}\}$ and $f^2 \in P$ implies that $f \in P$, for every prime ideal P . Hence $f \in \bigcap \{P : P \text{ is prime}\}$. Therefore $f \in (f^2)$ and it follows that there is $f_0 \in C_c(X)$ such that $f = f^2f_0$. So f has a Von Neumann inverse. Otherwise, if $(f^2) \subseteq M_{c\eta}$, then $f^2 \in M_{c\eta}$, so $\eta \in Z(f^2) = Z(f)$, thus $\eta \notin Z(1 - f)$. Hence $1 - f$ is a regular.

(9) \implies (8) It is clear.

(8) \implies (9) From (8), I is an intersection of prime ideals or $I \subset M_{c\eta}$. If all prime ideals are maximal we are done. Otherwise, if there is $P \in \text{Min}(I)$ such that P is not maximal by Proposition 2.3, $I \subset P \subset M_{c\eta}$, so $I \subset M_{c\eta}$. \square

An ideal $I \subset R$ is called pure if $I = mI$, where $mI = \{a \in R : I + A(a) = R\} = \{a \in R : a = ai, \text{ for some } i \in I\}$, see [1]. We remind the reader that the ring R is called a *SVNL*-ring, if for a nonempty subset S of R that $\langle S \rangle = R$, at least one element of S has Von Neumann inverse.

For a proper essential *CP*-space X with non *CP*-point η and $f \in C_c(X)$ if $\eta \notin Z(f)$, then f has a Von Neumann inverse. The next proposition is the counterpart of [1, Corollary 5.5].

Proposition 2.5. *$C_c(X)$ is a VNL-ring if and only if it is a SVNL-ring.*

Proof. If $C_c(X)$ is a *SVNL*-ring, then evidently it is a *VNL*-ring. Conversely, if $C_c(X)$ is a *VNL*-ring, then X is an (proper) essential *CP*-space. Suppose that $C_c(X) = \langle f_1, f_2, \dots, f_n \rangle$

for $f_i \in C_c(X)$, $1 \leq i \leq n$. So $\bigcap_{i=1}^n Z(f_i) = \emptyset$. Hence there is $1 \leq j \leq n$ such that $\eta \notin Z(f_j)$. Therefore f_j has a Von Neumann inverse and it follows that $C_c(X)$ is a *SVNL*-ring. \square

In [1, Theorem 2.6], it was shown that the ring R is an *SVNL*-ring if and only if all maximal ideals of R except may be one of them are pure. By using this fact and Proposition 2.5, we infer the next corollary.

Corollary 2.6. *$C_c(X)$ is a *VNL*-ring if and only if all maximal ideals of it except maybe one of them are pure.*

In case X is a *CP*-space, each ideal of $C_c(X)$ is a z_c -ideal. In the next proposition, it is shown that in an essential *CP*-space X with non *CP*-point η for each ideal I of $C_c(X)$, if each point of $\bigcap Z_c[I]$ be a *CP*-point, then I is a z_c -ideal, see [5].

Proposition 2.7. *Let X be an essential *CP*-space with non *CP*-point η and I be an ideal of $C_c(X)$. Whenever every point in $\bigcap Z_c[I]$ is a *CP*-point, then I is a z_c -ideal.*

Proof. Let η be the only non *CP*-point of X , then $\eta \notin \bigcap Z_c[I]$. So there exists $i \in I$ such that $\eta \notin Z(i)$. Now suppose that $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C_c(X)$, so $Z(f^2 + i^2) \subseteq Z(g)$. Each point of $Z(f^2 + i^2)$ is a *CP*-point. Therefore $Z(f^2 + i^2)$ is open and $Z(f^2 + i^2) \subseteq \text{int}_X Z(g)$. Hence g is a multiple of $f^2 + i^2$. Therefore $g \in I$ and I is a z_c -ideal. \square

In the next theorem which is the counterpart of [1, Theorem 5.6] some properties of (proper) essential *CP*-spaces are investigated.

Theorem 2.8. *If X is an essential *CP*-space with a non *CP*-point η , then the following statements hold.*

- (1) *The subspaces of X are essential *CP*-spaces.*
- (2) *Each continuous and open image of X is an essential *CP*-space.*
- (3) *If X is compact, there is an infinite discrete space Y such that X is an one-point compactification of it.*
- (4) *If X is a zero dimensional space and $\beta_0 X$ is an essential *CP*-space, then $X = \beta_0 X$.*
- (5) *If X is a zero dimensional space and $q \in \beta_0 X \setminus X$, M_c^q is a pure ideal.*
- (6) *If X is a zero dimensional space, then $v_0 X$ is an essential *CP*-space with a non *CP*-point η .*
- (7) *If $|Y| \geq 2$, then $X \times Y$ is not an essential *CP*-space.*

Proof. (1) It is evident.

(2) Let $\varphi : X \longrightarrow Y$ be an open and continuous surjection and Z_1, Z_2 be two disjoint zero

sets in Y . Since φ is continuous we infer that $\varphi^{-1}(Z_1)$ and $\varphi^{-1}(Z_2)$ are disjoint zero sets of X . But X is an essential CP -space, so by Proposition 2.4, one of them is open. Suppose that $\varphi^{-1}(Z_1)$ is open in X . Since φ is open and $\varphi[\varphi^{-1}(Z_1)] = Z_1$, we infer that Z_1 is open. So Y is an essential CP -space.

(3) Let U_x be a compact neighborhood of $x \neq \eta$, then it is a compact CP -space. So U_x is finite, i.e., x is an isolated point. If U is an open cover of X , it is a union of $\{x\}$ for each $x \neq \eta$ and a neighborhood of η , so it must have a finite subcover. So X is the one-point compactification of the discrete space $X \setminus \{\eta\}$. Therefore each neighborhood of η must be cofinite.

(4) Let $Y = X \setminus \{\eta\}$, then from (3), X and $\beta_0 X$ are the one point compactifications of Y .

(5) If $q \in \beta_0 X \setminus X$, then $q \notin X$, so $q \neq \eta$. Hence maximal ideal M_c^q must be pure.

(6) Since $C_c(X) \cong C_c(v_0 X)$ and $C_c(X)$ is a VNL -ring, we infer that $C_c(v_0 X)$ is a VNL -ring too, so $v_0 X$ is an essential CP -space.

(7) If $y_1 \neq y_2$, then $(\eta, y_1), (\eta, y_2)$ are two disjoint non CP -points of $X \times Y$. So $X \times Y$ is not an essential CP -space. \square

We recall that if X is an essential CP -space, then $v_0 X$ is an essential CP -space, see Theorem 2.8. Moreover, if X is a pseudocompact space, then by [14, Theorem 6.3] $v_0 X = \beta_0 X$. Hence $\beta_0 X$ is an essential CP -space. Therefore by Theorem 2.8, $X = \beta_0 X$ which implies that X is compact. So we have the next corollary.

Corollary 2.9. *Let X be a zero dimensional essential CP -space with a non CP -point η , then the following statements are equivalent.*

- (1) X is pseudocompact.
- (2) X is countably pseudocompact.
- (3) X is compact.

3. THE ESSENTIAL CP -SPACES VIA RELATED SPACES

In this section, we introduce a quasi F_c -space and investigate relations between quasi F -spaces, quasi F_c -spaces and essential CP -spaces. Also, we define c -basically disconnected spaces, almost CP -spaces and almost essential CP -spaces. It is shown that whenever X is an essential CP -space and $\{\eta\}$ is a G_δ , then F_c -spaces, quasi F_c -spaces and c -basically disconnected spaces coincide.

Definition 3.1. A space X is called a quasi F_c -space if each dense cozero-set in $C_c(X)$ is C_c^* -embedded.

It is clear that any quasi F -space is a quasi F_c -space, but the converse is not necessarily true. For example, \mathbb{R} with usual topology is a quasi F_c -space that is not a quasi F -space.

Remark 3.2. Each CP -space is a quasi F_c -space, but the converse is not necessarily true. For example, space $\Sigma = \mathbb{N} \cup \{\sigma\}$ in Example 2.2 is a F_c -space and an essential CP -space with a non CP -point σ . Since every F_c -space is a quasi F_c -space, we infer that Σ is a quasi F_c -space too.

Definition 3.3. A space X in Y is Z_c -embedded if for each $Z \in Z_c(X)$, there is a set H in $Z_c(Y)$ such that $H \cap X = Z$. A dense subspace X of Tychonoff space Y is $Z_c^\#$ -embedded in Y if for each $Z \in Z_c(X)$ there is H in $Z_c(Y)$ such that $\text{cl}_X(\text{int}_X Z) = X \cap \text{cl}_Y(\text{int}_Y H)$.

Theorem 3.4. *If X is an open or dense subspace of Y , Then the following are equivalent.*

- (1) X is $Z_c^\#$ -embedded in Y .
- (2) If $C \in \text{Coz}(X)$, then there exists $V \in \text{Coz}(Y)$ such that $\text{cl}_X C = X \cap \text{cl}_Y V$.

Proof. (1) \implies (2) Let $C \in \text{Coz}(X)$. So there exists $f \in C_c(X)$ that $C = X \setminus Z(f)$. From (1) there exists $g \in C_c(Y)$ such that $Z(f) = X \cap Z(g)$, therefore $X \setminus Z(f) = X \cap (Y \setminus Z(g))$. So $\text{cl}_X(X \setminus Z(f)) = \text{cl}_X(X \cap (Y \setminus Z(g))) = X \cap \text{cl}_X(Y \setminus Z(g))$. Now, if we put $Y \setminus Z(g) = V$, the proof is complete.

(2) \implies (1) It is evident. \square

The next proposition is the counterpart of [12, Proposition 3.2], that shows C_c^* -embedded, Z_c -embedded and $Z_c^\#$ -embedded are coincide for each dense subspace of a quasi F_c -space.

Proposition 3.5. *If X is a dense subspace of a quasi F_c -space Y , then the following statements are equivalent.*

- (1) X is C_c^* -embedded in Y .
- (2) X is Z_c -embedded in Y .
- (3) X is $Z_c^\#$ -embedded in Y .

Proof. (1) \implies (2) It is evident.

(2) \implies (3) Let $Z \in Z_c(X)$. So by (2) there exists $H \in Z_c(Y)$ such that $H \cap X = Z$. By Theorem 3.4, it is sufficient to show that $\text{cl}_X Z = X \cap \text{cl}_Y H$. Since $H \cap X = Z$, we infer that $\text{cl}_X Z = \text{cl}_X(H \cap X) = \text{cl}_Y H \cap X$.

(3) \implies (1) From [9], to show that X is c^* -embedded in Y , let Z_1 and Z_2 be two disjoint zero-sets in X . We show that $\text{cl}_Y Z_1 \cap \text{cl}_Y Z_2 = \emptyset$. Since each two disjoint zero-sets in X are completely separated, we infer that there exist disjoint zero-sets S_1 and S_2 in X such that $Z_1 \subseteq \text{int}_X S_1$ and $Z_2 \subseteq \text{int}_X S_2$. Since X is $Z_c^\#$ -embedded in Y , there exist V_1 and V_2 in $\text{Coz}(Y)$ such that $\text{cl}_X \text{int}_X S_1 = X \cap \text{cl}_Y \text{int}_Y (Y \setminus V_1)$ and $\text{cl}_X \text{int}_X S_2 = X \cap \text{cl}_Y \text{int}_Y (Y \setminus V_2)$ and from Theorem 3.4, we have $\text{cl}_X(X \setminus S_1) = X \cap \text{cl}_Y V_1$ and $\text{cl}_X(X \setminus S_2) = X \cap \text{cl}_Y V_2$. Since $S_1 \cap S_2 = \emptyset$, we infer that $\text{int}_X S_1 \cap \text{int}_X S_2 = \emptyset$, hence $X \setminus \text{int}_X S_1 = \text{cl}_X(X \setminus S_1) =$

$X \cap \text{cl}_Y V_1 \subseteq \text{cl}_Y V_1$. Therefore $\text{int}_Y(Y \setminus V_1) \subseteq \text{int}_X S_1$ and similarly $\text{int}_Y(Y \setminus V_2) \subseteq \text{int}_X S_2$, so $\text{int}_Y(Y \setminus V_1) \cap \text{int}_Y(Y \setminus V_2) = \emptyset$.

Since Y is a quasi F_c -space, we infer that $\text{cl}_Y(\text{int}_Y(Y \setminus V_1)) \cap \text{cl}_Y(\text{int}_Y(Y \setminus V_2)) = \emptyset$, see [18]. So $\text{cl}_X(\text{int}_X S_1) \cap \text{cl}_X(\text{int}_X S_2) = \emptyset$, therefore $\text{cl}_X Z_1 \cap \text{cl}_X Z_2 = \emptyset$. Hence $\text{cl}_Y Z_1 \cap \text{cl}_Y Z_2 = \emptyset$ and we are done. \square

Definition 3.6. A space X is called c -basically disconnected if for each $f \in C_c(X)$, $\text{cl}(\text{coz}(f))$ be open.

Definition 3.7. $p \in X$ is called an almost CP -point if $\text{int}(Z(f)) \neq \emptyset$ for each $f \in M_{cp}$. A topological space X is called an almost CP -space where each point of X is an almost CP -point.

Clearly each almost P -space is an almost CP -space, but the converse is not necessarily true. For instance, consider \mathbb{R} with usual topology. For each $x \in \mathbb{R}$, $\text{int}([x]) = \emptyset$ ($[x] \in Z(\mathbb{R})$), so x is not an almost P -point but \mathbb{R} is a CP -space and hence it is an almost CP -space. We note that for a zero dimensional space X , almost CP -space and almost P -space coincide, see [3].

Example 3.8. $\sigma \in \Sigma$ is not an almost P -point. From that $\{\sigma\}$ is a zero set in Σ , there is $f \in C(X)$ such that $Z(f) = \{\sigma\}$, but $\sigma \notin \text{int}(Z(f))$. If $\sigma \in \text{int}(Z(f))$, then there exists an open set G in Σ such that $\sigma \in G \subseteq Z(f)$, so $G = U \cup \{\sigma\}$ for $U \in \mathcal{U}$, thus $U = \emptyset$ which is a contradiction. Therefore σ is not an almost P -point.

Remark 3.9. Essential P -spaces and almost P -spaces may not be concluded each other. The space Σ in Example 2.2, is an essential P -space which is not an almost P -space. Consider $\beta\mathbb{N} \setminus \mathbb{N}$. Any nonempty G_δ -set in $\beta\mathbb{N} \setminus \mathbb{N}$ has a nonempty interior, see [9, 6S]. So $\beta\mathbb{N} \setminus \mathbb{N}$ is an almost P -space. But, $\beta\mathbb{N} \setminus \mathbb{N}$ has more than one non P -point, so it is not an essential P -space, see [9, 6T].

In the next definition we introduce spaces in which all points of them, maybe almost one of them are almost CP -points and characterize these spaces via essential CP -spaces and almost CP -spaces.

Definition 3.10. X is called an almost essential CP -space if all points of X except almost one of them are almost CP -point.

For example space $\Sigma = \mathbb{N} \cup \{\sigma\}$ is an almost essential P -space and hence it is an almost essential CP -space. Let $X = \Sigma \cup [-1, 0]$ in Example 2.2, then X is an almost essential CP -space that is not an almost essential P -space.

Let $mC_c(X)$ be the set of all minimal prime ideals of $C_c(X)$ that are not maximal, see [11]. $mC_c(X)$ need not to be compact. We can see that X is c -basically disconnected in case of $mC_c(X)$ is compact and any almost CP -point of X is CP -point.

Proposition 3.11. *For a topological space X where $CP(X)$ is dense the following statements hold.*

- (1) *If $CP(X)$ is Z_c -embedded in X , then $mC_c(X)$ is compact.*
- (2) *If $CP(X)$ is a cozero set in X , then $mC_c(X)$ is compact.*
- (3) *If $CP(X)$ is a Lindelöf subspace of X , then $mC_c(X)$ is compact.*

Proof. (1) Let $CP(X)$ be a CP -space, so $C_c(CP(X))$ is a Von Neumann regular ring. Therefore $mC_c(CP(X))$ is a compact space, see [11]. Since $CP(X)$ is dense and Z_c -embedded in X , we infer that the space $mC_c(CP(X))$ is homeomorphic with $m(C_c(X))$ and hence $m(C_c(X))$ is compact, see [17, Theorem 7.6].

(2) We show that each cozero-set in X is Z_c -embedded. If $g \in C_c(X)$ and $Y = \text{coz}(g)$, we put $h(x) = 0$ for each $x \in Z(g)$ and otherwise $h(x) = (f \wedge g)(x)$. Clearly, $h \in C_c(X)$ and $Z(f) = Z(h) \cap Y$. Hence Y is a Z_c -embedded in X . Now, by (1) the proof is complete.

(3) Suppose that S be a Lindelöf subspace of X . From (1) it is sufficient to show that S is Z_c -embedded in X . Let $Z \in Z_c(X)$, since $S \setminus Z$ is a F_σ -set and each F_σ -set in a Lindelöf space is Lindelöf, we infer that $S \setminus Z$ is a Lindelöf space. Let $F = \{(S \setminus Z) \cap Z' : Z' \in Z_c(X), Z \subseteq Z'\}$. Each element of F is closed in $S \setminus Z$. We show that $\bigcap F = \emptyset$. Since $S \setminus Z$ is open in S , for each $x \in S \setminus Z$ there exists open subset U in X such that $S \setminus Z = U \cap S$. It is evident that $U \cap Z = \emptyset$. So $x \notin \text{cl}_X Z$, hence there exists $f \in C_c(X)$ such that $f(\text{cl}_X Z) = \{0\}$ and $f(x) = 1$, therefore $x \notin Z(f) \cap (S \setminus Z)$. Therefore $\bigcap F = \emptyset$. Since $S \setminus Z$ is a Lindelöf space, we infer that F has not countable intersection property. So there exists a family of zero-sets of X such that $\{Z_n : n \in \mathbb{N}\}$ that $Z \subseteq Z_n$ for each $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} (Z_n \cap (S \setminus Z)) = \emptyset$. Let $Z^* = \bigcap_{n \in \mathbb{N}} Z_n$. Then $Z^* \in Z_c(X)$, $Z \subseteq Z^*$ and $Z^* \cap (S \setminus Z) = \emptyset$. Hence $Z^* \cap S = Z$, therefore S is Z_c -embedded in X and we are done. \square

We recall that X is basically disconnected if and only if βX is basically disconnected, see [9]. It is shown that for a zero dimensional space X , X is c -basically disconnected if and only if $\beta_0 X$ is c -basically disconnected. So if X is a zero dimensional CP -space, then X and $\beta_0 X$ are c -basically disconnected.

Theorem 3.12. *If X is a proper essential CP -space, then the following statements hold.*

- (1) *$mC_c(X)$ is compact if and only if $\{\eta\}$ is a G_δ .*
- (2) *If X is a zero dimensional quasi F_c -space and $\{\eta\}$ is a G_δ , then X is a c -basically disconnected.*

Proof. (1) Suppose that $\{\eta\}$ is a G_δ . So $CP(X) = X \setminus \{\eta\}$ is a cozero-set, hence from Proposition 3.11, $mC_c(X)$ is a compact space. Conversely suppose that $mC_c(X)$ be a compact space and $\{\eta\}$ does not be a G_δ -set. From [10, Corollary 2.6.6], $\{\eta\}$ is an almost CP -point

that is not a CP -point. So $mC_c(X)$ is not a compact space and it is a contradiction. Therefore $\{\eta\}$ need to be a G_δ -set.

(2) From Proof of (1), $CP(X)$ is a cozero-set of a quasi F_c -space X . So $CP(X)$ is C^* -embedded in X , therefore $\beta_0 X$ and so X is a c -basically disconnected space. \square

Corollary 3.13. *If X is a proper essential CP -space and $\{\eta\}$ is a G_δ , then the following statements are equivalent.*

- (1) X is a quasi F_c -space.
- (2) X is c -basically disconnected.
- (3) X is an F_c -space.
- (4) $C_c(X)$ has a unique prime ideal that is not maximal.

Proof. (1) \implies (2) It is evident.

(2) \implies (3) In each c -basically disconnected space every two disjoint cozero-set are completely separated. So for each $f \in C_c(X)$, $\text{neg}(f)$ and $\text{pos}(f)$ are completely separated, therefore from [4], X is a F_c -space.

(3) \implies (4) It is evident.

(4) \implies (1) It is evident. \square

We remind the reader that if X is a strongly zero dimensional space, then X is a quasi F -space if and only if whenever $f \in C(X)$ is regular, then there is a $k \in C(X)$ such that $f = k|f|$, see [18]. It is evident that the recent fact holds for quasi F_c -spaces too.

4. $C_c(X)$ AS A CSV -RING WHERE X IS AN ESSENTIAL CP -SPACE

In this section we investigate conditions that $\frac{C_c(X)}{P}$ is a valuation domain for prime ideal P of $C_c(X)$. We remind the reader that a commutative ring R is a valuation ring if for each nonzero elements a and b in R , $a|b$ or $b|a$. An integral domain D is called a valuation domain if it is a valuation ring. Any field F is a valuation domain and any valuation domain is a local ring. Each finitely generated ideal of a valuation ring is principal (i.e., any valuation ring is a Bezout domain). Also recall that a commutative ring R with identity is called an SV -ring if $\frac{R}{P}$ is a valuation domain for every proper prime ideal P . For each maximal ideal M of a ring R , number of minimal prime ideals of R that are contained in M is the rank of M and the rank of R is the supremum of the ranks of all maximal ideals of R , see [2]. A commutative integral domain D is called real-closed if a) it is totally ordered; b) nonnegative elements of it have square roots in D ; c) each monic polynomial of odd degree in $D[X]$ has a zero in D ; d) for a, b in D which $0 < a < b$, we have $b|a$, see [6].

Definition 4.1. A Tychonoff space X is called a CSV -space whenever $\frac{C_c(X)}{P}$ is a valuation domain for each prime ideal P . In this case $C_c(X)$ is called a CSV -ring.

Proposition 4.2. *Each zero dimensional F_c -space is a CSV -space.*

Proof. Suppose that X is a zero dimensional F_c -space. We show that $\frac{C_c(X)}{P}$ is a valuation ring for each minimal prime ideal P in $C_c(X)$. For given ideals I and J in $\frac{C_c(X)}{P}$ there are ideals A and B in $C_c(X)$ such that $I = \frac{A}{P}$ and $J = \frac{B}{P}$ and $P \subseteq A$, $P \subseteq B$. Since P is a prime ideal in $C_c(X)$ there exists $p \in \beta_0 X$ such that $O_{cp} \subseteq P$. So $O_{cp} \subseteq A$ and $O_{cp} \subseteq B$. Since X is a F_c -space we infer that the prime ideals of $C_c(X)$ contained in any given maximal ideal of $C_c(X)$ form a chain, from [4]. Therefore $A \subseteq B$ or $B \subseteq A$. Hence $I \subseteq J$ or $J \subseteq I$ and so $\frac{C_c(X)}{P}$ is a valuation ring. \square

Corollary 4.3. *Each zero dimensional c -basically disconnected space is a CSV -space.*

We remind that for a commutative ring R ideal P is called real-closed if and only if $\frac{R}{P}$ is real-closed. Let P be a real-closed ideal of $C_c(X)$. For each proper prime ideal Q where $P \subset Q$, then Q is a real-closed. From that we have the next proposition which is the counterpart of [13, Proposition 2.1].

Proposition 4.4. *A topological space X is CSV -space if and only if every minimal prime ideals of $C_c(X)$ are real-closed.*

For each maximal ideal M of $C_c(X)$, since $\frac{C_c(X)}{M}$ is a field and any field is a valuation domain, we infer the next proposition.

Proposition 4.5. *Each maximal ideal of $C_c(X)$ is real-closed.*

Since all prime ideals of a CP -space X are maximal, we infer that each prime ideals of X are real-closed, so $\frac{C_c(X)}{P}$ is valuation domain for each prime ideal P , therefore X is a CSV -space.

Definition 4.6. For each $x \in X$, the C -rank of x is the rank of M_{cx} , where $M_{cx} = \{f \in C_c(X) : x \in Z(f)\}$.

If the number of minimal prime ideals of $C_c(X)$ that are contained in M_{cx} are infinite then $C\text{-rank}(x) = \infty$.

Proposition 4.7. *The C -rank of $C_c(X)$ is the C -rank of $M_{c\eta}$ where X is a proper essential CP -space with non CP -point η .*

Proof. As noted in Proposition 2.3 each prime ideals P of $C_c(X)$ that are not maximal contained in $M_{c\eta}$. So the C -rank of $M_{c\eta}$ is the supremum of the C -rank of maximal ideals of $C_c(X)$. Therefore the C -rank of $C_c(X)$ is the C -rank of $M_{c\eta}$. \square

The next proposition is given from Proposition 2.3.

Proposition 4.8. *For a proper essential CP-space X , X is a CSV-space if and only if $\frac{C_c(X)}{P}$ is a valuation domain for each minimal prime ideal P contained in $M_{c\eta}$.*

From [16], the rank of $x \in X$ is k if there exist precisely k pairwise of disjoint cozero sets that x contained in their closure.

Corollary 4.9. *If X is a CP-space (F_c -space), then $C_c(X)$ has a finite C -rank and $C\text{-rank}(x) = 1$ for each $x \in X$.*

Proof. If X is a CP-space, then for each $x \in X$, $M_{cx} = O_{cx}$, so $C\text{-rank}(M_{cx}) = C\text{-rank}(x) = 1$. Thus $C_c(X)$ has a finite C -rank. Hence if X is a F_c -space, then from [4], every maximal ideal of $C_c(X)$ contains a unique minimal prime ideal, so $C\text{-rank}(M_{cx}) = 1$ for each $x \in X$. \square

By Corollary 3.13 and Proposition 4.7, we have the next proposition.

Proposition 4.10. *If X is a proper essential CP-space with a non CP-point η and $\{\eta\}$ is a G_δ , then $C\text{-rank}(C_c(X)) = C\text{-rank}(M_{c\eta}) = 1$ if and only if X is an F_c -space.*

From that $C_c(X) \cong C_c(v_0X)$ for a zero dimensional space X , see [14], we have the following proposition which is the counterpart of [13, Proposition 2.2].

Proposition 4.11. *For each zero dimensional Tychonoff space X , the following statements are equivalent.*

- (1) X is a CSV-space.
- (2) v_0X is a CSV-space.

By [14, Theorem 6.3], for a zero dimensional space X , X is pseudocompact if and only if $v_0X = \beta_0X$, so from Proposition 4.11 we have the next fact.

Corollary 4.12. *For a zero dimensional pseudocompact Tychonoff space X , X is a CSV-space if and only if β_0X is a CSV-space if and only if v_0X is a CSV-space.*

Proposition 4.13. *If $C_c^*(X)$ is a CSV-ring, then the homomorphism image of $C_c^*(X)$ is a CSV-ring.*

Proof. Let P be an arbitrary prime ideal of $C_c^*(Y)$ and $\varphi : C_c^*(X) \rightarrow C_c^*(Y)$ be an epimorphism. From that $\pi : C_c^*(Y) \rightarrow \frac{C_c^*(Y)}{P}$ is a homomorphism, $\pi \circ \varphi : C_c^*(X) \rightarrow \frac{C_c^*(Y)}{P}$ is an epimorphism and $\text{Ker}(\pi \circ \varphi) = \varphi^{-1}(P)$. So $\frac{C_c^*(X)}{\varphi^{-1}(P)} \cong \frac{C_c^*(Y)}{P}$. \square

Proposition 4.14. *If X is a CSV-space and Y is an C_c^* -embedded subspace of X , then Y is a CSV-space.*

Proof. Let $\varphi : C_c^*(X) \longrightarrow mC_c^*(Y)$ where $\varphi(\bar{f}) = \bar{f}|_Y$ for each $\bar{f} \in C_c^*(X)$ and $mC_c^*(Y)$ be a set of minimal prime ideals of $C_c^*(Y)$. For each $f \in C_c^*(Y)$, there exists $\bar{f} \in C_c^*(X)$ such that $\bar{f}|_Y = f$. So $C_c^*(Y)$ is a homomorphism image of $C_c^*(X)$. Hence Y is a CSV-space. \square

Remark 4.15. We know that each closed and compact subspace Y of X is C -embedded and therefore it is C_c -embedded. So if X is a compact and CSV-space and Y is a closed subspace of X , then Y is a CSV-space.

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REFERENCES

- [1] E. Abu Osba, M. Henriksen and O. Alkam, *Combining local and Von Neumann regular ring*, Comm. Algebra, **32** No. 7 (2004) 2639-2653.
- [2] E. Abu Osba, M. Henriksen and O. Alkam, *Essential P -spaces: a generalization of door spaces*, Comment. Math. Univ. Carol., **45** No. 3 (2004) 509-518.
- [3] F. Azarpanah, *On almost P -spaces*, Far East J. Math. Sci. special, **2000** (2000) 121-132.
- [4] F. Azarpanah, O. A. S. Karamzadeh, Z. Keshtkar and A. R. Olfati, *On maximal ideals of $C_c(X)$ and the uniformity of its localizations*, Rocky Mt. J. Math., **48** No. 2 (2018) 1-9.
- [5] F. Azarpanah, R. Mohamadian and P. Monjezi, *On P_F -spaces*, Topol. Appl., **302** (2021) 107821.
- [6] G. Cherlin and M. Dickmann, *Real closed rings. I*, Fund. Math., **126** (1986) 147-183.
- [7] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, *On the functionally countable subalgebra of $C(X)$* , Rend. Sem. Mat. Univ. Padova., **129** (2013) 47-69.
- [8] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, *$C(X)$ versus its functionally countable subalgebra*, Bull. Iranian Math. Soc., **45** (2019) 173-187.
- [9] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, 1976.
- [10] A. Hager and J. Martinez, *Fraction-dense algebras and spaces*, Canad. J. Math., **45** (1993) 997-996.
- [11] M. Henriksen and M. Jerison, *The space of minimal prime ideal of a commutative ring*, Trans. Amer. Math. Soc., **115** (1965) 110-130.
- [12] M. Henriksen, J. Vermeer and R. G. Woods, *Quasi F -covers of Tychonoff spaces*, Trans. Amer. Math. Soc., **303** (1987) 779-803.
- [13] M. Henriksen and R. G. Wilson, *When is $\frac{C(X)}{P}$ a valuation domain for every prime ideal P ?*, Topology Appl., **44** (1992) 175-180.
- [14] O. A. S. Karamzadeh and Z. Keshtkar, *On c -realcompact spaces*, Quaest. Math., **41** No. 8 (2018) 1135-1167.

- [15] O. A. S. Karamzadeh, M. Namdari and S. Soltanpour, *On the locally functionally countable subalgebra of $C(X)$* , Appl. Gen. Topol., **16** No. 2 (2015) 183-207.
- [16] S. Larson, *f -ring in which every maximal ideal contains finitely many minimal prime ideals*, Comm. Algebra, **25** (1997) 3859-3888.
- [17] R. Montgomery, *Structures determined by prime ideals of rings of functions*, Trans. Amer. Math. Soc., **147** (1970) 367-380.
- [18] Y. L. Park, *The quasi- F cover as a filter space*, Houston J. Math., **9** No. 1 (1983) 101-109.

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