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Research Paper

CHARACTERIZATION OF MONOIDS BY A GENERALIZATION OF WEAK FLATNESS PROPERTY

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Abstract. In [On a generalization of weak flatness property, Asian-European Journal of Mathematics, 14(1) (2021)] we introduce a generalization of weak flatness property, called $(WF)'$, and showed that a monoid S is absolutely $(WF)'$ if and only if S is regular and satisfies Conditions $(R_{(WF)'})$ and $(L_{(WF)'})$. In this paper we continue the characterization of monoids by this property of their (finitely generated, (mono)cyclic, Rees factor) right acts. Also we give a classification of monoids for which $(WF)'$ property of their (finitely generated, (mono)cyclic, Rees factor) right acts imply other properties and vise versa. The aim of this paper is to show that the class of absolutely $(WF)'$ monoids and absolutely (weakly) flat monids are coincide.

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1. Introduction

In studying weak pullback flatness of acts over monoids, Laan in [9], introduced Condition (E'), a generalization of Condition (E). After that Golchin and Mohammadzadeh in [5] gave a characterization of monoids by Condition (E') of acts. They also in [6] introduced a generalization of Condition (P) called Condition (P'), and gave a characterization of monoids by this property of their acts. In [2], we introduced a new flatness property of acts over monoids which is a generalization of weak flatness, called $(WF)'$, and gave a characterization of absolutely $(WF)'$ monoids. In this paper, first of all we recall some results of [2] , and then give a characterization of monoids by this property of their (finitely generated, (mono)cyclic, Rees factor) right acts.

Throughout this paper S will denote a monoid. We refer the reader to [8] for basic definitions and terminology relating to semigroups and acts over monoids and to [9] for definitions and results on flatness which are used here.

A monoid S is called *right (left) reversible* if for every $s, t \in S$, there exist $u, v \in S$ such that $us = vt(su = tv)$. A submonoid P of a monoid S is called *weakly right reversible* if for every $s, t \in P, z \in S, sz = tz$, implies the existence of $u, v \in P$ such that $us = vt$. A monoid S is called *left (right) collapsible* if for every $s, t \in S$ there exists $z \in S$ such that $zs = zt(sz = tz)$. A submonoid P of S is called *weakly left collapsible* if for every $s, t \in P, z \in S$, $sz = tz$ implies the existence of $u \in P$ such that $us = ut$. A right ideal K_S of a monoid S is called *left stabilizing* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

A nonempty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$, and all $s, t \in S$. A right S-act A_S satisfies *Condition* (E) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (E') if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (EP) if for all $a \in A_S, s, t \in S$, $as = at$ implies that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies *Condition* $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (P') if for all $a, a' \in A$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (PWP) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

Recall from [8] that an act A_S is called *weakly flat* if the functor $A_S \otimes_S -$ preserves all inclusoins of left ideals into S. This is equivalent to say that, $as = a't$ for $a, a' \in A_S, s, t \in S$ implies $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes {}_S(Ss \cup St)$.

Definition 1.1. A right S-act A_S is $(WF)'$ if $as = a't$ and $sz = tz$ for $a, a' \in A_S, s, t, z \in S$ imply $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes {}_S(Ss \cup St)$.

Lemma 1.2. [2] *Let S be a monoid. Then:*

(1) *if $\{A_i \mid i \in I\}$ is a chain of subacts of an act A_S, and every $A_i, i \in I$ is $(WF)'$, then $\bigcup_{i\in I} A_i$ is $(WF)'$;*
(2) *$A_S = \coprod_{i\in I} A_i$ is $(WF)'$, if and only if every $A_i, i \in I$, is $(WF)'$;*
(3) *the right S-act S_S is $(WF)'$.*

Definition 1.3. A right S-act A_S satisfies *Condition* $(W_{(WF)'})$, if $as = a't$ and $sz = tz$, for $a, a' \in A_S, s, t, z \in S$, imply that there exist $a'' \in A_S, w \in Ss \cap St$, such that $as = a't = a''w$.

Theorem 1.4. [2] *For any monoid S the following statements are equivalent:*

(1) *if $A_S = \prod_{i\in I} A_i$ is $(WF)'$, then every A_i, $i \in I$ is $(WF)'$;*
(2) *if $A_S = \prod_{i\in I} A_i$ is $(WF)'$, then every A_i, $i \in I$ satisfies Condition $(W_{(WF)'})$;*
(3) *the one-element right S-act Θ_S is $(WF)'$;*
(4) *the one-element right S-act Θ_S satisfies Condition $(W_{(WF)'})$;*
(5) *S is weakly right reversible;*
(6) *there exists a $(WF)'$ right S-act containing a zero;*
(7) *there exists a right S-act which containing a zero and satisfies Condition $(W_{(WF)'})$.*

Theorem 1.5. [2] *Any retract of a $(WF)'$ right S-act is $(WF)'$.*

Theorem 1.6. [2] *A right S-act A_S is $(WF)'$, if and only if it is principally weakly flat and satisfies Condition $(W_{(WF)'})$.*

Theorem 1.7. [2] *Let ρ be a right congruence on S. Then the right S-act S/ρ is $(WF)'$ if, and only if, for all $x, y, z, s, t \in S$ with $(xs)\rho(yt)$ and $sz = tz$, there exist $u, v \in S$, such that $x(\rho \vee ker\rho_s)u$, $y(\rho \vee ker\rho_t)v$ and $us = vt$.*

2. Characterization of monoids by $(WF)'$ property of right acts

In this section we give a characterization of monoids by $(WF)'$ of (finitely generated, (mono)cyclic) right acts, and also a classification of monoids for which $(WF)'$ of their (finitely generated, (mono)cyclic) right acts imply other properties. At first we recall two results from [2].

Theorem 2.1. [2] *Let $w, t \in S$, where $wt \neq t$. Then the following statements are equivalent:*

(1) *The right S-act $S/\rho(wt,t)$ is flat;*
(2) *The right S-act $S/\rho(wt,t)$ is weakly flat;*
(3) *The right S-act $S/\rho(wt,t)$ is $(WF)'$;*
(4) *The right S-act $S/\rho(wt,t)$ is principally weakly flat;*
(5) *t is a regular element in S.*

Definition 2.2. A monoid S satisfies *Condition $(R_{(WF)'})$, if for all $x,y,s,t,z \in S$, $sz = tz$ implies the existence of $w \in Ss \cap St$ such that $w\rho(xs,yt)xs$. Similarly it satisfies Condition $(L_{(WF)'})$, if for all $x,y,s,t,z \in S$, $zs = zt$ implies the existence of $w \in sS \cap tS$ such that $w\lambda(xs,yt)xs$.*

Theorem 2.3. [2] *For any monoid S the following statements are equivalent:*

(1) *all right S-acts are $(WF)'$;*
(2) *all finitely generated right S-acts are $(WF)'$;*
(3) *all cyclic right S-acts are $(WF)'$;*
(4) *all monocyclic right S-acts are $(WF)'$;*
(5) *S is regular and satisfies Condition $(R_{(WF)'})$.*

Theorem 2.4. *Let S be a right cancellative monoid. Then all principally weakly flat right S-acts are $(WF)'$.*

Proof. Suppose that A_S is principally weakly flat and let $as = a't$, $sz = tz$, for $a, a' \in A_S, s,t,z \in S$. Since S is right cancellative $s = t$, and so $as = a's$. Thus $a \otimes s = a' \otimes s$ in $A_S \otimes {}_SSs$. □

We recall from [8] that an element $t \in S$ is called *w-regular* for $w \in S$, if $wt \neq t$ and if for every right cancellable element $c \in S$ and every $u \in S, uc \in tS$ implies that $u\rho(wt,t)wu$.

Theorem 2.5. *Let S be a monoid. Then all torsion free monocyclic right S-acts of the form $S/\rho(wt,t), w,t \in S, wt \neq t$, are $(WF)'$ if and only if, every w-regular element of S is regular for every $w \in S$.*

Proof. Necessity. Suppose that $t \in S$ is w-regular for $w \in S$. Then $S/\rho(wt,t)$ is torsion free by ([8], III, 8.9). Thus $S/\rho(wt,t)$ is $(WF)'$, and so by Theorem 2.1, t is regular.

Sufficiency. Assume $S/\rho(wt,t), w,t \in S, wt \neq t$, is torsion free. Then t is w-regular by ([8], III, 8.9). Thus t is regular, and so $S/\rho(wt,t)$ is $(WF)'$ by Theorem 2.1. □

We recall from [11], that a right S-act A_S is $\Re$-*torsion free* if for every $a, a' \in A_S$ and every right cancellable element $c \in S$, $ac = a'c$ and $a\Re a'$ imply that $a = a'$. Also $\rho_{\Re TF}(s,t)$ is the smallest congruence containing (s,t), such that $S/\rho_{\Re TF}(s,t)$ is $\Re$-torsion free.

Note that since we have the following implications:

$$(WF)' \Rightarrow principal\ weak\ flatness \Rightarrow torsion\ freeness \Rightarrow \Re - torsion\ freeness$$

so $(WF)'$ of right S-acts implies $\Re$-torsion freeness, but the converse is not true in general, else $\Re$-torsion freeness implies torsion freeness, that is not the case (see [11], Example 1.1). For the converse see the following theorems.

Theorem 2.6. *Let S be a monoid. Then the following statements are equivalent:*
(1) all $\Re$-torsion free cyclic right S-acts are $(WF)'$;
(2) for every $x, y, s, t, z \in S$, where $sz = tz$, there exist $u, v \in S$ such that $us = vt$, $(u, x) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_s)$ and $(v, y) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_t)$.

Proof. (1) $\Rightarrow$ (2). Let $x, y, s, t, z \in S$, and $sz = tz$, since the cyclic right S-act $S/\rho_{\Re TF}(xs, yt)$ is $\Re$-torsion free, it is $(WF)'$. Thus by Theorem 1.7, there exist $u, v \in S$ such that $us = vt$, $(u, x) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_s)$ and $(v, y) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_t)$.

(2) $\Rightarrow$ (1). Suppose that S/τ is $\Re$-torsion free for the right congruence τ on S, and let for $x, y, s, t, z \in S$, $(xs, yt) \in \tau$ and $sz = tz$. Then by assumption there exist $u, v \in S$ such that $us = vt$, $(u, x) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_s)$ and $(v, y) \in (\rho_{\Re TF}(xs, yt) \vee ker\rho_t)$. Since $\rho_{\Re TF}(xs, yt) \subseteq \tau$, we have $(u, x) \in (\tau \vee ker\rho_s)$ and $(v, y) \in (\tau \vee ker\rho_t)$. Thus S/τ is $(WF)'$ by Theorem 1.7. □

Theorem 2.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) *all right S-acts are $(WF)'$;*
(2) *all $\Re$-torsion free right S-acts are $(WF)'$;*
(3) *all finitely generated $\Re$-torsion free right S-acts are $(WF)'$;*
(4) *all $\Re$-torsion free right S-acts generated by at most two elements are $(WF)'$;*
(5) *S is regular and satisfies Condition $(R_{(WF)'})$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. (1) $\Leftrightarrow$ (5) follows from Theorem 2.3.

(4) $\Rightarrow$ (1). Let $c \in S$ be a right cancellable element. If $cS \neq S$, then the right S-act $A(cS)$ satisfies Condition (E) by ([8], III, 14.3(3)), and so it is $\Re$-torsion free by ([11], Proposition 1.2). Thus by assumption it is $(WF)'$ and so is torsion free. Then the equality $(1, x)c = (1, y)c$ implies $(1, x) = (1, y)$, which is a contradiction. So for every $c \in S, cS = S$, which means that

all right cancellable elements are right invertible. That is, all right S-acts are torsion free, and so are $\Re$-torsion free, thus all right S-acts are $(WF)'$ as required. □

We recall from [12] that an act A_S is called *strongly torsion free* if for every $a, a' \in A_S$ and every $s \in S$, the equality $as = a's$ implies $a = a'$.

Theorem 2.8. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right S-acts are strongly torsion free;*
(2) *all finitely generated $(WF)'$ right S-acts are strongly torsion free;*
(3) *all cyclic $(WF)'$ right S-acts are strongly torsion free;*
(4) *all monocyclic $(WF)'$ right S-acts are strongly torsion free;*
(5) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Since S_S is $(WF)'$, as a monocyclic right S-act, it follows from ([12], Proposition 2.1(2)).

(5) $\Rightarrow$ (1). It follows from ([12], Proposition 2.1(7)). □

Lemma 2.9. *Let S be a right cancellative monoid. Then every right S-act A_S satisfies Condition $(W_{(WF)'})$.*

Proof. Let $as = a't, sz = tz$ for $a, a' \in A_S,\ s, t, z \in S$. Since S is right cancellative, $s = t$, and so obviously A_S satisfies Condition $(W_{(WF)'})$. □

Corollary 2.10. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition $(W_{(WF)'})$ are strongly torsion free;*
(2) *all finitely generated right S-acts satisfying Condition $(W_{(WF)'})$ are strongly torsion free;*
(3) *all cyclic right S-acts satisfying Condition $(W_{(WF)'})$ are strongly torsion free;*
(4) *all monocyclic right S-acts satisfying Condition $(W_{(WF)'})$ are strongly torsion free;*
(5) *S is group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By assumption all monocyclic $(WF)'$ right S-acts are strongly torsion free, and so by Theorem 2.8, S is right cancellative. Thus by Lemma 2.9 every right S-act satisfies Condition $(W_{(WF)'})$, which by assumption implies that all monocyclic right S-acts are strongly

torsion free. Let $s \in S$, since $(s^2, s) \in \rho(s^2, s)$ and $S/\rho(s^2, s)$ is strongly torsion free, it easily follows that $(1, s) \in \rho(s^2, s)$. By ([8], III, 8.6), there exists $t \in S$ such that $st = 1$, and so S is a group.

(5) $\Rightarrow$ (1). It is obvious. □

Theorem 2.11. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) *all strongly torsion free right S-acts are $(WF)'$;*
(2) *all strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*
(3) *all finitely generated strongly torsion free right S-acts are $(WF)'$;*
(4) *all finitely generated strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*
(5) *all cyclic strongly torsion free right S-acts are $(WF)'$;*
(6) *all cyclic strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*
(7) *S is weakly right reversible.*

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (6), (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7). The one-element right S-act Θ_S is strongly torsion free by ([12], Proposition 2.1(1)), and so by assumption it satisfies Condition $(W_{(WF)'})$. Thus S is weakly right reversible by Theorem 1.4.

(7) $\Rightarrow$ (1). Suppose that the right S-act A_S is strongly torsion free. Then the equality $ae = (ae)e$, for $a \in A_S$ and $e^2 = e \in S$, implies that $ae = a$. Thus $aS = \{a\}$, for every $a \in A_S$. Let $as = a't$, $sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Then $a = a'$. Since S is weakly right reversible, there exists $u, v \in S$ such that $us = vt$. If $w = us, a'' = a$, then $as = a't = a''w$, and so A_S satisfies Condition $(W_{(WF)'})$. Since by [12], strong torsion freeness implies principal weak flatness, A_S is $(WF)'$ by Theorem 1.6. □

Note that Condition (E) does not imply $(WF)'$, otherwise Condition (E) would imply principal weak flatness, which is not the case, (see [8], III, 14.4). Since $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$, it is obvious that Conditions $(E'), (EP)$ and $(E'P)$ does not imply $(WF)'$ too. Now it is natural to ask for monoids over which Conditions (E), $(E'), (EP)$ and $(E'P)$ imply $(WF)'$.

Theorem 2.12. *For any monoid S, the following statements are equivalent:*

(1) *S is regular;*
(2) *all right S-acts satisfying Condition $(E'P)$ are $(WF)'$;*
(3) *all right S-acts satisfying Condition (EP) are $(WF)'$;*

(4) *all right S-acts satisfying Condition (E') are $(WF)'$;*

(5) *all right S-acts satisfying Condition (E) are $(WF)'$.*

Proof. Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2). Suppose that the right S-act A_S satisfies Condition $(E'P)$ and let $as = a't,\ sz = tz$, for $a, a' \in A_S,\ s, t, z \in S$. Since S is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$, and so $a't = ass's = a'ts's$ and $ts' = ts'ss'$. Since A_S satisfies Condition $(E'P)$, there exist $a'' \in A_S$, and $u, v \in S$, such that $a' = a''u = a''v$ and $ut = vts's$. If $w = ut$, then $as = a't = a''w$, where $w \in Ss \cap St$, that is, A_S satisfies Condition $(W_{(WF)'})$. Since S is regular A_S is principally weakly flat by ([8], IV, 6.6) and so A_S is $(WF)'$ by Theorem 1.6.

(5) $\Rightarrow$ (1). By Theorem 1.6, all right S-acts satisfying Condition (E) are principally weakly flat, and so S is regular by ([8], IV, 8.5). □

We recall from [8] that a right S-act A_S is *divisible* if $Ac = A$ for any left cancellable element $c \in S$. A_S is called *completely reducible* if it is a disjoint union of simple subacts. By Lemma 1.2 for any monoid S, the right S-act S_S is $(WF)'$, but it is not divisible (completely reducible) in general.

Theorem 2.13. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right S-acts are divisible;*

(2) *all finitely generated $(WF)'$ right S-acts are divisible;*

(3) *all cyclic $(WF)'$ right S-acts are divisible;*

(4) *all monocyclic $(WF)'$ right S-acts are divisible;*

(5) *all left cancellable elements of S are left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By Lemma 1.2, S_S is $(WF)'$, and so it is divisible. Thus $Sc = S$, for any left cancellable element $c \in S$. That is, there exists $x \in S$ such that $xc = 1$.

(5) $\Rightarrow$ (1). It is clear from ([8], III, 2.2). □

Theorem 2.14. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right S-acts are completely reducible;*

(2) *all finitely generated $(WF)'$ right S-acts are completely reducible;*

(3) *all cyclic $(WF)'$ right S-acts are completely reducible;*

(4) *all monocyclic $(WF)'$ right S-acts are completely reducible;*

(5) *S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By Lemma 1.2, S_S is $(WF)'$ and so by assumption it is completely reducible. Thus S is a group by ([8], I, 5.33).

(5) $\Rightarrow$ (1). It follows from ([8], I, 5.34). $\square$

From Theorem 2.14 and ([8], I, 5.34) we have the following result.

Corollary 2.15. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition $(W_{(WF)'})$ are completely reducible;*
(2) *all finitely generated right S-acts satisfying Condition $(W_{(WF)'})$ are completely reducible;*
(3) *all cyclic right S-acts satisfying Condition $(W_{(WF)'})$ are completely reducible;*
(4) *all monocyclic right S-acts satisfying Condition $(W_{(WF)'})$ are completely reducible;*
(5) *S is a group.*

Theorem 2.16. *Let S be a monoid and A_S be a right S-act. If A_S satisfies Condition (P') then it is $(WF)'$.*

Proof. Let $as = a't,\ sz = tz$, for $a, a' \in A_S,\ s, t, z \in S$. Since A_S satisfies Condition (P') there exist $a'' \in A_S, u, v \in S$, such that $a = a''u,\ a' = a''v$ and $us = vt$. Then

$$a \otimes s = a''u \otimes s = a'' \otimes us = a'' \otimes vt = a''v \otimes t = a' \otimes t$$

in $A_S \otimes {}_S(Ss \cup St)$ $\square$

Theorem 2.17. *Let S be a regular monoid. Then a right S-act A_S is $(WF)'$ if and only if, for every $a \in A_S$ and $s, t, z \in S$, if $as = at, sz = tz$ then there exists $w \in Ss \cap St$ such that $as = at = aw$.*

Proof. Suppose that A_S is $(WF)'$ and let $as = at, sz = tz$ for $a \in A_S$ and $s, t, z \in S$. Then by Condition $(W_{(WF)'})$ there exist $a'' \in A_S$ and $w \in Ss \cap St$ such that $as = at = a''w$. Since S is regular there exists $w' \in S$ such that $ww'w = w$. Then $u = sw'w \in Ss \cap St$ and $au = asw'w = a''ww'w = a''w = as$ as required.

Conversely, suppose that A_S is a right S-act and $as = a't, sz = tz$, for $a, a' \in A_S$ and $s, t, z \in S$. Let $t' \in S$ be such that $tt't = t, t'tt' = t'$, so $as = a'tt't = ast't$ and $st' = st'tt'$. By assumption there exists $w \in Ss \cap St$ such that $as = ast't = a't = aw$, Thus A_S satisfies

Condition $(W_{(WF)'})$. Since S is regular A_S is principally weakly flat and so A_S is $(WF)'$, as required. □

Definition 2.18. A right S-act A_S satisfies *Condition* $(W_{(WF)'})'$, if $as = a't$, $sz = tz$ and $Ss \cap St \neq \varphi$ for $a, a' \in A_S, s, t, z \in S$, imply that there exist $a'' \in A_S$, and $w \in Ss \cap St$, such that $as = a't = a''w$. A right S-act A_S is called almost $(WF)'$, if A_S is principally weakly flat and satisfies Condition $(W_{(WF)'})'$.

Lemma 2.19. [8] *A right S-act A_S is a generator if and only if there exists an epimorphism $\pi : A_S \to S_S$.*

Lemma 2.20. [8] *Let $(P, (p_i)), i \in I$ be the product of $(A_i), i \in I$ in a category C and let $j \in I$. If $Mor_C(A_i, A_j) \neq \emptyset$ for every $i \in I$, then A_j is a retract of P.*

Theorem 2.21. *For any monoid S the following statements are equivalent:*

(1) *all generators are $(WF)'$;*
(2) *$S \times A_S$ is $(WF)'$, for each right S-act A_S;*
(3) *a right S-act A_S is $(WF)'$, if $Hom(A_S, S_S) \neq \emptyset$;*
(4) *all right S-acts are almost $(WF)'$.*

Proof. (1) $\Rightarrow$ (2). Suppose that all generators are $(WF)'$, and let A_S be a right S-act. Since by Lemma 2.19, $S \coprod (S \times A_S)$ is a generator, it is $(WF)'$ and so $S \times A_S$ is $(WF)'$ by Lemma 1.2.

(2) $\Rightarrow$ (3). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$. In view of Lemma 2.20, A_S is a retract of $S \times A_S$, which by assumption is $(WF)'$. Thus A_S is $(WF)'$ by Theorem 1.5.

(3) $\Rightarrow$ (1). Let A_S be a generator. Then $Hom(A_S, S) \neq \emptyset$ by Lemma 2.19, and so by assumption A_S is $(WF)'$.

(1) $\Rightarrow$ (4). Let A_S be a right S-act. Since by assumption all generators are principally weakly flat, S is regular and so A_S is principally weakly flat by ([8], IV, 6.6). Suppose now that $as = a't, sz = tz$ and $Ss \cap St \neq \emptyset$ for $a, a' \in A_S, s, t, z \in S$. Thus $xs = yt$ for some $x, y \in S$. So $(x, a)s = (y, a')t$ in $S \times A_S$. Since (1) $\Leftrightarrow$ (2), $S \times A_S$ is $(WF)'$. Thus $S \times A_S$ satisfies Condition $(W_{(WF)'})'$, which implies that A_S satisfies Condition $(W_{(WF)'})'$. Therefore A_S is almost $(WF)'$.

(4) $\Rightarrow$ (1). Suppose that A_S is a generator and $\pi : A_S \to S_S$ is an epimorphism. Let $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since $\pi(a)s = \pi(a')t$ we get $Ss \cap St \neq \varphi$, and so by the assumption there exist $a'' \in A_S$, and $w \in Ss \cap St$ such that $as = a't = a''w$. That is A_S is $(WF)'$ as required. □

Remark 2.22. In ([8], IV, 7.5) it was proved that all right S-act are weakly flat if and only if S is regular and satisfies Condition

$$(R) : (\forall s, t \in S)(\exists w \in Ss \cap St) : w\rho(s,t)s.$$

It is clear that Condition (R) implies Condition $(R_{(WF)'})$, but let

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}, a \neq 0 \right\},$$

then S is a right cancellative monoid, and so it satisfies Condition $(R_{(WF)'})$. Now let $s = \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}$, then for every $a, b, c, d \in \mathbb{Z}$ with $a, c \neq 0$,

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix} \neq \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}.$$

In following theorem we prove that all right S-acts are weakly flat if and only if S is regular and satisfies Condition $(R_{(WF)'})$.

Theorem 2.23. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts are weakly flat;*
(2) *all right S-acts are $(WF)'$;*
(3) *S is regular and satisfies Condition $(R_{(WF)'})$.*

Proof. Implication (1) $\Rightarrow$ (2) are obvious, and (2) $\Leftrightarrow$ (3) follows from Theorem 2.3.

(3) $\Rightarrow$ (1). Suppose that A_S is a right S-act and let $as = a't$, for $a, a' \in A_S, s, t \in S$. Since S is regular, there exists $t' \in S$ such that $t = tt't, t' = t'tt'$. Thus $as = a'tt't = ast't$, and $st' = st'tt'$. Since (3) and (2) are already equivaent, A_S is $(WF)'$ and so it satisfies Condition $(W_{(WF)'})$ by Theorem 1.6. Thus there exist $a'' \in A_S, w \in Ss \cap St$ such that $as = a't = a''w$. That is A_S satisfies Condition (W). Since S is regular A_S is principally weakly flat by ([8], IV, 6.6), and so it is weakly flat by ([8], III, 11.4) as required. □

Definition 2.24. A monoid S is said to be *right (left) absolutely* $(WF)'$ if all right (left) acts over it are $(WF)'$, and it is said to be *absolutely* $(WF)'$ if it is both right and left absolutely $(WF)'$.

From Theorem 2.23 and ([8], IV, 8.12) we have the following important result.

Theorem 2.25. *For any monoid S the following statements are equivalent:*

(1) *S is absolutely flat;*

(2) *S is weakly absolutely flat;*

(3) *S is absolutely $(WF)'$;*

(4) *S is regular and satisfies Conditions $(R_{(WF)'})$ and $(L_{(WF)'})$;*

(5) *S is regular and satisfies Conditions (R) and (L).*

3. Characterization of monoids by $(WF)'$ property of right Rees factor acts

Theorem 3.1. *Let S be a monoid and K_S be a right ideal of S. Then S/K_S is $(WF)'$ if and only if, S is weakly right reversible and K_S is left stabilizing.*

Proof. Necessity. Suppose that S/K_S is $(WF)'$ for the right ideal K_S of S. Then there are two cases as follow:

Case 1. $K_S = S$. Then by Theorem 1.4, S is weakly right reversible.

Case 2. $K_S \neq S$. By Theorem 1.6, S/K_S is principally weakly flat and so K_S is left stabilizing by ([8], III, 10.11). To show that S is weakly right reversible, suppose that $sz = tz$, for $s, t, z \in S$ and let $k \in K$. Since $[k]_{\rho_K} s = [k]_{\rho_K} t$, Condition $(W_{(WF)'})$ implies that there exist $u, v \in S$ such that $us = vt$.

Sufficiency. Suppose that S is weakly right reversible and K_S is a left stabilizing right ideal of S. Then there are two cases as follow:

Case 1. $K_S = S$. Since S is weakly right reversible, $S/K_S \cong \Theta_S$ is $(WF)'$ by Theorem 1.4.

Case 2. $K_S \neq S$. Let $(xs)\rho_K(yt)$ and $sz = tz$, for $x, y, s, t, z \in S$. Then there are two possibilities that can arise:

(1). $xs = yt$. If $u = x$ and $v = y$, then by Theorem 1.7, S/K_S is $(WF)'$.

(2). $xs \neq yt$. Then $xs, yt \in K_S$, and so there exist $l_1, l_2 \in K_S$ such that $l_1 xs = xs$, and $l_2 yt = yt$. That is, $(l_1 x) ker\rho_s (x)$, and $(l_2 y) ker\rho_t (y)$. Since $sz = tz$, there exist $u', v' \in S$ such that $u's = v't$. Let $u = l_1 u'$, and $v = l_1 v'$, then $(x) ker\rho_s (l_1 x)\rho_K(u)$, and so $x(\rho_K \vee ker\rho_s)u$. Similarly, $y(\rho_K \vee ker\rho_t)v$, and $us = l_1 u's = l_1 v't = vt$, and so S/K_S is $(WF)'$ by Theorem 1.7. $\square$

Since there exist monoids S which are not weakly right reversible and the one element act Θ_S satisfies Condition (PWP) for any S, we conclude that Condition (PWP) does not imply $(WF)'$, and so principal weak flatness does not imply $(WF)'$. Now it is natural to ask for monoids over which Condition (PWP) (principal weak flatness) of right Rees factor acts implies $(WF)'$.

Theorem 3.2. *For any monoid S the following statements are equivalent:*

(1) *all principally weakly flat right Rees factor S-acts are $(WF)'$;*

(2) *all right Rees factor S-acts satisfying Condition (PWP) are $(WF)'$;*

(3) *S is weakly right reversible.*

Proof. Since Condition $(PWP) \Rightarrow$ principal weak flatnes, implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). By ([9], Corollary 2.9) the one-element right S-act Θ_S, satisfies Condition (PWP), and so it is $(WF)'$ by the assumption. Thus S is weakly right reversible by Theorem 1.4.

(3) $\Rightarrow$ (1). Suppose that S is weakly right reversible, and let S/K_S be principally weakly flat. Then there are two cases that can arise:

Case 1. $K = S$. Then $S/K_S \cong \Theta_S$, and so by Theorem 1.4, it is $(WF)'$.

Case 2. $K \neq S$. Then by ([8], III, 10.11), the right ideal K_S is left stabilizing and so by Theorem 3.1, S/K_S is $(WF)'$. □

Theorem 3.2 together with ([8], IV, 6.5) imply the following:

Theorem 3.3. *All torsion free right Rees factor S-acts are $(WF)'$ if and only if S is a weakly right reversible left almost regular monoid.*

From Theorem 3.2 and ([8], IV, 6.6) one obtains the following:

Theorem 3.4. *All right Rees factor S-acts are $(WF)'$ if and only if S is a weakly right reversible regular monoid.*

Let S be the monoid which mentioned in Remark 2.22. Then Θ_S is $(WF)'$ while it is not weakly flat. See the following:

Theorem 3.5. *All $(WF)'$ right Rees factor S-acts are (weakly)flat if and only if S is not weakly right reversible or S is right reversible.*

Proof. Necessity. Suppose that all $(WF)'$ right Rees factor S-acts are (weakly) flat, and let S be weakly right reversible. Then by Theorem 1.4, $S/S_S \cong \Theta_S$ is $(WF)'$, and so by assumption it is (weakly)flat, then S is right reversible by ([8], III, 11.2.(2)).

Sufficiency. Suppose that S/K_S is $(WF)'$ for the right ideal K_S of S. Then there are two cases as follow:

Case 1. $K_S = S$. Then by Theorem 1.4, S is weakly right reversible and so by assumption S is right reversible. Hence $S/K_S \cong \Theta_S$ is (weakly)flat by ([8], III, 11.2.(2)).

Case 2. $K_S \neq S$. Since S/K_S is $(WF)'$, by Theorem 3.1, K_S is left stabilizing and S is weakly right reversible. Thus, S is right reversible by assumption, and so S/K_S is (weakly)flat by ([8], III, 12.17). □

Lemma 3.6. [12] *Let S be a monoid and K_S be a right ideal of S. Then the right Rees factor S-act S/K_S is strongly torsion free if and only if $K_S = S$.*

Theorem 3.7. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts are strongly torsion free;*

(2) *S is not weakly right reversible or S has no left stabilizing proper right ideal;*

(3) *S is not weakly right reversible or*

$$(\forall x_1, x_2, ... \in S)((\forall i \in N)(\exists t_{i+1} \in S)(x_i = x_{i+1}t_{i+1}x_i) \Rightarrow (\exists i_0 \in N)$$

$$(\forall j > i_0, x_j t_j = 1));$$

(4) *S is not weakly right reversible or*

$$(\forall x_1, x_2, ... \in S)((\forall i \in N)(x_{i+1}x_i = x_i) \Rightarrow (\exists i_0 \in N)(\forall j > i_0, x_j = 1));$$

(5) *S is not weakly right reversible or*

$$(\forall x, x_1, x_2, ... \in S)((\forall i \in N)(x_{i+1}x_i = x_i) \Rightarrow (\exists i_0 \in N)$$

$$(xx_{i_0} = x_{i_0} \Rightarrow x = 1)).$$

Proof. Implications (2) ⇔ (3) ⇔ (4) ⇔ (5) follow from ([12], Theorem 5.14).

(1) ⇒ (2). Suppose that all $(WF)'$ right Rees factor S-acts are strongly torsion free, S is weakly right reversible and K_S is a left stabilizing right ideal of S. By Theorem 3.1, S/K_S is $(WF)'$ and so it is strongly torsion free. Now by Lemma 3.6, $K_S = S$.

(2) ⇒ (1). From ([12], Proposition 2.1) we know that the one-element right S-act Θ_S is strongly torsion free. Suppose that for the proper right ideal K_S of S, S/K_S is $(WF)'$. By Theorem 3.1, S is weakly right reversible and K_S is left stabilizing. Now by the assumption $K_S = S$, and so by Lemma 3.6, S/K_S is strongly torsion free. □

The following example shows that $(WF)'$ property of right Rees factor acts does not imply Condition (PWP).

Example 3.8. Let $S = \{1, e, f, 0\}$ be a semmilattice, where $ef = 0$. Consider the right ideal $K_S = eS = \{0, e\}$ of S. Since S is weakly right reversible and K_S is a left stabilizing right ideal, the right Rees factor act S/K_S is $(WF)'$ by Theorem 3.1. Since $1, f \in S \setminus K_S$, $1e, fe \in K_S$, and $1e \neq fe$, K_S is not left annihilating, and so S/K_S does not satisfy Condition (PWP) by ([9], Lemma 2.8).

Theorem 3.9. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts satisfy Condition (PWP);*

(2) *S is not weakly right reversible or every left stabilizing right ideal of S is left annihilating;*

(3) *S is not weakly right reversible or*

$$(\forall\ t, x, y, x_0, y_0, x_1, y_1, x_2, y_2, ... \in S)$$

$$((x_0 = xt \wedge (\forall i \in N_0)(x_{i+1}x_i = x_i) \wedge y_0 = yt \wedge (\forall i \in N_0)(y_{i+1}y_i = y_i)$$

$$\wedge x_0 \neq y_0) \Rightarrow (\exists p \in \{x_0, x_1, ...\} \cup \{y_0, y_1, ...\})(\exists z \in S)(x = pz \vee y = pz)).$$

Proof. Implication $(2) \Leftrightarrow (3)$ follows from ([9], Proposition 3.6).

$(1) \Rightarrow (2)$. Suppose that all $(WF)'$ right Rees factor S-acts satisfy Condition (PWP). Let S be weakly right reversible, and K_S be a left stabilizing right ideal of S. Then the right Rees factor S-act S/K_S is $(WF)'$ by Theorem 3.1, and so by assumption it satisfies Condition (PWP). Now it follows from ([9], Lemma 2.8) that K_S is left annihilating.

$(2) \Rightarrow (1)$. Suppose that for the right ideal K_S of S, S/K_S is $(WF)'$. Then there are two cases as follow:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ satisfies Condition (PWP) by ([9], Corollary 2.9).

Case 2. $K_S \neq S$. Then S is weakly right reversible and K_S is a left stabilizing right ideal of S by Theorem 3.1. Thus by assumption K_S is left annihilating, and S/K_S satisfies Condition (PWP) by ([9], Lemma 2.8). □

We recall from [9] that a right ideal K_S of a monoid S is *strongly left annihilating* if for all $s, t \in S \setminus K_S$ and for all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$, $f(s), f(t) \in K_S$ imply that $f(s) = f(t)$.

Theorem 3.10. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts satisfy Condition (WP);*

(2) *S is not weakly right reversible or S is right reversible and every left stabilizing right ideal of S is strongly left annihilating;*

(3) *S is not weakly right reversible or S is right reversible and for all homomorphisms $f : {}_S(Sx \cup Sy) \to {}_SS$ such that $x_0 = f(x) \neq f(y) = y_0$, and for all $x_1, y_1, x_2, y_2, ... \in S$ such that*

$$(\forall i \in N_0)(x_{i+1}x_i = x_i) \wedge (\forall i \in N_0)(y_{i+1}y_i = y_i)$$

there exist $p \in \{x_0, x_1, ...\} \cup \{y_0, y_1, ...\}$ and $z \in S$ such that either $x = pz$ or $y = pz$.

Proof. Implication (2) ⇔ (3) follows from ([9], Corollary 3.16).

(1) ⇒ (2). Suppose that all $(WF)'$ right Rees factor S-acts satisfy Condition (WP), S is weakly right reversible, and K_S is a left stabilizing right ideal of S. Then S/K_S is $(WF)'$ by Theorem 3.1, and so by assumption it satisfies Condition (WP). Then by ([9], Lemma 2.13) S is right reversible and K_S is strongly left annihilating.

(2) ⇒ (1). Suppose that S/K_S is $(WF)'$ for the right ideal K_S of S. Then there are two cases as follow:

Case 1. $K_S = S$. Then S is weakly right reversible by Theorem 1.4, and so by assumption S is right reversible. Now the right S-act $S/K_S \cong \Theta_S$ satisfies Condition (WP) by ([9], Lemma 2.14).

Case 2. $K_S \neq S$. Then S is weakly right reversible and K_S is a left stabilizing right ideal of S by Theorem 3.1. Thus S is right reversible and K_S is strongly left annihilating by the assumption. So S/K_S satisfies Condition (WP) by ([9], Lemma 2.13). □

From ([8], IV, 9.2), we have the following:

Lemma 3.11. *For any monoid S the following statements are equivalent:*

(1) *there is no proper left stabilizing right ideal K_S of S, with $|K_S| \geq 2$;*

(2) *S contains at most two idempotents* (1, *and maybe* 0) *and satisfies Condition*

(ALU) : S does not contain any infinite sequence of pairwise distinct elements $s_1, s_2, ...,$ where $s_{i+1}s_i = s_i$, for any $i \in N$.

Theorem 3.12. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts satisfy Condition (P);*

(2) *S is not weakly right reversible, or S is right reversible and there is no proper left stabilizing right ideal K_S of S, with $|K_S| \geq 2$;*

(3) *S is not weakly right reversible, or S is right reversible and S contains at most two idempotents* (1, *and maybe* 0) *and satisfies Condition (ALU).*

Proof. Implication (2) $\Leftrightarrow$ (3) follows from Lemma 3.11.

(1) $\Rightarrow$ (2). Suppose that all $(WF)'$ right Rees factor S-acts of S satisfy Condition (P) and S is weakly right reversible. Then $S/S_S \cong \Theta_S$ is $(WF)'$ by Theorem 1.4, and so by assumption it satisfies Condition (P). Thus S is right reversible by ([8], III, 13.7). Now let K_S be a proper left stabilizing right ideal of S, then S/K_S is $(WF)'$ and so it satisfies Condition (P). Thus $|K_S| = 1$, by ([8], III, 13.9).

(2) $\Rightarrow$ (1). Suppose that S/K_S is $(WF)'$, for the right ideal K_S of S. Then there are two cases as follow:

Case 1. $K_S = S$. Then S is weakly right reversible by Theorem 1.4, and so by the assumption S is right reversible. Hence $S/K_S \cong \Theta_S$ satisfies Condition (P) by ([8], III, 13.7).

Case 2. $K_S \neq S$. Since S/K_S is $(WF)'$, K_S is left stabilizing and S is weakly right reversible, by Theorem 3.1. Thus $|K_S| = 1$, and so S/K_S satisfies Condition (P) by ([8], III, 13.9). □

We recall from [9] that a right $S-$act A_S is *weakly pullback flat* if, and only if it satisfies Conditions (P) and (E').

A similar argument as the proof of Theorem 3.12 will show the following theorems:

Theorem 3.13. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts are weakly pullback flat;*
(2) *S is not weakly right reversible, or S is right reversible and weakly left collapsible and there is no proper left stabilizing right ideal K_S of S, with $|K_S| \geq 2$;*
(3) *S is not weakly right reversible, or S is right reversible and weakly left collapsible and S contains at most two idempotents $(1,$ and maybe, $0)$ and satisfies Condition (ALU).*

Theorem 3.14. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts are strongly flat;*
(2) *S is not weakly right reversible, or S is left collapsible and there is no proper left stabilizing right ideal K_S of S, with $|K_S| \geq 2$;*
(3) *S is not weakly right reversible, or S is left collapsible and S contains at most two idempotents $(1,$ and maybe $0)$ and satisfies Condition (ALU).*

Theorem 3.15. *For any monoid S the following statements are equivalent:*

(1) *all $(WF)'$ right Rees factor S-acts are projective;*

(2) *S is not weakly right reversible, or S contains a left zero and there is no proper left stabilizing right ideal K_S of S, with $|K_S| \geq 2$;*

(3) *S is not weakly right reversible, or S contains a left zero and S contains at most two idempotents $(1$, and maybe $0)$ and satisfies Condition (ALU).*

Theorem 3.16. *All $(WF)'$ right Rees factor S-acts are free, if and only if S is not weakly right reversible or $|S| = 1$.*

Note that the above theorem is also valid for projective generators.

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