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Research Paper

SOME SEPARATION AXIOMS IN TOPOFRAMES

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Abstract. This paper is about the extension of some classical separation axioms Hausdorffness, regularity and complete regularity to topoframes. We show that they agree with those in frames except perhaps for complete regularity. The interesting results are about complete regularity, in particular when and how these differ from the frame results. These together with the results about B-filters are the focus of the paper.

1. Introduction

Let $\mathcal{P}(X)$ denote the power set of a topological space X, and $\mathcal{O}(X)$ denote the set of all open sets in X. When we study a topological space X, we have actually the pair $(\mathcal{P}(X),\mathcal{O}(X))$, for which all subset of X, their interiors, closures, boundaries and... should be considered. The theory of frames (pointfree topology) is a version of topology in which the lattice of open sets $\mathcal{O}(X)$ of a topological space X are studied, rather than their points. It is natural to ask the question: is a lattice in possession of both open elements and closed ones in itself?

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The answer is affirmative; in fact, in [14], [15] and [26], we talk about a new pointfree form of topology entitled topoframes and their homomorphisms that is, in fact, "a generalization of pointfree topology" in which the lattice of all sets (open, closed and...) of a topological space are studied. The definition of a topoframe emanates from the fact that every frame is isomorphic to a subframe of a complete Boolean algebra; more generally,

"*Every frame is isomorphic to a subframe τ of a frame L all of whose elements are complemented in L*"

mentioned in [21, p. 48-53]. In case τ is a frame, it is natural to identify τ with a pair (L, τ) for some frames L. The notion of a topoframe has been published in [14, 27]. By a **topoframe** we mean a pair (L, τ), abbreviated L_τ, consisting of a frame L and a subset τ satisfying the following conditions:

(1) every element p of τ has a complement p' in L,
(2) the supremum of any subfamily of τ belongs to τ, and
(3) the infimum of any finite subfamily of τ belongs to τ.

It follows from [23, Lemma 4.4.1] that τ is a frame. It is obvious that a topological space $(\mathcal{P}(X), \mathcal{O}(X))$ is a topoframe. Each member of τ in (L, τ) is called an **open element** and dually, each member of $\tau' := \{t' \mid t \in \tau\}$ is called a **closed element**. Let (L, τ) be a topoframe, and let $p \in L$. Then the **closure** and **interior** of $p \in L$ are defined by

$$\mathrm{Cl}(p) = \overline{p} := \bigwedge\{q \in \tau' \mid p \leq q\} \text{ and } \mathrm{Int}(p) = p^\circ := \bigvee\{t \in \tau \mid t \leq p\},$$

respectively. It is easy to see that p is a closed element if and only if $p = \overline{p}$, and p is an open element if and only if $p = p^\circ$. Moreover, $\mathrm{Cl}(-)$ and $\mathrm{Int}(-)$ are closure and kernel operators on a topoframes, respectively. A **topoframe map** f from a topoframe (L_1, τ_1) to a topoframe (L_2, τ_2) is a frame map f from L_1 to L_2 with the property $f(\tau_1) \subseteq \tau_2$. It is clear that the composition of topoframe maps are topoframe maps. Also, the identity function on a topoframe is a topoframe map. Hence, the class of all topoframes and their maps form a category. This category is denoted by **TFrm**. All notions of interior, closure, boundary, regularly closed and regularly open elements, z-elements, z-filters, z-ideals, and fixed and free B-filters can be defined in **TFrm** but not in **Frm** (see [26]). There are two functors:

(1) F : **TFrm** $\to$ **Frm** sending (L, τ) to L, and a morphism to its underlying frame homomorphism, and
(2) G : **Frm** $\to$ **TFrm** defined by $G(L) = (L, \mathbf{2})$, and $G(L \overset{h}{\to} M) = (L, \mathbf{2}) \overset{(h,i)}{\to} (M, \mathbf{2})$, where $\mathbf{2}$ is denoted the trivial frame and i is the identity map.

In a similar method as we deal with general topology, Wang Guo-Jun in [19] and later in [20] construct a model of the topological space on a completely distributive lattice. This view was followed in [1] and later in [2]. In [1], a new structure of pointless topology $(L; \tau)$ (called

LG-space) was introduced as the model of topological space ($\mathcal{P}(X)$,$\mathcal{O}(X)$), where L is a frame and τ is a subframe of L. In this new structure, "τ" plays the role of open sets and "τ^*" plays the role of closed sets. Therefore, it is clear that if every element of τ has a complement in L, then the LG-space will be the same as the topoframe.

Let X be any topological space and let $\mathbb{R}$ be the space of real numbers with its usual topology. It is known that $C(X)$, the set of all real-valued continuous functions with domain X and codomain $\mathbb{R}$, is an f-ring. Completely regular spaces were first discussed by Tychonoff [25]. We recall from [17, Theorem 3.9] that for every topological space X, there exists a completely regular Hausdorff space Y and a continuous mapping θ of X onto Y, such that the mapping $g \mapsto go\theta$ is an isomorphism of $C(Y)$ onto $C(X)$, which the reduction to completely regular spaces is due to Stone [24, P. 460] and Čech [6, P. 826]. As usual, let $\mathcal{R}(\tau)$ denote the f-ring of real-valued continuous functions on a frame τ. This f-ring usually denoted $\mathcal{C}(\tau)$ by some authors (see [3, 4], and [7, 8, 9, 10]). If $M \subseteq \tau$ is the largest completely regular subframe of the frame τ, then any $\alpha \in \mathcal{R}(\tau)$ factors through M by the complete regularity of $\mathcal{O}(\mathbb{R})$ and consequently $\mathcal{R}(\tau)$ is isomorphic to $\mathcal{R}(M)$, for a completely regular frame M (see [5]). For a topoframe L_τ, the subset of $HomFrm(\mathcal{P}(\mathbb{R}); L)$ which consists of those homomorphisms that map open sets of $\mathbb{R}$ to members of τ is a sub-f-ring of $\mathcal{R}(\tau)$ and denoted by $\mathcal{R}(L_\tau)$ (see [14]). Just as in the case of $C(X)$ and $\mathcal{R}(\tau)$, we lose no generality from an algebraic perceptive when we restrict our study of the ring of continuous functions on a topoframe being completely regular. The study of $\mathcal{R}(L_\tau)$ is easier than $\mathcal{R}(\tau)$. The existence of both zero elements and cozero elements in **TFrm** makes $\mathcal{R}(L_\tau)$ more similar to the historic f-ring $C(X)$. These topics motivated us to define and study the concepts of Hausdorff, regular, and completely regular in topoframes.

2. Background

A partially ordered set $(L, \leq)$ is a **lattice** (bounded lattice) if every finite subset (including the empty set) has both a meet (greatest lower bound) and a join (least upper bound). Throughout this paper, we denote the top element and the bottom element of a lattice by 1 and 0, respectively. An element $0 < p$ in a lattice L is called a **coprime** iff $p \leq a \vee b$ implies that $p \leq a$ or $p \leq b$. To express the relation $a \leq b$ in words, we say that a is **below** b, and that b is **above** a, or that b **dominates** a. An element $0 < a$ in a lattice L is called an **atom** if it is minimal above the bottom.

Recall that a **frame** τ is a complete lattice in which the first infinite distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$$

holds for all $a \in \tau$ and $S \subseteq \tau$. A **frame map** is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. For a frame τ, we say that an element $a \in \tau$ has a **pseudo-complement** if there exists the largest element a^* of L such that $a \wedge a^* = 0$. τ is said to be pseudo-complemented if every element of τ has a pseudo-complement. A **Heyting algebra** is a lattice L with an additional binary operation, $\rightarrow$, sometimes referred to as the Heyting arrow, that satisfies for all $a, b, c \in L$,

$$c \leq (a \rightarrow b) \Leftrightarrow a \wedge c \leq b.$$

An element a of τ is said to be **rather below** an element b, written as $a \prec b$, if there is an element $s \in \tau$ such that $a \wedge s = 0$ and $s \vee b = 1$, in other words, $a^* \vee b = 1$. τ is said to be **regular** iff $a = \bigvee_{b \prec a} b$, for every $a \in \tau$. For any $x, t \in \tau$, we say that x **interpolates** t (or x is **completely below** t) and write $x \prec\!\!\prec t$, if there exists a trail $\{a_i\}_{i \in [0,1] \cap \mathbb{Q}} \subseteq \tau$ such that $a_0 = x$, $a_1 = t$ and for every $p, q \in [0, 1] \cap \mathbb{Q}$ with $p < q$, we have $a_p \prec a_q$. τ is said to be **completely regular** iff $a = \bigvee_{b \prec\!\!\prec a} b$, for every $a \in \tau$. It is proved in [5] that τ is a completely regular frame if and only if for every $t \in \tau$, there exists $\{\alpha_i\}_{i \in I} \subseteq \mathcal{R}(\tau)$ such that

$$t = \bigvee_{\operatorname{coz}(\alpha_i) \leq t} \operatorname{coz}(\alpha_i)\,,$$

where $\operatorname{coz}(\alpha_i) := \alpha_i(-\infty, 0) \vee f(0, +\infty)$ for all $i \in I$.

The first De Morgan law ([23, p. 332]): In a Heyting algebra,

$$\bigwedge_i a_i^* = (\bigvee_i a_i)^*$$

whenever the supremum $\bigvee_i a_i$ exists.

Now, we present some notions and notations on topoframes that will be required in the main results. The proofs of the respective results exist in [15] and [26].

In the sequel, if L_τ is a topoframe, then τ is called a topoframe on L.

Definition 2.1. [26]. Let τ be a topoframe on L. Then for any $a \in L$, we define

$$a^\perp := \bigvee \{t \in \tau \mid a \wedge t = 0\} \quad \text{and} \quad a_\perp := \bigwedge \{t' \mid t \in \tau,\, a \vee t' = 1\}$$

Lemma 2.2. [26]. *Let τ be a topoframe on L. Then the following statements hold.*

(1) $a^\perp = (\overline{a})'$ *and* $a_\perp = (a^\circ)'$, *where a stands for an arbitrary element in L.*
(2) *If a is complemented in L, then* $a^\perp = (\overline{a})' = (a')^\circ$ *and* $a_\perp = \overline{a'} = (a^\circ)'$.
(3) *If $a \in \tau$, then* $a^* = (\overline{a})' = (a')^\circ$, *where the pseudo-complement of a is formed in τ.*

Definition 2.3. [15]. For every $f \in \mathcal{R}(L_\tau)$, $f(\{0\})$ is called a **zero-element** of f and denoted by $z(f)$. Also, for every $f \in \mathcal{R}(L_\tau)$, a **cozero-element** of f is defined by

$$\operatorname{coz}(f) := f(-\infty, 0) \vee f(0, +\infty)\,.$$

Obviously, $z(f) = (\mathrm{coz}(f))'$.

Proposition 2.4. [15]. *For every $f, g \in \mathcal{R}(L_\tau)$, we have*

(1) $z(f+g) = z(f) \wedge z(g)$, *while* $f, g \geq \mathbf{0}$;
(2) *if* $\mathbf{0} \leq f \leq g$, *then* $z(f) \geq z(g)$;
(3) *for every* $n \in \mathbb{N}$, $z(f) = z(-f) = z(|f|) = z(f^n)$;
(4) $z(fg) = z(f) \vee z(g)$;
(5) $z(f+g) \geq z(f) \wedge z(g)$;
(6) $z(f) \wedge z(g) = z(|f| + |g|) = z(f^2 + g^2)$;
(7) $z(\mathbf{1}) = 0$. *Moreover,* $z(f) = 1$ *if and only if* $f = \mathbf{0}$;
For every $c, c_1, c_2 \in \mathbb{R}$ *and* $f \in \mathcal{R}(L_\tau)$, *we have*
(8) $z(f - \mathbf{c}) = f(\{c\})$.
(9) $z((f-\mathbf{c})^+) = f(-\infty, c]$ *and* $z((f-\mathbf{c})^-) = z((\mathbf{c}-f)^+) = f[c, +\infty)$.
(10) $z(f^+) = f(-\infty, 0]$ *and* $z(f^-) = f[0, +\infty)$.

Theorem 2.5. [15]. *A frame map f is a unit of $\mathcal{R}(L_\tau)$ if and only if $z(f) = 0$.*

We refer the reader to [11], [12], [13], and [16] for more information.

3. Hausdorff and regular topoframes

Recall that a topological space X is regular if and only if for each $U \in \mathcal{O}(X)$ and every $x \in U$, there exists an open set V with $x \in V$ and $\overline{V} \subseteq U$. Therefore X is regular if and only if $U = \bigcup_{\bar{V} \subseteq U} V$ for each $U \in \mathcal{O}(X)$ if and only if $U = \bigcup_{(\bar{V})' \cup U = X} V$ for each $U \in \mathcal{O}(X)$. In **Frm**, for any set V in a topological space, the well-known pointfree form of the expression $(\overline{V})'$ is b^* for some element b; that is, a frame τ is regular if $a = \bigvee_{b^* \vee a = 1} b$, for every $a \in \tau$. Fortunately, for any set V in a topological space, it is possible to give a more explicit description of $\overline{V}$ (and $(\overline{V})'$) in **TFrm** analogously denoted by the expression $\overline{b}$ (and $(\overline{b})'$) for some element b. So we will describe our candidate for regularity in topoframes as follows.

Definition 3.1. A topoframe L_τ is said to be **regular** iff for every $t \in \tau$,

$$t = \bigvee \{a \in \tau \mid \overline{a} \leq t\},$$

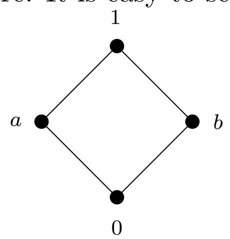
and L_τ (or may simply say that L) is said to be **τ-regular** iff for every $b \in L$,

$$b = \bigvee \{a \in L \mid \overline{a} \leq b\}.$$

Example 3.2. It is clear that the topoframe $(\mathcal{P}(X), \mathcal{O}(X))$ is $\mathcal{O}(X)$-regular if and only if X is a T_1-space.

Let X be any infinite set with the cofinite topology (in which the closed sets are the finite sets and X). It is well known that X is a T_1-space, which implies that the topoframe $(\mathcal{P}(X), \mathcal{O}(X))$ is $\mathcal{O}(X)$-regular

Example 3.3. Take $L = \{0, a, b, 1\}$. We define the binary relation $\leq$ on L in the following figure. It is easy to see that L is a frame.



If $\tau = \{0, 1\}$, then L_τ is a regular topoframe and if $\tau = \{0, a, 1\}$, then L_τ is not a regular topoframe.

To redefine the notions of regularity and τ- regularity for a topoframe L_τ, we give a new notion $\precsim$ in **TFrm** that is similar to the notion $\prec$ in **Frm** as follows.

Definition 3.4. Let τ be a topoframe on L and $a, b \in L$. We define a binary relation $\precsim$ on L by

$$a \precsim b \text{ iff } \overline{a} \leq b.$$

Example 3.5. Let L be a frame and $\tau = \{0, 1\} \subseteq L$. Then we have in topoframe L_τ,

$$\precsim \; = \{(x, 1) \colon x \in L\} \cup \{(0, 0)\}.$$

Proposition 3.6. *Let τ be a topoframe on L. Then for $a, b \in L$,*

(1) *$a \precsim b$ in L_τ if and only if $a^\perp \vee b = 1$ if and only if there exists $x \in \tau$ such that $x \wedge a = 0$ and $x \vee b = 1$;*

(2) *if $a, b \in \tau$, then $a \prec b$ in τ if and only if $\overline{a} \leq b$.*

Proof. 1. By Lemma 2.2(1), for $a \in L$, $a^\perp = (\overline{a})'$. So $a^\perp \vee b = 1$ if and only if $(\overline{a})' \vee b = 1$ if and only if $\overline{a} \leq b$ if and only if $a \precsim b$.

2. By Lemma 2.2(3), for any $a \in \tau$, $a^* = (\overline{a})'$, where the pseudo-complement of a is formed in τ. So for $a, b \in \tau$, $\overline{a} \leq b$ if and only if $(\overline{a})' \vee b = 1$ if and only if $a^* \vee b = 1$ if and only if $a \prec b$ in τ. □

Now, we show that regularity for topoframes agrees with that for frames.

Proposition 3.7. *L_τ is a regular topoframe if and only if τ is a regular frame.*

Proof. By Proposition 3.6, for any $a, b \in \tau$, $a \precsim b$ if and only if $a \prec b$ in τ. □

Next, we show that for any topoframe L_τ, the notion of τ-regularity is stronger than the regularity of the frame L. For example, a frame L is $\{0,1\}$-regular if and only if $L = \{0,1\}$; in this case, L is obviously a regular frame, but the reverse is not true. However, every complete Boolean algebra B is clearly B-regular which is the same as the regularity of B.

Proposition 3.8. *For any topoframe L_τ, the frame L is regular whenever L is τ-regular.*

Proof. Let L_τ is τ-regular. Then for every $a \in L$, $a = \bigvee\{x \in L \mid x^\perp \vee a = 1\}$, since $x^\perp \vee a = 1$ if and only if $\overline{x} \leq a$. But for every $x \in L$, $x^\perp \leq x^*$, where the pseudo-complement of x is formed in L, and hence

$$a = \bigvee\{x \in L \mid x^\perp \vee a = 1\} \leq \bigvee\{x \in L \mid x^* \vee a = 1\} \leq a\,,$$

that is L is a regular frame. □

The notion of the Hausdorffness of a frame goes back to [22] that a frame τ is said to be a T_2**-frame** if $t = \bigvee\{a \in \tau \mid a \leq t, a^* \nleq t\}$ holds for any $1 \neq t \in \tau$. T_2-frames coincide for topological spaces with Hausdorff spaces being described independently on points. To define T_2-axiom on topoframes, we use Lemma 2.2(3) and turn a^* into $(\overline{a})'$ for $a \in \tau$ in the definition of a T_2-frame mentioned above being the counterparts of the expression $(\overline{V})'$ for some set V in a topological space.

Definition 3.9. A topoframe L_τ is said to be **Hausdorff** iff for every $1 \neq t \in \tau$,

$$t = \bigvee\{a \in \tau \mid a \leq t, (\overline{a})' \nleq t\}\,.$$

Example 3.10. It is clear that if X be a Hausdorff space, then for every $G \in \mathcal{O}(X) \setminus \{X\}$,

$$G = \bigcup\{O \in \mathcal{O}(X) \mid O \subseteq G, (X \setminus \overline{O}) \nsubseteq G\}\,.$$

Hence, for every Hausdorff space X, the topoframe $(\mathcal{P}(X), \mathcal{O}(X))$ is Hausdorff.

Let X be any infinite set with the cofinite topology. Then the topoframe $(\mathcal{P}(X),\mathcal{O}(X))$ is not a Hausdorff topoframe.

Now, we show that Hausdorffness in topoframes agrees with that in frames.

Proposition 3.11. *A topoframe L_τ is Hausdorff if and only if τ is a T_2-frame.*

Proof. Since, by Lemma 2.2 for any $a \in \tau$, $a^* = a^\perp = (\overline{a})'$, we conclude that for every $1 \neq t \in \tau$,

$$t = \bigvee\{a \in \tau \mid a \leq t, (\overline{a})' \nleq t\} \text{ if and only if } t = \bigvee\{a \in \tau \mid a \leq t, a^* \nleq t\}\,,$$

where the pseudo-complement of a is formed in τ. This completes the proof. □

Proposition 3.12. *Let (L, τ) be a regular topoframe. Then (L, τ) is a Hausdorff topoframe.*

Proof. Let $1 \neq t \in \tau$. By assumption, we have $t = \bigvee\{a \in \tau \mid \overline{a} \leq t\}$, and $\overline{a} \leq t$ implies $(\overline{a})' \nleq t$ or else by contraposition the assumption $(\overline{a})' \leq t$ leads to the contradiction

$$1 = (\overline{a})' \vee \overline{a} \leq t \vee t = t\,.$$

Hence, for any $1 \neq t \in \tau$,

$$t = \bigvee\{a \in \tau \mid \overline{a} \leq t\} \leq \bigvee\{a \in \tau \mid \overline{a} \leq t, (\overline{a})' \nleq t\} \leq t\,.$$

This completes the proof. □

We close this section with some relations between compactness, locally compactness, and regularity in **TFrm**. For instance, we show that any compact regular topoframe is locally compact. Recall that a topological space X is **compact** iff every open covering $\{U_\alpha\}_{\alpha\in I}$ of X has a finite subcover. Since X is the largest element of the frame $\mathcal{O}(X)$ (or $\mathcal{P}(X)$), the following definition is the most reasonable counterpart of the notion of compactness for topoframes.

Definition 3.13. Let L_τ is a topoframe. An element $a \in L$ is said to be **compact** iff whenever $a \leq \bigvee S$, for $S \subseteq \tau$, then $a \leq \bigvee F$ for some finite subsets $F \subseteq S$. Also, L_τ is **compact** iff 1_L is a compact element.

Let B be a complete Boolean algebra. Suppose that $Atm(B)$ the set of all atoms of B and $Atm(B) \neq \emptyset$. We set

$$\tau = \left\{b \in B \mid \text{ there exists a finite subset } A \text{ of } Atm(B) \text{ such that } b' = \bigvee A\right\} \cup \{0\}$$

Then B_τ is a topoframe which is called the **cofinite** topoframe. Suppose that $S \subseteq \tau$ such that $1 = \bigvee S$. Let A be a finite subset of $Atm(B)$ such that $s_0 = (\bigvee A)' \in S$. Let $a \in A$ be given. If for every $s \in S$, $a \leq s'$, then

$$a = s' \wedge a = s' \wedge \bigvee_{s\in S} (s \wedge a) \leq s' \wedge s = 0.$$

This is a contradiction to the choice of a which is an atom element of B. Thus, for every $a \in A$, there exists an element $s_a \in S$ such that $a \leq s_a$, which implies that $1 = s_0 \vee \bigvee_{a\in A} s_a$. Therefore, B_τ is a compact topoframe.

Remark 3.14. Let L_τ be a compact topoframe. Then each closed element $k \in L$ is compact, because if $k \leq \bigvee S$ for some $S \subseteq \tau$, then $k' \vee \bigvee S = 1$ and so, by compactness of 1, there is $s_1, \ldots, s_n \in S$ such that $k' \vee (s_1 \vee \cdots \vee s_n) = 1$, and consequently, $k \leq s_1 \vee \cdots \vee s_n$ as desired.

Definition 3.15. Let L_τ be a topoframe and $a, b \in \tau$. We say that a is **τ-relatively compact with respect to** b and write $a \lll b$, iff there exists a compact element k such that $a \leq k \leq b$. A topoframe L_τ is said to be **locally compact** whenever, for each $t \in \tau$,

$$t = \bigvee \{x \mid x \in \tau, x \lll t\}\,.$$

Example 3.16. We observed in the previous example, if B_τ is a cofinite topoframe, then B_τ is compact, which implies that it is locally compact.

Example 3.17. It is well known that a topological space X is called a locally compact space if for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact subspace of X. Therefore, the topoframe $(\mathcal{P}(X),\mathcal{O}(X))$ is locally compact if and only if X is locally compact.

Proposition 3.18. (1) *In a compact topoframe L_τ, $a \prec b$ in τ implies $a \lll b$, in particular $a \lll 1$. So a compact regular topoframe is locally compact.*
(2) *In a regular topoframe L_τ, $a \lll b$ implies $a \prec b$ in τ.*
(3) *In a compact regular topoframe L_τ, $a \lll b$ if and only if $a \prec b$ in τ.*

Proof. 1. Since $\overline{a}$ is closed, $\overline{a}$ is compact, by Remark 3.14. Further, using Proposition 2.2, if $a \prec b$ in τ, then $a \leq \overline{a} \leq b$, and so $a \lll b$, by definition.

2. Let $a \lll b$. So there exists a compact element k such that $a \leq k \leq b$, and we get $a \leq k \leq b = \bigvee_{u \prec b} u$, by regularity. Thus, there exist $u_1, \ldots, u_n \in \tau$ such that $a \leq k \leq u_1 \vee \cdots \vee u_n \prec b$ in τ, by compactness and so $a \prec b$ in τ, by the properties of $\prec$.

3. It follows from parts (1) and (2). □

4. Completely regular topoframes

In this section, we define the notion of a completely regular topoframe. As we see, the case of regularity and Hausdorffness in topoframes are the same as those for frames. For the case of complete regularity, we have perhaps two different f-ring $\mathcal{R}(\tau)$ and $\mathcal{R}(L_\tau)$ in **Frm** and **TFrm**, respectively, and there are not as many functions in $\mathcal{R}(L_\tau)$ (as a sub-f-ring of $\mathcal{R}(\tau)$) as there are in $\mathcal{R}(\tau)$. So the set of all cozero elements in L_τ is contained in that of τ (see the proof of Lemma 4.3). If this inclusion is strict, the definition of complete regularity for L_τ may differ from that for τ.

Definition 4.1. A topoframe L_τ is said to be **completely regular** iff for every $t \in \tau$, there exists $\{f_i\}_{i \in I} \subseteq \mathcal{R}(L_\tau)$ such that

$$t = \bigvee_{\mathrm{coz}(f_i) \leq t} \mathrm{coz}(f_i)\,.$$

In fact, L_τ is completely regular if and only if $\tau =< \mathrm{Coz}(L_\tau) >$, where

$$\mathrm{Coz}(L_\tau) = \{\mathrm{coz}(f) \colon f \in \mathcal{R}(L_\tau)\}.$$

Example 4.2. Let $A \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$ be given. If $\tau = (\{\emptyset, A, \mathbb{N}\}, \subseteq)$, then $\mathcal{P}(\mathbb{N})_\tau$ is a completely regular topoframe.

Proposition 4.3. *Let L_τ be a completely regular topoframe. Then τ is a completely regular frame. In particular, every completely regular topoframe is regular.*

Proof. Let L_τ be a completely regular topoframe. Then for any $t \in \tau$ there exists $\{f_i\}_i \subseteq \mathcal{R}(L_\tau)$ such that

$$t = \bigvee_i \{\mathrm{coz}(f_i) \mid \mathrm{coz}(f_i) \leq t\},$$

and hence

$$t = \bigvee_i \{\mathrm{coz}(f_i|_{\mathcal{O}(\mathbb{R})}) \mid \mathrm{coz}(f_i|_{\mathcal{O}(\mathbb{R})}) \leq t\},$$

where $f_i|_{\mathcal{O}(\mathbb{R})} \in \mathcal{R}(\tau)$. Consequently, τ is a completely regular frame. □

In the following example we show that the inverse of Proposition 4.3 is not true.

Example 4.4. Suppose K is the topology space introduced in [28], which is in Exercise 18G on page 134. Then $(\mathcal{P}(K), \mathcal{O}(K))$ is a regular topoframe which is not completely regular topoframe.

Characterizing of complete regularity in **Frm** is a pointfree form of the method that is used to prove a normal space is completely regular and actually goes back to Urysohn who essentially employed a trail in $\mathcal{O}(X)$. That this method works in **TFrm** is left as an open problem. Counterexamples may exist out of the ring $C(X)$, because $C(X)$ is isomorphic to $\mathcal{R}(\mathcal{O}(X))$ (see [5, p.38]). This of course should not tempt one to think that properties of the former rings can, by and large, be deduced from those of the latter kind. As it is well known, the ring $C(X, \mathbb{Z})$, for a zero-dimensional space X, is (isomorphic to) a subring of $C(X)$, but, for instance, the ideal structure of $C(X, \mathbb{Z})$ is much more complex than that of $C(X)$.

Recall that **a base** B **of a frame** τ is a subset of τ such that every element of τ is a join of some elements in B. We say that B is **a base of a topoframe** L_τ if it is a base of frame τ. A **subbase** B of a frame τ is a subset of τ such that every element of τ is a join of finite meets of some elements in B. It is obvious that the set of all cozero elements in a completely regular topoframe L_τ is a base for L_τ. Also, the set of all interiors of zero elements in L_τ is a base for L_τ.

Proposition 4.5. *Let L_τ be a completely regular topoframe. Then every $t \in \tau$ is a supremum of (the interiors of) some zero elements in L_τ.*

Proof. By Definition 4.1, for every $t \in \tau$, there exists $\{f_i\}_{i\in I} \subseteq \mathcal{R}(L_\tau)$ such that

$$t = \bigvee_{\mathrm{coz}(f_i)\leq t} \mathrm{coz}(f_i)\,.$$

So

$$\begin{aligned} t &= \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{f_i(\frac{1}{n},+\infty) \vee f_i(-\infty,-\frac{1}{n}) \mid \mathrm{coz}(f_i)\leq t\} \\ &= \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{f_i(-\infty,-\frac{1}{n}]^\circ \vee f_i[\frac{1}{n},+\infty)^\circ \mid \mathrm{coz}(f_i)\leq t\} \\ &\leq \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{(f_i(-\infty,-\frac{1}{n}])^\circ \vee (f_i[\frac{1}{n},+\infty))^\circ \mid \mathrm{coz}(f_i)\leq t\} \\ &\leq \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{f_i(-\infty,-\frac{1}{n}] \vee f_i[\frac{1}{n},+\infty) \mid \mathrm{coz}(f_i)\leq t\} \\ &\leq \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{f_i(-\infty,-\frac{1}{n+1}) \vee f_i(\frac{1}{n+1},+\infty) \mid \mathrm{coz}(f_i)\leq t\} \\ &= t \end{aligned}$$

and let $h_{in} := (f_i + \mathbf{\frac{1}{n}}) \vee \mathbf{0}$ and $k_{in} := (f_i - \mathbf{\frac{1}{n}}) \wedge \mathbf{0}$ for all $i \in I, n \in \mathbb{N}$. Then, by Proposition 2.4(9), we have

$$\begin{aligned} t &= \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{f_i(-\infty,-\frac{1}{n}]) \vee f_i[\frac{1}{n},+\infty) \mid f_i(-\infty,-\frac{1}{n}] \leq t,\ f_i[\frac{1}{n},+\infty) \leq t\} \\ &= \textstyle\bigvee_{i\in I, n\in\mathbb{N}}\{z(h_{in}) \vee z(k_{in}) \mid z(h_{i,n}) \leq t,\ z(k_{in}) \leq t\}\,. \end{aligned}$$

Again, by Proposition 2.4(9), we obtain

$$t = \bigvee_{i\in I, n\in\mathbb{N}} \{(z(h_{in}))^\circ \vee (z(k_{in}))^\circ \mid (z(h_{i,n}))^\circ \leq t,\ (z(k_{in}))^\circ \leq t\}\,.$$

This completes the proof. □

Remark 4.6. To give an example of completely regular topoframes, consider a topoframe L_τ. Suppose w is a subframe of L generated by the set B of all cozero elements

$$f(-\infty,r) = \mathrm{coz}((f-\mathbf{r})\wedge 0)\,, \quad f(r,+\infty) = \mathrm{coz}((f-\mathbf{r})\vee 0)\,,$$

where $r \in \mathbb{R}$ and $f \in \mathcal{R}(L_\tau)$. It follows from Proposition 2.4 that the meet of a finite family of cozero elements is also a cozero element. Hence, every $a \in w$ can be written in the form

$$a = \bigvee_i \mathrm{coz}(f_i)$$

for some $\{f_i\}_i \subseteq \mathcal{R}(L_\tau)$. Also, by the first De Morgan law for Heyting algebras and that $\bigvee_i \mathrm{coz}(f_i) \in \tau$ (is complemented), we have

$$a' := (\bigvee_i \mathrm{coz}(f_i))' = \bigwedge_i (\mathrm{coz}(f_i))' = \bigwedge_i z(f_i)\,;$$

that is every element of w is complemented in L. So the subframe w is a topoframe on L generated by the subbase B and we shall say that w is weaker than τ. Since any morphism in $\mathcal{R}(L_\tau)$ is an $(\mathcal{O}(\mathbb{R}), w)$-homomorphism belonging to $\mathcal{R}(L_w)$, we conclude that L_w is a completely regular topoframe.

The next theorem eliminates any reason for considering rings of real-continuous functions on topoframes on other than completely regular topoframes.

Theorem 4.7. *For any topoframe L_τ, there exists a completely regular topoframe M_w such that the set $w \cup w'$ is a subbase for M and $\mathcal{R}(L_\tau)$ is isomorphic to $\mathcal{R}(M_w)$ as f-rings, where $w' := \{t' \mid t \in w\}$.*

Proof. Let w be the generated subframe of L by the set

$$B = \{f(-\infty, r) \mid f \in \mathcal{R}(L_\tau),\ r \in \mathbb{R}\} \cup \{f(r, +\infty) \mid f \in \mathcal{R}(L_\tau),\ r \in \mathbb{R}\}$$

and take M to be the generated subframe of L by $w \cup w'$. Then, every τ-real continuous function $f : \mathcal{P}(\mathbb{R}) \to L$ takes value in M, since

$$f(X) = \bigvee_{x\in X}^{L} f(\{x\}) = \bigvee_{x\in X}^{L} (f(-\infty, x)' \wedge f(x, +\infty)') \in \langle w' \rangle_L \subseteq M$$

for every $X \subseteq \mathbb{R}$. This is also an $(\mathcal{O}(\mathbb{R}), w)$-homomorphism since w contains the range of $f|_{\mathcal{O}(\mathbb{R})}$. So M_w is a completely regular topoframe. We denote the function f, when we replace L with M, by $f|^M$ and define θ from $\mathcal{R}(L_\tau)$ to $\mathcal{R}(M_w)$ by

$$\theta(f) := f|^M\,.$$

It is clear that θ is a one-one map. To show that θ is onto, let $g \in \mathcal{R}(M_w)$ and define $f : \mathcal{P}(\mathbb{R}) \to L$ by $f(S) := g(S)$. Since $w \subseteq \tau$, f is also an $(\mathcal{O}(\mathbb{R}), \tau)$-homomorphism and hence $f|^M = g$. Further, for every $f, g \in \mathcal{R}(L_\tau)$, and $\diamond \in \{+, ., \vee, \wedge\}$,

$$\theta(f \diamond g) = (f \diamond g)|^M = f|^M \diamond g|^M = \theta(f) \diamond \theta(g)$$

and hence $\theta : \mathcal{R}(L_\tau) \to \mathcal{R}(M_w)$ is an f-ring isomorphism with M_w completely regular and $w \cup w'$ a subbase for M. □

Now, it is helpful to look at some auxiliary facts about particles. Let L be a frame. The element $0 < p \in L$ is called a **particle** iff $p \leq \bigvee_i a_i$ implies that $p \leq a_i$ for some i (see [15]). Note that a particle of a frame is obviously is a coprime element. Also, every atom is a particle. For this, argue by contraposition. Let a be the supremum of a family $\{a_i\}_i$ in a frame L, and suppose x is below a but not below any element a_i. Then $x \wedge a_i = 0$ for each index i. So $x \wedge a = 0$, by the first infinite distributive law in the frame L, and hence $x = 0$. This contradicts the fact that x is an atom. There are, however, a number of characterizations of particles in complete Boolean algebras; if a frame L happens to be a complete Boolean algebra, then, by Lemma 1 in page 117 of [18], atoms, coprimes and particles coincide.

Proposition 4.8. *Let L_τ be completely regular. Suppose that p is a particle of L and $k \in \tau'$ such that $p \nleq k$. Then there exists $f \in \mathcal{R}(L_\tau)$ such that $k \leq z(f)$, and $p \leq \operatorname{coz}(f)$.*

Proof. Since $p \nleq k$, $p \leq 1 = k \vee k'$ and p is a particle, we conclude that $p \leq k'$. Since $k' \in \tau$ and L_τ is completely regular, there exists $\{f_i\}_{i \in I} \subseteq \mathcal{R}(L_\tau)$ such that

$$p \leq k' = \bigvee_{\operatorname{coz}(f_i) \leq t} \operatorname{coz}(f_i)\,.$$

Since p is a particle, there exists $i \in I$, $p \leq \operatorname{coz}(f_i) \leq k'$, so that $k \leq z(f_i)$ and $p \leq \operatorname{coz}(f_i)$ for some $f_i \in \mathcal{R}(L_\tau)$. □

Definition 4.9. Let L_τ be a topoframe. A **neighborhood** of $a \in L$ is an element u whose interior dominates a.

The following lemma just expresses the fact that in a completely regular topoframe L_τ, every neighborhood of a particle $p \in L$ dominates an open-neighborhood $\operatorname{Int}(z(h))$ of p for some $h \in \mathcal{R}(L_\tau)$.

Lemma 4.10. *Let L_τ be a completely regular topoframe with some particles. Then for each open element t and every particle p of L with $p \leq t$, there exists $h \in \mathcal{R}(L_\tau)$ such that*

$$p \leq \operatorname{Int}(z(h)) \leq z(h) \leq t\,.$$

Proof. By Proposition 4.5 and the definition of particle, the proof is evident. □

Let L_τ be a topoframe. We showed in [15] that for each particle $p \in L$, if it exists, $O_p := \{f \in \mathcal{R}(L_\tau) \mid p \leq (z(f))^\circ\}$ is a z-ideal and $M_p := \{f \in \mathcal{R}(L_\tau) \mid p \leq z(f)\}$ is a maximal z-ideal containing O_p. Here, we show that in the case of complete regularity, M_p is not only a maximal ideal but also the only maximal ideal of $\mathcal{R}(L_\tau)$ which contains O_p.

Proposition 4.11. *Let L_τ be a completely regular topoframe with some particle p in L. Then M_p is the only maximal ideal of $\mathcal{R}(L_\tau)$ which contains O_p.*

Proof. Clearly, O_p and M_p are ideals, and $O_p \subseteq M_p$. To prove M_p is the unique maximal ideal containing O_p, argue by contraposition. Suppose that M is another maximal ideal containing O_p. Then, $M \nsubseteq M_p$ and hence there exists $g \in M$ such that $g \notin M_p$, i.e., $p \nleq z(g)$, so that $p \leq \mathrm{coz}(g)$ because $p \leq 1 = \mathrm{coz}(g) \vee z(g)$ and p is a particle. Since $\mathrm{coz}(g)$ is open, by Lemma 4.10, there exists $h \in \mathcal{R}(L_\tau)$ such that

$$p \leq \mathrm{Int}(z(h)) \leq z(h) \leq \mathrm{coz}(g)\,.$$

So $z(h^2+g^2) = z(h) \wedge z(g) = 0$, and consequently, using Theorem 2.5, h^2+g^2 is invertible. Since $p \leq \mathrm{Int}(z(h))$, we have $h \in O_p \subseteq M$. On the other hand, $g \in M$. Thus $h^2+g^2 \in M$, and hence $\mathbf{1}_{\mathcal{R}(L_\tau)} \in M$, namely $M = \mathcal{R}(L_\tau)$ which is a contradiction to the maximality of M.

□

Example 4.12. Consider the topoframe of $(\mathcal{P}(\mathbb{R}), \mathfrak{O}(\mathbb{R}))$. Let $r \in \mathbb{R}$ be given. Then r is a particle element of $\mathcal{P}(\mathbb{R})$. If $f(x \mapsto x-r)\colon \mathbb{R} \to \mathbb{R}$, then $f \in M_r \setminus O_r$. Also, M_r is the only maximal ideal of $\mathcal{R}(\mathcal{P}(\mathbb{R})_{\mathfrak{O}(\mathbb{R})})$ which contains O_r.

Finally, in this section, we encounter a new notion such as the notion of "completely below" on frames being used to introduce the notion of the τ-complete regularity of a topoframe L_τ. Let τ be a topoframe on L and $a, b \in L$. We define a binary relation $\precapprox$ on L by $a \precapprox b$, if there exists a trail $\{a_i\}_{i\in\mathbb{Q}} \subseteq L$ such that $a_0 = a$, $a_1 = b$ and for every $p, q \in [0,1] \cap \mathbb{Q}$ with $p < q$, $a_p \precsim a_q$, $a_p = 0$ if $p \in (-\infty, 0) \cap \mathbb{Q}$ and $a_p = 1$ if $p \in (1, +\infty) \cap \mathbb{Q}$.

Definition 4.13. Let L_τ be a topoframe. Then L_τ (or may simply say that L) is said to be **τ-completely regular** iff for every $a \in L$,

$$a = \bigvee\{x \in L \mid x \precapprox a\}\,.$$

Now, we show that for any topoframe L_τ, the notion of τ-complete regularity is stronger than complete regularity.

Proposition 4.14. *For any topoframe L_τ, L is a completely regular frame whenever L is τ-completely regular.*

Proof. For every $x \in L$, $x^\perp \leq x^*$, where the pseudo-complement of x is formed in L, and hence, by Proposition 3.6, for every $x, a \in L$, $x \precapprox a$ implies $x \prec\!\!\prec a$ in L, so that

$$a = \bigvee\{x \in L \mid x \precapprox a\} \leq \bigvee\{x \in L \mid x \prec\!\!\prec_L a\} \leq a\,.$$

Thus, L is a completely regular frame. □

5. B-filters on topoframes

Using the machinery developed in the foregoing section, we now describe some results about B-filters (specially z-filters) on topoframes containing fixed and free ones, the convergent ones, and an outline of the theory of convergence of z-filters on completely regular topoframes.

Let B be a bounded distributive lattice. It is known that

(1) $\emptyset \neq G \subseteq B$ is **a B-filter base** provided that $0 \notin G$ and for every $a, b \in G$, there exists $0 \neq c \in G$ that $c \leq a \wedge b$; and

(2) a B-filter base G is a B**-filter** if whenever $b \in G$, $a \in B$, and $b \leq a$, then $a \in G$.

A subset of a poset $(P, \leq)$ is called an **ideal** if it is a lower set which is also directed. This definition needs not purely interest us unless we know some of their important properties on topoframes collecting them here. Let L_τ be a topoframe and let B be a distributive sublattice of L. Let $\mathbf{F}$ be a B-filter base on L. We define

$$\widetilde{\mathbf{F}} = \{a \in B \mid b \leq a \ \text{ for some } \ b \in \mathbf{F}\}.$$

Evidently, $\widetilde{\mathbf{F}}$ is a B-filter, and say that $\mathbf{F}$ generates $\widetilde{\mathbf{F}}$. Obviously, $\bigwedge_{b\in\widetilde{\mathbf{F}}} \overline{b} = \bigwedge_{b\in\mathbf{F}} \overline{b}$. A B-filter base $\mathbf{F}$ is **fixed** iff $\bigwedge_{b\in\mathbf{F}} \overline{b} \neq 0$ and **free** iff $\bigwedge_{b\in\mathbf{F}} \overline{b} = 0$. To show that every B-filter base on a compact topoframe L_τ is fixed, let $\mathbf{F}$ be an arbitrary B-filter base on a distributive sublattice B of L. Assume that, on the contrary, $\bigwedge_{b\in\mathbf{F}} \overline{b} = 0$. Then $\bigvee_{b\in\mathbf{F}} \overline{b}' = 1$ and hence, by compactness of 1, there exist $b_1, \ldots, b_n \in \mathbf{F}$ such that $\overline{b_1}' \vee, \ldots, \vee \overline{b_n}' = 1$, so $\overline{b_1} \wedge, \ldots, \wedge \overline{b_n} = 0$. This contradicts the fact that $\mathbf{F}$ is a proper B-filter base.

If I is a proper ideal in $\mathcal{R}(L_\tau)$, then the family $z[I] = \{z(f) \mid f \in I\}$ is a z-filter on L_τ (see [15, Proposition 5.2]). An ideal I of $\mathcal{R}(L_\tau)$ (or the z-filter $z[I]$) is said to be **fixed** iff $\bigwedge_{f\in I} z(f) \neq 0$ and **free** iff $\bigwedge_{f\in I} z(f) = 0$. Clearly, if L_τ is a compact topoframe, then every proper ideal in $\mathcal{R}(L_\tau)$ is fixed.

Let $a \in B$ and $\mathbf{F}$ be a B-filter base; we say a **meets** $\mathbf{F}$ iff $a \wedge b \neq 0$, for all $b \in \mathbf{F}$. Two B-filter base $\mathbf{F}$ and $\mathbf{G}$ **meet each other** iff each member of $\mathbf{G}$ meets $\mathbf{F}$ and vice versa. A coprime element $x \in L$ is said to be an **accumulation point** (a **cluster point**) of $\mathbf{F}$, when every neighborhood of x meets $\widetilde{\mathbf{F}}$, equivalently, when every neighborhood of x meets $\mathbf{F}$. $\mathbf{F}$ **converges** to an element $x \in L$, iff every neighborhood of x dominates at least a member of $\widetilde{\mathbf{F}}$. By a B**-ultrafilter** on L, we mean a maximal B-filter, i.e., one not contained in any other B-filter.

For $A \subseteq \mathcal{R}(L_\tau)$, we write $z[A]$ to designate the family of zero-elements

$$\{z(f) \mid f \in A\}.$$

Recall from [15] that a proper filter of $z[\mathcal{R}(L_\tau)]$ (simply denoted by $z[L_\tau]$) is called a **z-filter** on L_τ. It is worth mentioning that for a topoframe L_τ, $z[L_\tau]$ is obviously a distributive sublattice of L, and consequently every z-filter on L_τ is considered as a $z[L_\tau]$-filter. So all facts about B-filters on topoframes are true about z-filters in topoframes.

Lemma 5.1. *Let τ be a topoframe on L and $x, b \in L$. If x is a coprime element (specially a particle) of L, then $x \leq \overline{b}$ if and only if every neighborhood u of x meets b, that is $u \wedge b \neq 0$.*

Proof. Let $x \leq \overline{b}$ and suppose, to the contrary, that there exists $u \in \tau$ such that $x \leq u$ and $u \wedge b = 0$, it follows that $b \leq u'$. Since u' belongs to τ', by the properties of closure, we infer that $x \leq \overline{b} \leq \overline{u'} = u'$. Therefore, $x \leq u' \wedge u = 0$, which contradicts the definition of a coprime element. To demonstrate the sufficiency of the condition, assume, to the contrary, that $x \not\leq \overline{b}$. Since x is coprime, we have $x \leq \overline{b}'$. But, by the properties of closure, $\overline{b}' \wedge b \leq \overline{b}' \wedge \overline{b} = 0$. This contradicts the assumption that every neighborhood of x meets b. □

Corollary 5.2. *Let L_τ be a topoframe and B a distributive sublattice of L. Suppose x is a coprime element of L. Then x is a cluster point of a B-filter base $\mathbf{F}$ if and only if $x \leq \bigwedge_{b \in \mathbf{F}} \overline{b}$.*

Proof. If x is a cluster point of $\mathbf{F}$, using Lemma 5.1, we have $x \leq \overline{b}$, for any $b \in \mathbf{F}$. Hence, $x \leq \bigwedge_{b \in \mathbf{F}} \overline{b}$. Conversely, let x be a coprime element which $x \leq \bigwedge_{b \in \mathbf{F}} \overline{b}$. Then for any $b \in \mathbf{F}$, $x \leq \overline{b}$ and by Lemma 5.1, the claim that x is a cluster point of $\mathbf{F}$ is trivially true. □

Lemma 5.3. *Let L_τ be a topoframe and B a bounded distributive sublattice of L. Suppose that $\mathbf{F}$ is a B-filter base and $\widetilde{\mathbf{F}} = \{a \in B \mid b \leq a \text{ for some } b \in \mathbf{F}\}$. Then the following conditions hold.*

(1) $\widetilde{\mathbf{F}} = \bigcap \{\mathbf{G} \mid \mathbf{G}$ *is a B-filter,* $\mathbf{F} \subseteq \mathbf{G}\}$.
(2) $\mathbf{F}$ *is contained in a B-ultrafilter.*
(3) $\mathbf{F}$ *is a B-ultrafilter if and only if for every $a \in B$, if a meets $\mathbf{F}$, then $a \in \mathbf{F}$.*
(4) *If $\mathcal{U}$ and $\mathcal{V}$ are disjoint B-ultrafilter, then there exist elements $u \in \mathcal{U}$ and $v \in \mathcal{V}$ such that $u \wedge v = 0$.*
(5) *If $\{\mathbf{F}_i \mid i \in I\}$ is a nonempty collection of B-filters, then $\bigcap_{i \in I} \mathbf{F}_i$ is a B-filter.*

Proof. Trivial. □

Proposition 5.4. *Let $\mathbf{F}$ be a z-filter on a completely regular topoframe L_τ, and p be a particle of L. Then*

(1) *p is a cluster point of $\mathbf{F}$ if and only if $p \leq \bigwedge \mathbf{F}$;*
(2) *if $\mathbf{F}$ converges to p, then p is a cluster point of $\mathbf{F}$;*

(3) $\mathbf{F}$ *converges to* p *if and only if* $z[O_p] \subseteq \mathbf{F}$;

(4) *if* p *is a cluster point of* $\mathbf{F}$, *then at least one* z*-ultrafilter containing* $\mathbf{F}$ *converges to* p.

Proof. 1. By Corollary 5.2 and the fact that every B-filter is a B-filter base, the proof is evident.

2. Let u be a neighbourhood of p. The assumption implies that there is $z \in \mathbf{F}$ such that $z \leq u$ and z, by the definition of a z-filter, meets $\mathbf{F}$. Thus,

$$0 < z \wedge y \leq u \wedge y$$

for all $y \in \mathbf{F}$ and consequently, p is a cluster point of $\mathbf{F}$.

3. Let $z \in z[O_p] = \{z \in z[L_\tau] \mid p \leq \mathrm{Int}(z)\}$. Suppose $\mathbf{F}$ converges to p. $\mathrm{Int}(z)$ is a neighborhood of p. So, by assumption, there exists $y \in \mathbf{F}$ such that $y \leq \mathrm{Int}(z)$. Since $y \leq \mathrm{Int}(z) \leq z$, by the definition of a z-filter, $z \in \mathbf{F}$; that is $z[O_p] \subseteq \mathbf{F}$. For the converse, let v be an open-neighbourhood of p. Since L_τ is a completely regular topoframe, by Lemma 4.10, there exists $f \in \mathcal{R}(L_\tau)$ such that

$$p \leq \mathrm{Int}(z(f)) \leq z(f) \leq v\,,$$

and hence, by assumption, $z(f) \in \mathbf{F}$. So $\mathbf{F}$ converges to p.

4. Let $\mathbf{F}_1 := \mathbf{F} \cup \{z \in z[L_\tau] \mid p \leq \mathrm{Int}(z)\}$. Since p is a cluster point of $\mathbf{F}$, for any $z \in z[L_\tau]$, the open element $\mathrm{Int}(z)$ meets $\mathbf{F}$, and so does z. Thus, for any $a, b \in \mathbf{F}_1$, $0 < a \wedge b \in \mathbf{F}_1$, that is $\mathbf{F}_1$ is a z-filter base and hence, by Lemma 5.3(2), $\mathbf{F}_1$ is contained in a z-ultrafilter $\mathbf{E}$. Since $z[O_p] \subseteq \mathbf{E}$, by part (3), $\mathbf{E}$ converges to p. □

Remark 5.5. Let L_τ be a completely regular topoframe. Then every z-ultrafilter on L_τ converges to their cluster points, if exist, that are particles of L. For this, let $\mathcal{U}$ be a z-ultrafilter on L_τ, and p a cluster point of $\mathcal{U}$ which is a particle. Then, by Proposition 5.4(4), at least one z-ultrafilter $\mathcal{V}$ containing $\mathcal{U}$ converges to p. Thus, by maximality we have $\mathcal{U} = \mathcal{V}$, and hence $\mathcal{U}$ converges to p.

By a **prime** z**-filter**, we mean a z-filter $\mathbf{F}$ with the following property: whenever the join of two zero-elements belongs to $\mathbf{F}$, then at least one of them belongs to $\mathbf{F}$. The converse of statement (2) of Proposition 5.4 is generally not true (see [17, p. 46]) but it holds for all prime z-filters as follows.

Proposition 5.6. *Let* $\mathbf{P}$ *be a prime (or a maximal) z-filter on a completely regular topoframe* L_τ *and* p *a particle of* L. *Then,* $\mathbf{P}$ *converges to* p *if and only if* p *is a cluster point of* $\mathbf{P}$.

Proof. If $\mathbf{P}$ converges to p, then, by Proposition 5.4(2), p is a cluster point of $\mathbf{P}$. For the converse, assume that p is a cluster point of $\mathbf{P}$, and $z \in z(L_\tau)$ such that $p \leq \mathrm{Int}(z)$. Then, by the complete regularity of L_τ, there is $\{f_i\}_{i\in I} \subseteq \mathcal{R}(L_\tau)$ such that

$$p \leq \mathrm{Int}(z) = \bigvee \{\mathrm{coz}(f_i) \mid \mathrm{coz}(f_i) \leq \mathrm{Int}(z)\},$$

and since p is a particle, there is an $i \in I$ with $p \leq \mathrm{coz}(f_i) \leq \mathrm{Int}(z)$, and thus $\mathrm{coz}(f_i) \leq z$. Therefore $z(f_i) \vee z = 1 \in \mathbf{P}$, and hence $z(f_i) \in \mathbf{P}$ or $z \in \mathbf{P}$, because P is prime. But if $z(f_i) \in \mathbf{P}$, then, using Proposition 5.4(1), $p \leq \bigwedge \mathbf{P} \leq z(f_i)$, and consequently $p \not\leq \mathrm{coz}(f_i)$. This contradicts the early conclusion. Thus $z \in \mathbf{P}$, and therefore $z[O_p] \subseteq \mathbf{F}$. Hence, by Proposition 5.4(3), $\mathbf{P}$ converges to p. □

Finally, we characterize all z-ultrafilters converging to particles.

Proposition 5.7. *Let L_τ be a completely regular topoframe. Then for a particle p, if it exists,*

(1) *p is a cluster point of a z-filter $\mathbf{F}$ on L_τ if and only if $\mathbf{F} \subseteq z[M_p]$;*
(2) *$z[M_p]$ is the unique z-ultrafilter converging to p;*
(3) *distinct z-ultrafilters can not have a common cluster point which is also a particle;*
(4) *if $\mathbf{F}$ is a z-filter converging to p, then $z[M_p]$ is the unique z-ultrafilter containing $\mathbf{F}$.*

Proof. 1. Let $\mathbf{F} \subseteq z[M_p]$. Then $p \leq z$, for every $z \in \mathbf{F}$, and hence $p \leq \bigwedge \mathbf{F}$. Thus, by Proposition 5.4(1), p is a cluster point of a z-filter $\mathbf{F}$. The reverse is easier.

2. Since $z[O_p] \subseteq z[M_p]$, by Proposition 5.4(3) , $z[M_p]$ is a z-filter converging to p. Let now $\mathbf{A}$ be a z-ultrafilter converging to p. Then, by Proposition 5.4(2), p is a cluster point of $\mathbf{A}$, and, by part (1), $\mathbf{A} \subseteq z[M_p]$. So $\mathbf{A} = z[M_p]$, since $\mathbf{A}$ is a z-ultrafilter. So $z[M_p]$ is the unique z-ultrafilter converging to p.

3. Let z-ultrafilters $\mathbf{U}$ and $\mathbf{V}$ have a common cluster point p which is also a particle. So, by Remark 5.5, they converge to the particle p. Hence, $\mathbf{U} = \mathbf{V} = z[M_p]$, by part (1).

4. Let $\mathbf{A}$ be a z-ultrafilter containing $\mathbf{F}$. Thus, $\mathbf{A}$ converges to p. Using part (2), we have $\mathbf{A} = z[M_p]$, as desired. □

6. Conclusion

We introduced Hausdorff, compact, locally compact, regular and completely regular topoframes. We proved that L_τ is a regular topoframe if and only if τ is a regular frame. Also, if L_τ is a regular topoframe, then L_τ is a Hausdorff topoframe. We concluded that every open element of a topoframe, is a supremum of (the interiors of) some zero elements in topoframe. We showed that for every topoframe L_τ, there exists a completely regular topoframe M_w such that $\mathcal{R}(L_\tau) \cong \mathcal{R}(M_w)$ as f-rings. The previous fact eliminates any reason for considering rings of continuous functions on other than completely regular topoframes. We described the notion of B-filters on topoframes and for instance, we discussed some results such as the convergency of B-filters being analogous to the standard theory of convergence of filters or filter bases on an arbitrary topological space.

The following can be suggested future work on this topic.

(1) What is the classification of maximal ideals of $\mathcal{R}(L_\tau)$?
(2) What is the classification of prime ideals of $\mathcal{R}(L_\tau)$?
(3) What is the product of objects in category **TFrm**?
(4) What is the coproduct of objects in category **TFrm**?
(5) What is the equalizer of each pair of morphisms in category **TFrm**?
(6) What is the coequalizer of each pair of morphisms in category **TFrm**?

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