

Research Paper

GE-DERIVATIONS

YOUNG BAE JUN AND RAVIKUMAR BANDARU\*

ABSTRACT. The notions of xi-inside GE-derivation and xi-outside GE-derivation on a GE-algebra are introduced and its properties are investigated. Conditions for a self-map on GE-algebra to be a xi-inside GE-derivation and a xi-outside GE-derivation are provided. The xi-inside GE-derivation or the xi-outside GE-derivation rho are used to form two sets X\_{(rho=xi)} and ker(rho), and GE-subalgebra and GE-filter are studied for these two sets.

1. INTRODUCTION

Y. Imai and K. Iséki (see [8, 9]) introduced BCK-algebras in 1966 as the algebraic semantics for a non-classical logic with only implication. Various scholars have studied the generalized concepts of BCK-algebras since then. L. Henkin and T. Skolem introduced Hilbert algebras in the 1950s for research into intuitionistic and other non-classical logics. A. Diego demonstrated that Hilbert algebras constitute a locally finite variety (see [6]). Later, several researchers expanded on the theory of Hilbert algebras (see [4, 5, 7, 10, 11]). The notion of BE-algebra

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\*Corresponding author

was introduced by H. S. Kim and Y. H. Kim as a generalization of a dual BCK-algebra (see [13]). A. Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see [16]). Y. B. Jun et al. (see [12]) introduced the notion of left-right (resp. right-left) derivation of a BCI-algebra and some related properties are investigated. Using the idea of regular derivation they gave characterizations of a p-semisimple BCI-algebra. They also gave a condition for a derivation to be regular. K. H. Kim et al. extended the concept of derivation to BE-algebras (see [14]) and investigated its related properties. In the study of algebraic structures, the generalization process is also an important topic. As a generalization of Hilbert algebras, R. K. Bandaru et al. introduced the notion of GE-algebras, and investigated several properties (see [1]). For the general development of GE-algebras, the filter theory plays an important role. With this motivation, R. K. Bandaru et al. introduce the notion of belligerent GE-filters in GE-algebras and studied its properties (see [2]). A. Rezaei et al. introduced the concept of prominent GE-filters in GE-algebras and discussed its properties (see [17]). M. A. Öztürk et al. introduced the concept of Strong GE-filters, GE-ideals of bordered GE-algebras and investigated its properties (see [15]). A. Borumand Saeid et al. introduced the concept of voluntary GE-filters of GE-algebras and investigated its properties (see [3]). S. Z. Song et al. introduced the concept of Imploring GE-filters of GE-algebras and discussed its properties (see [18]).

In this paper, we introduce the notions of  $\xi$ -inside GE-derivation and  $\xi$ -outside GE-derivation and study their properties. We provide conditions for a self-map on GE-algebra to be a  $\xi$ -inside GE-derivation and a  $\xi$ -outside GE-derivation. We explore conditions under which the two self-maps  $\varrho$  and  $\xi$  on GE-algebra become equal. We give conditions for a  $\xi$ -inside GE-derivation and a  $\xi$ -outside GE-derivation to be order preserving. We make two sets  $X_{(\varrho=\xi)}$  and  $\ker(\varrho)$  using the  $\xi$ -inside GE-derivation or the  $\xi$ -outside GE-derivation, study their properties, and also find the conditions under which they become GE-subalgebra and/or GE-filter.

## 2. Preliminaries

**Definition 2.1** ([1]). A *GE-algebra* is a non-empty set  $X$  with a constant 1 and a binary operation “ $*$ ” satisfying the following axioms:

$$(GE1) \quad u * u = 1,$$

$$(GE2) \quad 1 * u = u,$$

$$(GE3) \quad u * (v * w) = u * (v * (u * w))$$

for all  $u, v, w \in X$ .

In a GE-algebra  $X$ , a binary relation “ $\leq$ ” is defined by

$$(1) \quad (\forall u, v \in X) (u \leq v \Leftrightarrow u * v = 1).$$

**Definition 2.2** ([1, 2]). A GE-algebra  $X$  is said to be

- *transitive* if it satisfies:

$$(2) \quad (\forall u, v, w \in X) (u * v \leq (w * u) * (w * v)).$$

- *commutative* if it satisfies:

$$(3) \quad (\forall u, v \in X) ((u * v) * v = (v * u) * u).$$

**Proposition 2.3** ([1]). *Every GE-algebra  $X$  satisfies the following items.*

$$(4) \quad (\forall u \in X) (u * 1 = 1).$$

$$(5) \quad (\forall u, v \in X) (u * (u * v) = u * v).$$

$$(6) \quad (\forall u, v \in X) (u \leq v * u).$$

$$(7) \quad (\forall u, v, w \in X) (u * (v * w) \leq v * (u * w)).$$

$$(8) \quad (\forall u \in X) (1 \leq u \Rightarrow u = 1).$$

$$(9) \quad (\forall u, v \in X) (u \leq (v * u) * u).$$

$$(10) \quad (\forall u, v \in X) (u \leq (u * v) * v).$$

$$(11) \quad (\forall u, v, w \in X) (u \leq v * w \Leftrightarrow v \leq u * w).$$

If  $X$  is transitive, then

$$(12) \quad (\forall u, v, w \in X) (u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w).$$

$$(13) \quad (\forall u, v, w \in X) (u * v \leq (v * w) * (u * w)).$$

$$(14) \quad (\forall u, v, w \in X) (u \leq v, v \leq w \Rightarrow u \leq w).$$

### 3. INSIDE AND OUTSIDE GE-DERIVATIONS

In what follows, let  $X$  denote a GE-algebra unless otherwise specified, and the execution of operation “ $*$ ” precedes the execution of operation “ $\widehat{+}$ ”. Given a self-map  $\varrho : X \rightarrow X$ , the image of  $x \in X$  under  $\varrho$  is denoted by  $x\varrho$ .

**Definition 3.1.** An *inside* (resp. *outside*) *GE-derivation* on  $X$  is defined as a self-map  $\varrho : X \rightarrow X$  that satisfies the identical equation

$$(x * y)\varrho = x * y\varrho \widehat{+} x\varrho * y \quad (\text{resp.}, (x * y)\varrho = x\varrho * y \widehat{+} x * y\varrho)$$

where  $x \widehat{+} y = (x * y) * y$  for all  $x, y \in X$ .

If  $\varrho : X \rightarrow X$  is both an inside GE-derivation and an outside GE-derivation, we say it is a *GE-derivation*.

**Example 3.2.** 1. Let  $X = \{1, a, b, c, d, e, f\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$	$e$	$f$
1	1	$a$	$b$	$c$	$d$	$e$	$f$
$a$	1	1	$e$	1	1	$e$	1
$b$	1	1	1	1	1	1	1
$c$	1	$d$	$f$	1	$d$	1	$f$
$d$	1	$c$	$c$	$c$	1	1	1
$e$	1	1	$f$	1	1	1	$f$
$f$	1	$a$	$c$	$c$	$d$	1	1

Then  $X$  is a GE-algebra. Define a self-map:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, d, e\}, \\ b & \text{if } x = b, \\ c & \text{if } x \in \{a, c\}, \\ f & \text{if } x = f. \end{cases}$$

Then  $\varrho$  is an inside GE-derivation on  $X$ . But  $\varrho$  is not an outside GE-derivation because of

$$a\varrho * b \hat{+} a * b\varrho = c * b \hat{+} a * b = f \hat{+} e = e \neq 1 = e\varrho = (a * b)\varrho.$$

2. Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	$d$	1	$d$
$b$	1	$a$	1	1	1
$c$	1	$a$	$b$	1	$b$
$d$	1	$a$	1	1	1

Then  $X$  is a GE-algebra. Define a self-map:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, c\}, \\ b & \text{if } x = b, \\ d & \text{if } x = d. \end{cases}$$

Then  $\varrho$  is an outside GE-derivation on  $X$ . But  $\varrho$  is not an inside GE-derivation on  $X$  because of

$$a * b\varrho \hat{+} a\varrho * b = a * b \hat{+} 1 * b = d \hat{+} b = b \neq d = d\varrho = (a * b)\varrho.$$

3. Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	$b$	$d$	$d$
$b$	1	1	1	$c$	$c$
$c$	1	$a$	$b$	1	1
$d$	1	1	$b$	1	1

Then  $X$  is a GE-algebra. Define a self-map:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x = a, \\ b & \text{if } x = b. \end{cases}$$

Then  $\varrho$  is a GE-derivation on  $X$ .

**Proposition 3.3.** *If  $\varrho : X \rightarrow X$  is either an inside GE-derivation or an outside GE-derivation, then  $1\varrho = 1$ .*

*Proof.* Let  $\varrho : X \rightarrow X$  be an inside GE-derivation. Using (GE1) and (4), we have

$$1\varrho = (x * 1)\varrho = x * 1\varrho \hat{+} x\varrho * 1 = x * 1\varrho \hat{+} 1 = 1$$

for all  $x \in X$ . If  $\varrho : X \rightarrow X$  is an outside GE-derivation, then

$$1\varrho = (x * 1)\varrho = x\varrho * 1 \hat{+} x * 1\varrho = 1 \hat{+} x * 1\varrho = 1$$

by (GE1), (GE2) and (4).  $\square$

**Corollary 3.4.** *If  $\varrho : X \rightarrow X$  is a GE-derivation, then  $1\varrho = 1$ .*

**Definition 3.5.** Given a self-map  $\varrho : X \rightarrow X$ , if there exists a GE-endomorphism  $\xi : X \rightarrow X$  that satisfies:

$$(15) \quad (\forall x, y \in X)((x * y)\varrho = x\xi * y\varrho \hat{+} x\varrho * y\xi),$$

then we say that  $\varrho$  is a  $\xi$ -outside GE-derivation.

**Definition 3.6.** Given a self-map  $\varrho : X \rightarrow X$ , if there exists a GE-endomorphism  $\xi : X \rightarrow X$  that satisfies:

$$(16) \quad (\forall x, y \in X)((x * y)\varrho = x\varrho * y\xi \hat{+} x\xi * y\varrho),$$

then we say that  $\varrho$  is a  $\xi$ -inside GE-derivation.

**Example 3.7.** (1) Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$c$	1
$b$	1	1	1	$c$	$d$
$c$	1	$a$	$a$	1	1
$d$	1	$b$	$b$	$c$	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, b, d\}, \\ c & \text{if } x = c. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, d\}, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -outside GE-derivation on  $X$ .

(2) Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$c$	$c$
$b$	1	1	1	$d$	$d$
$c$	1	$a$	$b$	1	1
$d$	1	$a$	$b$	1	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, b, c, d\}, \\ a & \text{if } x = a. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ b & \text{if } x = a, \\ a & \text{if } x = b. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$ .

**Proposition 3.8.** *If a self-map  $\varrho : X \rightarrow X$  is a GE-endomorphism, then it is both a  $\varrho$ -inside GE-derivation and a  $\varrho$ -outside GE-derivation.*

*Proof.* Assume that  $\varrho : X \rightarrow X$  is a GE-endomorphism and let  $x, y \in X$ . Then  $(x * y)\varrho = x\varrho * y\varrho = x\varrho * y\varrho \hat{+} x\varrho * y\varrho$  for all  $x, y \in X$ . This completes the proof.  $\square$

**Lemma 3.9.** *If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation or a  $\xi$ -inside GE-derivation, then  $1\varrho = 1$ .*

*Proof.* Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation. Then  $\xi$  is a GE-endomorphism, and so  $1\xi = 1$ . Hence

$$1\varrho = (x * 1)\varrho = x\xi * 1\varrho \hat{+} x\varrho * 1\xi = x\xi * 1\varrho \hat{+} x\varrho * 1 = x\xi * 1\varrho \hat{+} 1 = 1$$

for all  $x \in X$  by (GE1) and (4). Now, if  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation, then

$$1\varrho = (x * 1)\varrho = x\varrho * 1\xi \hat{+} x\xi * 1\varrho = x\varrho * 1 \hat{+} x\xi * 1\varrho = 1 \hat{+} x\xi * 1\varrho = 1$$

by (GE1), (GE2) and (4).  $\square$

**Corollary 3.10.** *Every  $\xi$ -outside GE-derivation or  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  satisfies:*

$$(17) \quad (\forall x \in X)((x \hat{+} 1)\varrho = 1 = (1 \hat{+} x)\varrho).$$

**Proposition 3.11.** *Every  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  satisfies:*

$$(18) \quad (\forall x \in X)(x\varrho = x\varrho \hat{+} x\xi).$$

*Also, if  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation, then*

$$(19) \quad (\forall x \in X)(x\varrho = x\xi \hat{+} x\varrho).$$

*Proof.* If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation, then  $\xi$  is a GE-endomorphism and hence  $1\xi = 1$ . It follows from (GE2) and Lemma 3.9 that

$$x\varrho = (1 * x)\varrho = 1\xi * x\varrho \hat{+} 1\varrho * x\xi = 1 * x\varrho \hat{+} 1 * x\xi = x\varrho \hat{+} x\xi$$

for all  $x \in X$ . Now, suppose that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation. Then  $1\xi = 1$  because  $\xi$  is a GE-endomorphism. The combination of (GE2) and Lemma 3.9 builds the following.

$$x\varrho = (1 * x)\varrho = 1\varrho * x\xi \hat{+} 1\xi * x\varrho = 1 * x\xi \hat{+} 1 * x\varrho = x\xi \hat{+} x\varrho$$

for all  $x \in X$ .  $\square$

Some  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  does not satisfy the condition (18) as seen in the next example.

**Example 3.12.** Consider the GE-algebra  $X$  in Example 3.7(2) and define self-maps  $\varrho$  and  $\xi$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ b & \text{if } x = a, \\ a & \text{if } x = b. \end{cases}$$

Then  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$ . But it does not satisfy (18) because of

$$a\varrho \hat{+} a\xi = a \hat{+} b = b \neq a = a\varrho.$$

**Proposition 3.13.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -outside GE-derivation. Then the next assertion is valid.*

$$(20) \quad (\forall x \in X)(x\xi \leq x\varrho).$$

Moreover, if  $X$  is transitive, then

$$(21) \quad (\forall x, y \in X)(x\varrho * y\xi \leq x\xi * y\varrho).$$

*Proof.* Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation. Using (GE1), (GE3), (4) and (18), we have

$$\begin{aligned} x\xi * x\varrho &= x\xi * (x\varrho \hat{+} x\xi) = x\xi * ((x\varrho * x\xi) * x\xi) \\ &= x\xi * ((x\varrho * x\xi) * (x\xi * x\xi)) \\ &= x\xi * ((x\varrho * x\xi) * 1) = x\xi * 1 = 1, \end{aligned}$$

that is,  $x\xi \leq x\varrho$  for all  $x \in X$ . Moreover, we suppose that  $X$  is transitive. Applying (12) to (20) leads to the following.

$$x\varrho * y\xi \leq x\xi * y\xi \leq x\xi * y\varrho,$$

and so  $x\varrho * y\xi \leq x\xi * y\varrho$  for all  $x, y \in X$ .  $\square$

If  $X$  is not transitive in Proposition 3.13, then the condition (21) is not valid as seen in the following example.



**Example 3.14.** Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$d$	$d$
$b$	1	$a$	1	$a$	1
$c$	1	1	$b$	1	1
$d$	1	$a$	1	$a$	1

Then  $X$  is a GE-algebra which is not transitive because of

$$(a * b) * ((c * a) * (c * b)) = 1 * (1 * b) = 1 * b = b \neq 1.$$

Define self-maps  $\varrho$  and  $\xi$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, d\}, \\ a & \text{if } x = a, \\ b & \text{if } x = b, \\ c & \text{if } x = c. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ c & \text{if } x \in \{a, c\}, \\ b & \text{if } x \in \{b, d\}. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -outside GE-derivation on  $X$ . Based on the following calculation:

$$(a\varrho * b\xi) * (a\xi * b\varrho) = (a * b) * (c * b) = 1 * b = b \neq 1,$$

we know that the condition (21) is not valid

**Proposition 3.15.** *Every  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  satisfies the condition (20).*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation. The combination of (GE1), (GE3), (4), and (19) leads to the following.

$$\begin{aligned} x\xi * x\varrho &= x\xi * (x\xi \widehat{+} x\varrho) = x\xi * ((x\xi * x\varrho) * x\varrho) \\ &= x\xi * ((x\xi * x\varrho) * (x\xi * x\varrho)) = x\xi * 1 = 1, \end{aligned}$$

that is,  $x\xi \leq x\varrho$  for all  $x \in X$ . Hence (20) is valid.  $\square$

**Proposition 3.16.** *If  $X$  is a transitive GE-algebra, then every  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  satisfies the condition (21).*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation on a transitive GE-algebra  $X$ . Then  $x\xi \leq x\varrho$  for all  $x \in X$  by Proposition 3.15. It follows from (12) that

$$x\varrho * y\xi \leq x\xi * y\xi \leq x\xi * y\varrho$$

for all  $x, y \in X$ . Hence  $x\varrho * y\xi \leq x\xi * y\varrho$  for all  $x, y \in X$  by (13).  $\square$

The following example shows that if  $X$  is not transitive, then Proposition 3.16 is not valid.

**Example 3.17.** Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	1	$d$
$b$	1	1	1	$c$	1
$c$	1	1	$b$	1	$d$
$d$	1	$a$	1	$c$	1

Then  $X$  is a GE-algebra which is not transitive because of

$$(a * c) * ((b * a) * (b * c)) = 1 * (1 * c) = 1 * c = c \neq 1.$$

Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ b & \text{if } x = a, \\ c & \text{if } x = b. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x = a, \\ a & \text{if } x = b, \\ d & \text{if } x = c, \\ c & \text{if } x = d. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$ . Based on the following calculation:

$$(a\varrho * b\xi) * (a\xi * b\varrho) = (b * a) * (b * c) = 1 * c = c \neq 1,$$

we know that the condition (21) is not valid.

Given two self-maps  $\varrho$  and  $\xi$  on  $X$ , consider the next equality:

$$(22) \quad (\forall x, y \in X)((x * y)\varrho = x\xi * y\varrho).$$

**Question 3.18.** If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation, is the equality (22) valid?

The next example shows that the answer to Question 3.18 is negative.

**Example 3.19.** Let  $X = \{1, a, b, c, d, e\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	e	c	d	e
b	1	a	1	d	d	1
c	1	a	1	1	1	1
d	1	a	b	b	1	b
e	1	a	1	d	d	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, d\}, \\ b & \text{if } x \in \{b, c, e\}, \\ a & \text{if } x = a. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, d\}, \\ c & \text{if } x \in \{b, c, e\}, \\ a & \text{if } x = a. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -outside GE-derivation on  $X$ . Based on the following calculation:

$$(a * b)\varrho = e\varrho = b \neq e = a * b = a\xi * b\varrho.$$

we know that the condition (22) is not valid.

Now we provide condition(s) for the answer to Question 3.18 to be positive.

**Theorem 3.20.** *If  $X$  is a commutative GE-algebra, then every  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  satisfies the equality (22).*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -outside GE-derivation on a commutative GE-algebra  $X$  and let  $x, y \in X$ . Then  $X$  is transitive, and so  $x\varrho * y\xi \leq x\xi * y\varrho$  by Proposition 3.13. It follows from (GE2) and (3) that

$$\begin{aligned} (x * y)\varrho &= x\xi * y\varrho \hat{+} x\varrho * y\xi = ((x\xi * y\varrho) * (x\varrho * y\xi)) * (x\varrho * y\xi) \\ &= ((x\varrho * y\xi) * (x\xi * y\varrho)) * (x\xi * y\varrho) \\ &= 1 * (x\xi * y\varrho) = x\xi * y\varrho \end{aligned}$$

Therefore the equality (22) is valid.  $\square$

**Question 3.21.** If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation, is the equality (22) valid?

The next example shows that the answer to Question 3.21 is negative.

**Example 3.22.** Let  $X = \{1, a, b, c, d, e\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	1	c
b	1	a	1	1	a	1
c	1	a	1	1	a	1
d	1	1	1	c	1	c
e	1	a	1	1	d	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c\}, \\ a & \text{if } x \in \{a, d\}, \\ c & \text{if } x = b, \\ b & \text{if } x = e. \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x = b, \\ a & \text{if } x \in \{a, d\}, \\ c & \text{if } x \in \{c, e\}. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$ . Based on the following calculation:

$$(a * b)\varrho = 1\varrho = 1 \neq c = a * c = a\xi * b\varrho.$$

we know that the condition (22) is not valid.

Now we provide condition(s) for the answer to Question 3.21 to be positive. In a commutative GE-algebra,  $\xi$ -inside GE-derivation and  $\xi$ -outside GE-derivation coincide, so we obtain the next result as a corollary of Theorem 3.20.

**Corollary 3.23.** *If  $X$  is a commutative GE-algebra, then every  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  satisfies the equality (22).*

We now have a question: If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation, then are the two self-maps  $\varrho$  and  $\xi$  the same? Namely Is  $x\varrho = x\xi$  for all  $x \in X$ ?

In the next example, we can see the answer to this question is negative.

**Example 3.24.** In Example 3.7(1), we can observe that the  $\xi$ -outside GE-derivation  $\varrho$  does not satisfy  $x\varrho = x\xi$  for all  $x \in X$  since  $b\varrho = 1 \neq a = b\xi$ . Also, the  $\xi$ -inside GE-derivation  $\varrho$  in Example 3.7(2) does not satisfy  $x\varrho = x\xi$  for all  $x \in X$  since  $b\varrho = 1 \neq a = b\xi$ .

We explore the conditions under which the two self-maps  $\varrho$  and  $\xi$  on  $X$  become equal.

We first give an example to show that some  $\xi$ -inside GE-derivations (or some  $\xi$ -outside GE-derivations)  $\varrho : X \rightarrow X$  does not satisfy the next equality:

$$(23) \quad (\forall x, y \in X)((x * y)\varrho = x\varrho * y\xi).$$

**Example 3.25.** In Example 3.7(2), we can observe that  $\varrho$  is a  $\xi$ -inside derivation on  $X$  and it does not satisfy (23) because of  $(c * b)\varrho = b\varrho = 1 \neq a = 1 * a = c\varrho * b\xi$ . Also, the  $\xi$ -outside derivation  $\varrho$  in Example 3.7(1) does not satisfy (23) because of  $(c * b)\varrho = a\varrho = 1 \neq a = c * a = c\varrho * b\xi$ .

**Theorem 3.26.** *If a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  satisfies the condition (23), then  $\varrho$  and  $\xi$  are the same.*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation. If it satisfies the condition (23), then

$$x\varrho = (1 * x)\varrho = 1\varrho * x\xi = 1 * x\xi = x\xi$$

for all  $x \in X$  by (GE2) and Lemma 3.9. Hence  $\varrho$  and  $\xi$  are the same.  $\square$

The following example shows that a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  does not satisfy the next equality.

$$(24) \quad (\forall x, y \in X)(x\xi * y\varrho = x\varrho * y\xi).$$

**Example 3.27.** In Example 3.14, the  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  does not satisfy (24) because of  $a\xi * b\varrho = c * b = b \neq 1 = a * b = a\varrho * b\xi$ . Also, the  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  in Example 3.22 does not satisfy (24) since  $a\xi * b\varrho = a * c = c \neq 1 = a * b = a\varrho * b\xi$ .

**Lemma 3.28.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation. If it satisfies the condition (24), then it also satisfies the condition (23).*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation satisfying the equality (24). Then

$$\begin{aligned} (x * y)\varrho &= x\varrho * y\xi \hat{+} x\xi * y\varrho = ((x\varrho * y\xi) * (x\xi * y\varrho)) * (x\xi * y\varrho) \\ &= ((x\xi * y\varrho) * (x\xi * y\varrho)) * (x\xi * y\varrho) = 1 * (x\xi * y\varrho) \\ &= x\xi * y\varrho = x\varrho * y\xi \end{aligned}$$

for all  $x, y \in X$ . If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation satisfying the equality (24), then

$$\begin{aligned} (x * y)\varrho &= x\xi * y\varrho \hat{+} x\varrho * y\xi = ((x\xi * y\varrho) * (x\varrho * y\xi)) * (x\varrho * y\xi) \\ &= ((x\varrho * y\xi) * (x\varrho * y\xi)) * (x\varrho * y\xi) = 1 * (x\varrho * y\xi) \\ &= x\varrho * y\xi \end{aligned}$$

for all  $x, y \in X$ .  $\square$

**Theorem 3.29.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation that satisfies the equality (24). Then  $\varrho$  and  $\xi$  are the same.*

*Proof.* Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation that satisfies the equality (24). The combination of (GE2), Lemma 3.9, and Lemma 3.28 leads to the following:

$$x\varrho = (1 * x)\varrho = 1\varrho * x\xi = 1 * x\xi = x\xi$$

for all  $x \in X$ . Hence  $\varrho$  and  $\xi$  are the same.  $\square$

**Proposition 3.30.** *Every  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  satisfies:*

$$(25) \quad (\forall x \in X)((x\xi * x\varrho)\varrho = 1).$$

*Proof.* If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation, then  $x\varrho = x\xi \hat{+} x\varrho$  for all  $x \in X$  by (19), and so

$$\begin{aligned} (x\xi * x\varrho)\varrho &= (x\xi * (x\xi \hat{+} x\varrho))\varrho = (x\xi * ((x\xi * x\varrho) * x\varrho))\varrho \\ &= (x\xi * ((x\xi * x\varrho) * (x\xi * x\varrho)))\varrho \\ &= (x\xi * 1)\varrho = 1\varrho = 1 \end{aligned}$$

for all  $x \in X$ . If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation, then  $x\varrho = x\varrho \hat{+} x\xi$  for all  $x \in X$  by (18), and so

$$\begin{aligned} (x\xi * x\varrho)\varrho &= (x\xi * (x\varrho \hat{+} x\xi))\varrho = (x\xi * ((x\varrho * x\xi) * x\xi))\varrho \\ &= (x\xi * ((x\xi * x\varrho) * (x\xi * x\xi)))\varrho \\ &= (x\xi * ((x\xi * x\varrho) * 1))\varrho \\ &= (x\xi * 1)\varrho = 1\varrho = 1 \end{aligned}$$

for all  $x \in X$ .  $\square$

**Question 3.31.** If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation, then the following assertions are valid or not?

(26)  $\varrho$  is order preserving, i.e.,  $(\forall x, y \in X)(x \leq y \Rightarrow x\varrho \leq y\varrho)$ .

(27)  $(\forall x, y \in X)(x\varrho \hat{+} y\varrho \leq (x \hat{+} y)\varrho)$ .

The following example shows that there exists a  $\xi$ -inside GE-derivation  $\varrho : X \rightarrow X$  such that the conditions (26) and (27) are not valid.

**Example 3.32.** Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$c$	1
$b$	1	$a$	1	1	$a$
$c$	1	$a$	1	1	$a$
$d$	1	1	1	$c$	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c\}, \\ a & \text{if } x \in \{a, d\} \\ c & \text{if } x = b, \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, d\}, \\ b & \text{if } x = b, \\ c & \text{if } x = c. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$ . Based on the following calculations:

$$a * b = 1 \text{ and } a\varrho * b\varrho = a * c = c \neq 1$$

and

$$\begin{aligned} (a\varrho \hat{+} b\varrho) * ((a \hat{+} b)\varrho) &= ((a\varrho * b\varrho) * b\varrho) * ((a * b) * b)\varrho \\ &= ((a * c) * c) * (1 * b)\varrho \\ &= (c * c) * b\varrho \\ &= 1 * b\varrho = 1 * c = c \neq 1, \end{aligned}$$

we know that the conditions (26) and (27) are not valid.

**Theorem 3.33.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation that satisfies the condition (27). If  $X$  is transitive, then  $\varrho$  is order preserving.*

*Proof.* Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation that satisfies the condition (27) and suppose that  $X$  is transitive. Let  $x, y \in X$  be such that  $x \leq y$ . Then  $x \hat{+} y = (x * y) * y = 1 * y = y$ , which implies from (9) and (27) that

$$x\varrho \leq x\varrho \hat{+} y\varrho \leq (x \hat{+} y)\varrho = y\varrho$$

Hence  $x\varrho \leq y\varrho$  for all  $x, y \in X$  with  $x \leq y$  by (14). Therefore  $\varrho$  is order preserving.  $\square$

The following example shows that there exists a  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  in which the conditions (26) and (27) are not established.



**Example 3.34.** Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	1	$d$
$b$	1	1	1	1	$d$
$c$	1	$a$	$b$	1	1
$d$	1	$a$	$b$	1	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, d\}, \\ a & \text{if } x = b, \\ c & \text{if } x = c, \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -outside GE-derivation on  $X$ . We know that the conditions (26) and (27) are not valid by the following calculations:

$$a * b = 1 \text{ and } a\varrho * b\varrho = 1 * a = a \neq 1$$

and

$$\begin{aligned} (a\varrho \hat{+} b\varrho) * ((a \hat{+} b)\varrho) &= ((a\varrho * b\varrho) * b\varrho) * ((a * b) * b)\varrho \\ &= ((1 * a) * a) * (1 * b)\varrho \\ &= (a * a) * b\varrho \\ &= 1 * b\varrho = 1 * a = a \neq 1. \end{aligned}$$

By the same way to the proof of Theorem 3.33, we have the following theorem.

**Theorem 3.35.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -outside GE-derivation that satisfies the condition (27). If  $X$  is transitive, then  $\varrho$  is order preserving.*

4. GE-SUBALGEBRAS AND GE-FILTERS WHICH ARE RELATED TO  $\xi$ -INSIDE (OR  $\xi$ -OUTSIDE)  
GE-DERIVATION

Let  $\varrho : X \rightarrow X$  be a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation. We define two sets:

$$(28) \quad X_{(\varrho=\xi)} := \{x \in X \mid x\varrho = x\xi\},$$

$$(29) \quad \ker(\varrho) := \{x \in X \mid x\varrho = 1\}.$$

**Theorem 4.1.** *If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation, then the set  $X_{(\varrho=\xi)}$  and  $\ker(\varrho)$  are GE-subalgebras of  $X$  containing the constant 1.*

*Proof.* It is clear that  $1 \in X_{(\varrho=\xi)} \cap \ker(\varrho)$ . Let  $x, y \in X_{(\varrho=\xi)}$ . Then  $x\varrho = x\xi$  and  $y\varrho = y\xi$ . Hence

$$\begin{aligned} (x * y)\varrho &= x\varrho * y\xi \hat{+} x\xi * y\varrho \\ &= ((x\varrho * y\xi) * (x\xi * y\varrho)) * (x\xi * y\varrho) \\ &= ((x\xi * y\varrho) * (x\xi * y\varrho)) * (x\xi * y\xi) \\ &= 1 * (x\xi * y\xi) \\ &= x\xi * y\xi = (x * y)\xi. \end{aligned}$$

Let  $x, y \in \ker(\varrho)$ . Then  $x\varrho = 1$ , and  $y\varrho = 1$ . Hence

$$\begin{aligned} (x * y)\varrho &= x\varrho * y\xi \hat{+} x\xi * y\varrho \\ &= ((x\varrho * y\xi) * (x\xi * y\varrho)) * (x\xi * y\varrho) \\ &= ((1 * y\xi) * (x\xi * 1)) * (x\xi * 1) \\ &= (y\xi * 1) * 1 = 1. \end{aligned}$$

Hence  $x * y \in X_{(\varrho=\xi)} \cap \ker(\varrho)$ . Thus  $X_{(\varrho=\xi)}$  and  $\ker(\varrho)$  are GE-subalgebras of  $X$  containing the constant 1.  $\square$

**Proposition 4.2.** *If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation or a  $\xi$ -inside GE-derivation, then the set  $X_{(\varrho=\xi)}$  is closed under the operation “ $\hat{+}$ ”.*

*Proof.* Let  $x, y \in X_{(\varrho=\xi)}$ . Then  $x\varrho = x\xi$  and  $y\varrho = y\xi$ . If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation, then

$$\begin{aligned} (x \hat{+} y)\varrho &= ((x * y) * y)\varrho = (x * y)\xi * y\varrho \hat{+} (x * y)\varrho * y\xi \\ &= (x * y)\xi * y\varrho \hat{+} (x\xi * y\varrho \hat{+} x\varrho * y\xi) * y\xi \\ &= (x * y)\xi * y\xi \hat{+} (x\xi * y\xi \hat{+} x\xi * y\xi) * y\xi \\ &= (x * y)\xi * y\xi \hat{+} (x\xi * y\xi) * y\xi \\ &= (x * y)\xi * y\xi \hat{+} (x * y)\xi * y\xi \\ &= (x * y)\xi * y\xi = ((x * y) * y)\xi \\ &= (x \hat{+} y)\xi, \end{aligned}$$

and thus  $x \hat{+} y \in X_{(\varrho=\xi)}$ . Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation. Then

$$\begin{aligned} (x \hat{+} y)\varrho &= ((x * y) * y)\varrho = (x * y)\varrho * y\xi \hat{+} (x * y)\xi * y\varrho \\ &= ((x\varrho * y\xi) \hat{+} (x\xi * y\varrho)) * y\xi \hat{+} (x * y)\xi * y\varrho \\ &= ((x\xi * y\xi) \hat{+} (x\xi * y\xi)) * y\xi \hat{+} (x * y)\xi * y\varrho \\ &= (x\xi * y\xi) * y\xi \hat{+} (x * y)\xi * y\xi \\ &= (x * y)\xi * y\xi \hat{+} (x * y)\xi * y\xi \\ &= (x * y)\xi * y\xi = ((x * y) * y)\xi \\ &= (x \hat{+} y)\xi, \end{aligned}$$

and thus  $x \hat{+} y \in X_{(\varrho=\xi)}$ .  $\square$

**Proposition 4.3.** *If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation, then the set  $\ker(\varrho)$  satisfies:*

$$(30) \quad (\forall x, y \in X)(x \in \ker(\varrho) \Rightarrow y * x \in \ker(\varrho)).$$

*Proof.* Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation and let  $x, y \in X$ . If  $x \in \ker(\varrho)$ , then  $x\varrho = 1$ , and so

$$(y * x)\varrho = y\xi * x\varrho \hat{+} y\varrho * x\xi = y\xi * 1 \hat{+} y\varrho * x\xi = 1 \hat{+} y\varrho * x\xi = 1,$$

that is,  $y * x \in \ker(\varrho)$ . Suppose that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation and let  $x \in \ker(\varrho)$  for every  $x \in X$ . Then  $x\varrho = 1$ , and thus

$$(y * x)\varrho = y\varrho * x\xi \hat{+} y\xi * x\varrho = y\varrho * x\xi \hat{+} y\xi * 1 = y\varrho * x\xi \hat{+} 1 = 1$$

for all  $y \in X$ . Hence  $y * x \in \ker(\varrho)$ .  $\square$

**Proposition 4.4.** *If  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation, then the set  $\ker(\varrho)$  satisfies:*

$$(31) \quad (\forall x, y \in X)(x \in \ker(\varrho) \Rightarrow y \hat{+} x \in \ker(\varrho)).$$

*Proof.* For every  $x, y \in X$ , let  $x \in \ker(\varrho)$ . Then  $x\varrho = 1$ . Assume that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation. Then

$$\begin{aligned} (y \hat{+} x)\varrho &= ((y * x) * x)\varrho = (y * x)\varrho * x\xi \hat{+} (y * x)\xi * x\varrho \\ &= (y\varrho * x\xi \hat{+} y\xi * x\varrho) * x\xi \hat{+} (y * x)\xi * x\varrho \\ &= (y\varrho * x\xi \hat{+} y\xi * 1) * x\xi \hat{+} (y * x)\xi * 1 \\ &= (y\varrho * x\xi \hat{+} 1) * x\xi \hat{+} (y * x)\xi * 1 \\ &= 1, \end{aligned}$$

and so  $y \hat{+} x \in \ker(\varrho)$ . If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation, then

$$\begin{aligned} (y \hat{+} x)\varrho &= ((y * x) * x)\varrho = (y * x)\xi * x\varrho \hat{+} (y * x)\varrho * x\xi \\ &= (y * x)\xi * x\varrho \hat{+} (y\xi * x\varrho \hat{+} y\varrho * x\xi) * x\xi \\ &= (y * x)\xi * 1 \hat{+} (y\xi * 1 \hat{+} y\varrho * x\xi) * x\xi \\ &= 1 \hat{+} (1 \hat{+} y\varrho * x\xi) * x\xi \\ &= 1. \end{aligned}$$

Hence  $y \hat{+} x \in \ker(\varrho)$ .  $\square$

Given a self-map  $\varrho : X \rightarrow X$ , consider the following condition:

$$(32) \quad (\forall x, y \in X)(x \leq y, x \in \ker(\varrho) \Rightarrow y \in \ker(\varrho)).$$

The following example shows that there exists a  $\xi$ -inside GE-derivation or a  $\xi$ -outside GE-derivation  $\varrho : X \rightarrow X$  which does not satisfy the condition (32).

**Example 4.5.** (1) Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$d$	$d$
$b$	1	1	1	$c$	$c$
$c$	1	$a$	$b$	1	1
$d$	1	$a$	$b$	1	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a\}, \\ a & \text{if } x = b, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ d & \text{if } x \in \{c, d\}. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -outside GE-derivation on  $X$  and  $\ker(\varrho) = \{1, a\}$ .

We can observe that

$$a \leq b \text{ and } a \in \ker(\varrho) \text{ since } a * b = 1 \text{ and } a\varrho = 1.$$

But  $b \notin \ker(\varrho)$  since  $b\varrho = a \neq 1$ . Thus (32) is not valid.

(2) Let  $X = \{1, a, b, c, d\}$  be a set with the binary operation “ $*$ ” given in the following table:

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$c$	$d$
$b$	1	1	1	$c$	$d$
$c$	1	1	1	1	$d$
$d$	1	1	1	$c$	1

Then  $X$  is a GE-algebra. Define self-maps  $\varrho$  and  $\xi$  on  $X$  as follows:

$$\varrho : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, c\}, \\ a & \text{if } x = b, \\ d & \text{if } x = d, \end{cases}$$

and

$$\xi : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d. \end{cases}$$

Then  $\xi$  is a GE-endomorphism and  $\varrho$  is a  $\xi$ -inside GE-derivation on  $X$  and  $\ker(\varrho) = \{1, a, c\}$ .

We can observe that

$$a \leq b \text{ and } a \in \ker(\varrho) \text{ since } a * b = 1 \text{ and } a\varrho = 1.$$

But  $b \notin \ker(\varrho)$  since  $b\varrho = a \neq 1$ . Thus (32) is not valid.

**Proposition 4.6.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -outside GE-derivation or a  $\xi$ -inside GE-derivation. If  $X$  is commutative, then  $\varrho$  satisfies the condition (32).*

*Proof.* Assume that  $X$  is commutative and let  $x, y \in X$  be such that  $x \leq y$  and  $x \in \ker(\varrho)$ . Then  $x * y = 1$  and  $x\varrho = 1$ . Suppose that  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation. Then

$$\begin{aligned} y\varrho &= (1 * y)\varrho = ((x * y) * y)\varrho = ((y * x) * x)\varrho \\ &= (y * x)\xi * x\varrho \hat{+} (y * x)\varrho * x\xi \\ &= (y * x)\xi * x\varrho \hat{+} (y\xi * x\varrho \hat{+} y\varrho * x\xi) * x\xi \\ &= (y * x)\xi * 1 \hat{+} (y\xi * 1 \hat{+} y\varrho * x\xi) * x\xi \\ &= 1 \hat{+} (1 \hat{+} y\varrho * x\xi) * x\xi = 1 \end{aligned}$$

by (GE2), (3), (4) and (15). Hence  $y \in \ker(\varrho)$ . Suppose that  $\varrho : X \rightarrow X$  is a  $\xi$ -inside GE-derivation. Then

$$\begin{aligned} y\varrho &= (1 * y)\varrho = ((x * y) * y)\varrho = ((y * x) * x)\varrho \\ &= (y * x)\varrho * x\xi \hat{+} (y * x)\xi * x\varrho \\ &= (y\varrho * x\xi \hat{+} y\xi * x\varrho) * x\xi \hat{+} (y * x)\xi * x\varrho \\ &= (y\varrho * x\xi \hat{+} y\xi * 1) * x\xi \hat{+} (y * x)\xi * 1 \\ &= (y\varrho * x\xi \hat{+} 1) * x\xi \hat{+} (y * x)\xi * 1 \\ &= 1 * x\xi \hat{+} 1 = 1, \end{aligned}$$

and so  $y \in \ker(\varrho)$ .  $\square$

**Question 4.7.** If  $\varrho : X \rightarrow X$  is a  $\xi$ -outside GE-derivation or a  $\xi$ -inside GE-derivation, is the set  $\ker(\varrho)$  a GE-filter of  $X$ ?

The example below shows that the answer to Question 4.7 is negative.

**Example 4.8.** In Example 4.5(1),  $\varrho$  is a  $\xi$ -outside GE derivation and  $\ker(\varrho) = \{1, a\}$ . We can observe that

$$a * b = 1 \in \ker(\varrho) \text{ and } a \in \ker(\varrho)$$

But  $b \notin \ker(\varrho)$  since  $b\varrho = a \neq 1$ . Hence  $\ker(\varrho)$  is not a GE-filter of  $X$ . Also, in Example 4.5(2),  $\varrho$  is a  $\xi$ -inside GE derivation and  $\ker(\varrho) = \{1, a, c\}$ . We can observe that

$$a * b = 1 \in \ker(\varrho) \text{ and } a \in \ker(\varrho)$$

But  $b \notin \ker(\varrho)$  since  $b\varrho = a \neq 1$ . Hence  $\ker(\varrho)$  is not a GE-filter of  $X$ .

We provide the conditions under which the answer to Question 4.7 can be positive.

**Theorem 4.9.** *Let  $\varrho : X \rightarrow X$  be a  $\xi$ -outside GE-derivation or a  $\xi$ -inside GE-derivation. If  $\varrho$  is a GE-endomorphism, then the set  $\ker(\varrho)$  is a GE-filter of  $X$ .*

*Proof.* Assume that  $\varrho$  is a GE-endomorphism. It is clear that  $1 \in \ker(\varrho)$ . Let  $x, y \in X$  be such that  $x \in \ker(\varrho)$  and  $x * y \in \ker(\varrho)$ . Then  $x\varrho = 1$  and  $(x * y)\varrho = 1$ . Hence

$$1 = (x * y)\varrho = x\varrho * y\varrho = 1 * y\varrho = y\varrho,$$

and so  $y \in \ker(\varrho)$ . Therefore  $\ker(\varrho)$  is a GE-filter of  $X$ .  $\square$

## 5. CONCLUSION

A derivation is a function on an algebra which generalizes certain features of the derivative operator. It belongs to one important field in the study of algebraic structures and is being studied by several mathematicians in various algebraic structures. In this paper, we have introduced the notions of  $\xi$ -inside GE-derivation and  $\xi$ -outside GE-derivation and have studied several properties. We have provided conditions for a self-map on GE-algebra to be a  $\xi$ -inside GE-derivation and a  $\xi$ -outside GE-derivation, and have explored conditions under which the two self-maps  $\varrho$  and  $\xi$  on GE-algebra become equal. We have given conditions for a  $\xi$ -inside GE-derivation and a  $\xi$ -outside GE-derivation to be order preserving. We have made two sets  $X_{(\varrho=\xi)}$  and  $\ker(\varrho)$  using the  $\xi$ -inside GE-derivation or the  $\xi$ -outside GE-derivation, have studied their properties, and also have found the conditions under which they become GE-subalgebra and/or GE-filter. Based on the results in this paper, we will seek more generalized derivation in GE-algebra in the future. In addition, we will apply the ideas and results in this paper to other forms of logical algebra to conduct research.

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**Young Bae Jun**

Department of Mathematics Education,  
 Gyeongsang National University,  
 Jinju 52828, Korea.  
 skywine@gmail.com



**Ravikumar Bandaru**

Department of Mathematics,  
GITAM (Deemed to be University),  
Hyderabad Campus,  
Telangana-502329, India.  
ravimaths83@gmail.com