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Research Paper

ω-FILTERS OF DISTRIBUTIVE LATTICES

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Abstract. The notion of ω-filters is introduced in distributive lattices and their properties are studied. A set of equivalent conditions is derived for every maximal filter of a distributive lattice to become an ω-filter which leads to a characterization of quasi-complemented lattices. Some sufficient conditions are derived for proper D-filters of a distributive lattice to become an ω-filter. Finally, ω-filters of a distributive lattice are characterized with the help of minimal prime D-filters.

1. Introduction

Many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [11] and W.H. Cornish [5] made an extensive study of annihilators in distributive lattices. In [4], some properties of minimal prime filters are studied in distributive lattices and the properties of dense elements and D-filters are studied in MS-algebras [9]. In [2], the notion of D-filters

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was introduced in pseudo-complemented semilattices. Later it was generalized by the author [9] in MS-algebras. In [10], the authors investigated certain important properties of prime D-filters of distributive lattices. In this paper, some properties of minimal prime D-filters of distributive lattices are derived with respect to congruences.

The main aim of this paper is to introduce the notion of ω-filters and to study certain properties of these filters with the help of maximal filters and minimal prime D-filters of distributive lattices. These of ω-filters are the dual of O-ideals of distributive lattices [6] whenever $D = \{1\}$. In this paper, we initially observe the properties of prime D-filters in quasi-complemented lattices and Boolean algebras in order to elevate their properties in distributive lattices. A necessary and sufficient condition is derived for a proper ω-filter of a distributive lattice to become a prime ω-filter. A set of equivalent conditions is derived for every maximal filter of a distributive lattice to become an ω-filter which leads to a characterization of quasi-complemented lattice. Some equivalent conditions are derived for the class of all ω-filters to become a sublattice of the lattice of all filters of a distributive lattices. Finally, the role of minimal prime D-filters is observed in the characterization of ω-filters of distributive lattices.

2. Preliminaries

The reader is referred to [1] and [3] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results of [9] and [10] are presented for the ready reference of the reader.

Definition 2.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal (filter) M of a lattice is called *maximal* if there exists no proper ideal(filter) N such that $M \subset N$.

The set $(a] = \{x \in L \mid x \leq a\}$ is called a *principal ideal* generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually the set $[a) = \{x \in L \mid a \leq x\}$ is called a *principal filter* generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called *prime* if for all $a, b \in L$, $a \wedge b \in P$ ($a \vee b \in P$) then $a \in P$ or $b \in P$. Every maximal(ideal) filter is prime. For any element a of a distributive lattice L, the *annihilator* of a is defined as the set $(a)^* = \{\, x \in L \mid x \wedge a = 0 \,\}$. An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a lattice L forms a filter of L. A distributive lattice L with 0 is called *quasi-complemented* [6] if to each $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$.

Definition 2.2. [10] A filter F of a lattice L is called a D-filter if $D \subseteq F$.

The set D of all dense elements of a distributive lattice is the smallest D-filter of the lattice. For any subset A of a distributive lattice L, define $A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}$. Clearly $L^\circ = D$ and $D^\circ = L$. It can also be observed that $D \subseteq A^\circ$ for any subset A of a lattice L. For any $a \in L$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$. Then it is obvious that $(1)^\circ = L$. For any subset A of L, A° is a D-filter of L. It is clear that $((x])^\circ = (x)^\circ$. Then clearly $(0)^\circ = D$.

Proposition 2.3. [10] *Let L be a distributive lattice. For any $a, b, c \in L$, we have*

(1) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$,

(2) $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,

(3) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,

(4) $(a)^\circ = L$ *if and only if* $a \in D$.

Let F be a D-filter and P be a prime D-filter of a lattice L such that $F \subseteq P$. Then P is called a *minimal prime D-filter belonging to F* if there is no prime D-filter Q such that $F \subseteq Q \subset P$. A prime D-filter belonging to D is simply called *minimal prime D-filter*. A prime D-filter P of a lattice L is minimal prime D-filter belonging to F [10] if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in F$. Throughout this article, all lattices are bounded distributive lattices unless otherwise mentioned.

3. ω-filters of lattices

In this section, we initially observe certain properties of prime D-filters. The notion of ω-filters is introduced in a distributive lattice. A characterization theorem of quasi-complemented lattices is derived. Finally, the class of all ω-filters are characterized in terms of minimal prime D-filters.

Proposition 3.1. *Let P be a prime filter of a quasi-complemented lattice L. Then the following assertions are equivalent:*

(1) *$D \subseteq P$;*

(2) *for any $x \in L$, $x \in P$ if and only if $(x)^\circ \nsubseteq P$;*

(3) *for any $x, y \in L$ with $(x)^\circ = (y)^\circ$, $x \in P$ implies that $y \in P$;*

(4) *$D \cap (L - P) = \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Assume that $D \subseteq P$. Suppose $x \in P$. Since L is quasi-complemented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. Hence $x' \in (x)^\circ$. Suppose $x' \in P$. Then $0 = x \wedge x' \in P$, which is a contradiction. Hence $x' \notin P$. Therefore $(x)^\circ \nsubseteq P$. Conversely, suppose that $(x)^\circ \nsubseteq P$. Then there exists $x' \in L$ such that $x' \in (x)^\circ$ and $x' \notin P$. Clearly $x \vee x' \in D \subseteq P$. Since P is prime and $x' \notin P$, we must have $x \in P$.
(2) $\Rightarrow$ (3): Assume the condition (2). Let $x, y \in L$ such that $(x)^\circ = (y)^\circ$. Suppose $x \in P$. Then by (2), we get $y \in P$.
(3) $\Rightarrow$ (4): Assume that condition (3) holds. Let $x \in L$. Suppose $x \in D \cap (L - P)$. Then $(x)^\circ = L$ and $x \notin P$. Hence $(x)^\circ = L = (d)^\circ$ for any $d \in D \subseteq P$. Since $d \in P$, by (3), we get $x \in P$ which is a contradiction. Therefore $D \cap (L - P) = \emptyset$.
(4) $\Rightarrow$ (1): It is obvious. □

Theorem 3.2. *Let L be a quasi-complemented lattice and $x \in L$. If x' is the quasi-complement of x, then every prime D-filter must contain exactly one of x or x'.*

Proof. Let P be a prime D-filter of L. Clearly $x \vee x' \in D \subseteq P$. Since P is prime, we get $x \in P$ or $x' \in P$. Suppose both x and x' are in P. Then $0 = x \wedge x' \in P$, which is a contradiction. Therefore P must contain exctly one of x or x'. □

In the following theorem, a set of equivalent conditions is derived, with the help of prime filters, for a quasi-complemented lattice to become a Boolean algebra.

Proposition 3.3. *Let L be a quasi-complemented lattice and $x \in L$. Then the following assertions are equivalent:*

(1) *L is a Boolean algebra;*

(2) *every prime filter contains exactly one of x or x';*

(3) *every prime filter is a D-filter;*

(4) *every minimal prime filter is a D-filter;*

where x' is the quasi-complement of x in L.

Proof. (1) $\Rightarrow$ (2): Assume that L is Boolean. Let P be a prime filter of L and $x \in P$. Since L is Boolean, we get $D = \{1\}$. Hence $x \wedge x' = 0$ and $x \vee x' = 1 \in P$. Since P is prime, we get $x \in P$ or $x' \in P$. If both x and x' are in P, then $0 = x \wedge x' \in P$ which is a contradiction. Therefore P must contain exactly one of x or x'.

(2) $\Rightarrow$ (3): Assume the condition (2). Let P be a prime filter of L. Let $x \in D$. Since L is quasi-complemented, we get that $x' \in (x)^* = \{0\}$. Hence $x' = 0 \notin P$. By the condition (2), we get $x \in P$. Thus $D \subseteq P$. Therefore P is a D-filter of L.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Assume that condition (4) holds. Let $x \in L$. Suppose $x \vee x' \neq 1$. Then there exists a maximal ideal M of L such that $x \vee x' \in M$. Then $L - M$ is a minimal prime filter such that $x \vee x' \notin M$. Hence $x \notin M$ and $x' \notin M$. By the hypothesis, we get $D \subseteq L - M$. Since L is quasi-complemented, by Theorem 3.2, $L - M$ must contain exactly one of x or x', which is a contradiction. Hence $x \vee x' = 1$. Therefore L is a Boolean algebra. □

Theorem 3.4. *Let M be a proper filter of a quasi-complemented lattice L. Then M is maximal if and only if M is prime D-filter.*

Proof. Assume that M is a maximal filter of L. Clearly M is prime. Let $x \in D$. Then $(x)^* = \{0\}$. Suppose $x \notin M$. Then $M \vee [x) = L$. Hence $0 = m \wedge x$ for some $0 \neq m \in M$. Then $m \in (x)^* = \{0\}$, which is a contradiction. Hence $x \in M$. Thus $D \subseteq M$. Therefore M is a prime D-filter of L.

Conversely, assume that M is a prime D-filter of L. Suppose M is not maximal. Let Q be a proper filter of L such that $M \subset Q$. Choose $x \in Q - M$. Since L is quasi-complemented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D \subseteq M$. Since M is prime and $x \notin M$, we get $x' \in M \subset Q$. Hence $0 = x \wedge x' \in Q$, which is a contradiction. Therefore M is maximal. □

From Theorem 3.4, one can notice that the class of all maximal filters and the class of all prime D-filters of a quasi-complemented lattice are the same. Since every prime D-filter is maximal, we can conclude that every prime D-filter is minimal in a quasi-complemented lattice. Therefore maximal filters, prime D-filter, and minimal prime D-filters are the same in a quasi-complemented lattice.

Definition 3.5. Let I be an ideal of a lattice L. Then define $\omega(I) = \{x \in L \mid x \vee a \in D \text{ for some } a \in I\}$. In other words, $\omega(I) = \bigcup_{x \in I} (x)^\circ$.

Proposition 3.6. *For any ideal I of a lattice L, the set $\omega(I)$ is a D-filter in L.*

Proof. Clearly $D \subseteq \omega(I)$. Let $x, y \in \omega(I)$. Then $x \vee a \in D$ and $y \vee b \in D$ for some $a, b \in I$. Now $(x \wedge y) \vee (a \vee b) = (x \vee a \vee b) \wedge (y \vee a \vee b) \in D$ and $a \vee b \in I$. Hence $x \wedge y \in \omega(I)$. Again, let $x \in \omega(I)$ and $x \leq y$. Then $x \vee a \in D$ for some $a \in I$. Since $x \vee a \leq y \vee a$, we get $y \vee a \in D$. Thus $y \in \omega(I)$. Therefore $\omega(I)$ is a D-filter of L. □

Lemma 3.7. *For any two ideals I, J of a lattice L, the following properties hold:*

(1) $I \cap \omega(I) \neq \emptyset$ *if and only if* $\omega(I) = L$,

(2) $I \subseteq J$ *implies* $\omega(I) \subseteq \omega(J)$,

(3) $\omega(I) \cap \omega(J) = \omega(I \cap J)$

Proof. (1) : Suppose $I \cap \omega(I) \neq \emptyset$. Choose $x \in I \cap \omega(I)$. Then $x \in I$ and $x \vee a \in D$ for some $a \in I$. By Proposition 2.3(4), we get $(x \vee a)^\circ = L$. Since $x \in I$ and $a \in I$, we get $x \vee a \in I$. Therefore $\omega(I) = \bigcup_{x \in I} (x)^\circ = L$. Conversely, assume that $\omega(I) = L$. Then $0 \in \omega(I)$. Hence $0 \in I \cap \omega(I)$, which means that $I \cap \omega(I) \neq \emptyset$.

(2) : Suppose $I \subseteq J$. Let $x \in \omega(I)$. Then $x \vee a \in D$ for some $a \in I \subseteq J$. Hence $x \in \omega(J)$. Therefore $\omega(I) \subseteq \omega(J)$.

(3) : Clearly $\omega(I \cap J) \subseteq \omega(I) \cap \omega(J)$. Conversely, let $x \in \omega(I) \cap \omega(J)$. Then $x \vee a \in D$ and $x \vee b \in D$ for some $a \in I$ and $b \in J$. Then $a \wedge b \in I \cap J$. Hence $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \in D$. Hence $x \in \omega(I \cap J)$. Therefore $\omega(I) \cap \omega(J) \subseteq \omega(I \cap J)$. □

Proposition 3.8. *Let I, J be two ideals of a lattice L such that $\omega(I) \cap J = \emptyset$, then there exists a prime D-filter P such that $\omega(I) \subseteq P$ and $P \cap J = \emptyset$.*

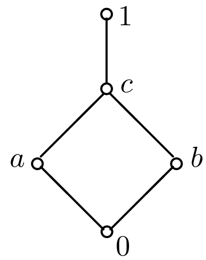
Proof. Let I and J be as mentioned in the hypothesis. Then there exists a maximal ideal Q such that $J \subseteq Q$ and $\omega(I) \cap Q = \emptyset$. Since Q is a prime ideal, we get that $L - Q$ is a prime filter. Since $\omega(I) \cap Q = \emptyset$, we get that $D \subseteq \omega(I) \subseteq L - Q$. Therefore $L - Q = P$ is a prime D-filter of L that is containing $\omega(I)$. □

We now introduce the concept of ω-filters of lattices.

Definition 3.9. Let F be a D-filter of a lattice L. Then F is called an ω-filter of L if $F = \omega(I)$ for some ideal I of L such that $I \cap D = \emptyset$.

From the above definition, it is an easy consequence that $\omega(\{0\}) = D$. Hence D is proper and the smallest ω-filter of the lattice L.

Example 3.10. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure:



Consider the filters $F_1 = \{a, c, 1\}, F_2 = \{b, c, 1\}$ and the ideals $I_1 = \{0, b\}, I_2 = \{0, a\}$. It is easily seen that $F_1 = \omega(I_1)$ and $F_2 = \omega(I_2)$ such that $I_1 \cap D = I_2 \cap D = \emptyset$. Therefore F_1 and F_2 are ω-filters of L.

Proposition 3.11. *Let L be a lattice and $x \in L - D$. Then $(x)^\circ$ is an ω-filter of L.*

Proof. Let $x \in L - D$. Clearly $(x] \cap D = \emptyset$. Let $t \in (x)^\circ$. Then $t \vee x \in D$. Since $x \in (x]$, we get $t \in \omega((x])$. Therefore $(x)^\circ \subseteq \omega((x])$. Conversely, let $t \in \omega((x])$. Then $t \vee a \in D$ for some $a \in (x]$. Hence $t \vee x \in D$, which implies that $t \in (x)^\circ$. Thus $\omega((x]) \subseteq (x)^\circ$. Hence $(x)^\circ = \omega((x])$. Therefore $(x)^\circ$ is an ω-filter of L. □

Theorem 3.12. *Every prime D-filter P of a lattice with $P^\circ \neq D$ is an ω-filter.*

Proof. Let P be a prime D-filter of a lattice L. Suppose $P^\circ \neq D$. Since $D \subset P^\circ$, there exists $x \notin D$ such that $x \in P^\circ$. Clearly $(x] \cap D = \emptyset$ and $x \notin P$. Then $P \subseteq P^{\circ\circ} \subseteq (x)^\circ$. On the other hand, let $a \in (x)^\circ$. Then $a \vee x \in D \subseteq P$. Since $x \notin P$, we must have $a \in P$. Thus $(x)^\circ \subseteq P$. Hence $P = (x)^\circ = \omega((x])$. Therefore P is an ω-filter of L. □

Theorem 3.13. *Every minimal prime D-filter of a lattice is an ω-filter.*

Proof. Let P be a minimal prime D-filter of a lattice L. Then $L - P$ is a prime ideal of L such that $D \cap (L - P) = \emptyset$. We now show that $P = \omega(L - P)$. Let $x \in P$. Since P is minimal, there exists $y \in L - P$ such that $x \vee y \in D$. Hence $x \in \omega(L - P)$. Therefore $P \subseteq \omega(L - P)$. Conversely, let $x \in \omega(L - P)$. Then, we get $x \vee a \in D \subseteq P$ for some $a \in L - P$. Since P is prime and $a \notin P$, we get $x \in P$. Thus $\omega(L - P) \subseteq P$. Hence $P = \omega(L - P)$. Therefore P is an ω-filter of L. □

We now turn our intension towards the converse of the above theorem. In general, every ω-filter of a lattice need not be a minimal prime D-filter. In fact it need not even be a prime D-filter. It can be observed in the following example:

Example 3.14. Let $X = \{1,2,3,4\}$ be a set and L the sublattice of the power set of X, which is generated by the sets $\{1\},\{2\}$ and $\{3\}$. That is, $L = \big\{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},$ $\{2,3\},\ \{1,3\},\ \{1,2,3\}\ \big\}$. Clearly $D = \{\{1,2,3\}\}$. Consider $F = \big\{\{1,2\},\{1,2,3\}\big\}$ and $I = \big\{\emptyset,\{2\},\{3\},\{2,3\}\big\}$. Clearly F is a filter and I is an ideal of L such that $I \cap D = \emptyset$. It can be easily verified that $\omega(I) = \big\{\{1,2\},\{1,2,3\}\big\} = F$. Therefore F is an ω-filter of L. Note that the filter F is not a prime D-filter of L, because $\{2,3\} \notin F$ and $\{1,3\} \notin F$ but $\{2,3\} \vee \{1,3\}\ = \{1,2,3\} \in F$.

Though every ω-filter need not be a prime D-filter, we derive a necessary and sufficient condition for an ω-filter of a lattice to become a prime D-filter.

Theorem 3.15. *Let F be a proper ω-filter of a lattice L. Then F is a prime D-filter if and only if F contains a prime D-filter.*

Proof. The necessary part is clear. For sufficiency, assume that F contains a prime D-filter, say P. Since $D \subseteq P \subseteq F$, F is a D-filter of L. Since F is an ω-filter, we get $F = \omega(I)$ for some ideal I of L with $I \cap D = \emptyset$. Choose $a, b \in L$ such that $a \notin F$ and $b \notin F$. Since $P \subseteq F$, we get $a \notin P$ and $b \notin P$. Since P is prime, we get $a \vee b \notin P$. Thus $(a \vee b)^{\circ} \subseteq P \subseteq F = \omega(I)$. Suppose $a \vee b \in F = \omega(I)$. Then $a \vee b \vee i \in D$ for some $i \in I$. Hence $i \in (a \vee b)^{\circ} \subseteq \omega(I)$. Thus $i \in I \cap \omega(I)$. Hence $I \cap \omega(I) \neq \emptyset$. By Lemma 3.7(1), $F = \omega(I) = L$ which is a contradiction. Thus F is a prime D-filter of L. □

We have already observed in Theorem 3.15 that every minimal prime D-filter is a prime ω-filter of L. Now we derive, in the following theorem, the equivalency between prime ω-filters and minimal prime D-filters of a lattices.

Theorem 3.16. *Every prime ω-filter of a lattice L is a minimal prime D-filter.*

Proof. Let P be a prime ω-filter of L. Then $P = \omega(I)$ for some ideal I of L with $I \cap D = \emptyset$. Let $x \in P = \omega(I)$. Then $x \vee y \in D$ for some $y \in I$. Suppose $y \in P$. Then $y \in I \cap \omega(I)$. Hence $I \cap \omega(I) \neq \emptyset$. By Lemma 3.7(1), $P = \omega(I) = L$ which is a contradiction. Hence $y \notin P$. Therefore P is a minimal prime D-filter. □

Theorem 3.17. *The following assertions are equivalent in a lattice L:*

(1) *L is quasi-complemented;*

(2) *every prime D-filter is an ω-filter;*

(3) *every prime D-filter is minimal;*

(4) *every maximal filter is a minimal prime D-filter;*

(5) *every maximal filter is an ω-filter.*

Proof. (1) $\Rightarrow$ (2): Assume that L is quasi-complemented. Let P be a prime D-filter L. Then $L-P$ is a prime ideal of L such that $(L-P)\cap D=\emptyset$. We now prove that $P=\omega(L-P)$. Let $x\in P$. Since L is quasi-complemented, there exists $y\in L$ such that $x\wedge y=0$ and $x\vee y\in D$. If $y\in P$, then $0=x\wedge y\in P$, which is a contradiction. Hence $y\notin P$, which gives that $y\in L-P$. Since $x\vee y\in D$, we get $x\in\omega(L-P)$. Thus $P\subseteq\omega(L-P)$. Conversely, let $x\in\omega(L-P)$. Then $x\vee y\in D$ for some $y\in L-P$. Since $x\vee y\in D\subseteq P$ and $y\notin P$, we must have $x\in P$. Hence $\omega(L-P)\subseteq P$. Therefore P is an ω-filter of L.

(2) $\Rightarrow$ (3): Assume that every prime D-filter is an ω-filter. Let P be a prime D-filter of L. By (2), P will be a prime ω-filter. By Theorem 3.16, P is minimal.

(3) $\Rightarrow$ (4): Since every maximal filter is a prime D-filter, it is clear.

(4) $\Rightarrow$ (5): Since every minimal prime D-filter is an ω-filter, it is clear.

(5) $\Rightarrow$ (1): Assume that every maximal filter is an ω-filter. Let $x\in L$. Suppose $0\notin[x)\vee(x)^{\circ}$. Then there exists a maximal filter M such that $[x)\vee(x)^{\circ}\subseteq M$. Hence $x\in M$ and $(x)^{\circ}\subseteq M$. By the assumption, M is an ω-filter. Since M is prime, by Theorem 3.16, M is minimal prime D-filter. Hence $x\notin M$, which is a contradiction. Therefore $0\in[x)\vee(x)^{\circ}$. Hence $x\wedge a=0$ for some $a\in(x)^{\circ}$. Since $a\in(x)^{\circ}$, we get $a\vee x\in D$. Therefore L is quasi-complemented. □

We conclude this paper with a characterization theorem of ω-filters in terms of minimal prime D-filters. For this, we first need the following results.

Lemma 3.18. *Let I be an ideal of a lattice L such that $I\cap D=\emptyset$. If P is a minimal prime D-filter containing $\omega(I)$, then $I\cap P=\emptyset$.*

Proof. Let P be a minimal prime D-filter containing $\omega(I)$. Suppose $x\in I\cap P$. Then $x\in P$ and $x\in I$. Since P is minimal, there exists $y\notin P$ such that $x\vee y\in\omega(I)$. Then $x\vee y\vee i\in D$ for some $i\in I$. Hence $y\vee(x\vee i)\in D$ and $x\vee i\in I$. Thus $y\in\omega(I)\subseteq P$, which is a contradiction. Therefore $I\cap P=\emptyset$. □

Lemma 3.19. *Every minimal prime D-filter of a lattice L containing an ω-filter is a minimal prime D-filter in L.*

Proof. Let F be an ω-filter of L. Then $F=\omega(I)$ for some ideal I of L such that $I\cap D=\emptyset$. Let P be a minimal prime D-filter containing $F=\omega(I)$. By the above lemma, $I\cap P=\emptyset$. Let $x\in P$. Then there exists $y\notin P$ such that $x\vee y\in\omega(I)$. Hence $x\vee y\vee i\in D$ for some $i\in I$. Thus $x\vee(y\vee i)\in D\subseteq P$ and $y\vee i\notin P$(since $I\cap P=\emptyset$, we get that $i\notin P$ and also $y\notin P$). Hence P is a minimal prime D-filter of L. □

Now, ω-filters are characterized in terms of minimal prime D-filters.

Theorem 3.20. *Every ω-filter of a lattice L is the intersection of all minimal prime D-filters containing it.*

Proof. Let F be an ω-filter of L. Then $F = \omega(I)$ for some ideal I of L such that $I \cap D = \emptyset$. Let $F_0 = \bigcap\{P \mid P \text{ is a minimal prime } D\text{-filter containing } F\}$. Clearly $F \subseteq F_0$. Conversely, let $a \notin F = \omega(I)$. Then $a \vee t \notin D$ for all $t \in I$. Then there exists a minimal prime D-filter P such that $a \vee t \notin P$. Hence $a \notin P$ and $t \notin P$. Since P is prime, $(t)^\circ \subseteq P$ for all $t \in I$. Therefore $F = \omega(I) \subseteq P$. Thus P is minimal such that $F \subseteq P$ and $a \notin P$. Hence $a \notin F_0$, which yields $F_0 \subseteq F$. Therefore $F = F_0$. □

4. Lattice of ω-filters

In this section, we derive the lattice-theoretic properties of ω-filters of lattices. A set of equivalent conditions is derived for the class of all ω-filters of a distributive lattice to become a sublattice of the lattice of all filters.

Theorem 4.1. *Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a family of ω-filters of a lattice L. Then $\bigcap\limits_{\alpha\in\Delta} F_\alpha$ is again an ω-filter of L.*

Proof. For each $\alpha \in \Delta$, let $F_\alpha = \omega(I_\alpha)$ where I_α is an ideal of L such that $I_\alpha \cap D = \emptyset$. Then $\{I_\alpha\}_{\alpha\in\Delta}$ will be an arbitrary family of ideals in L such that $I_\alpha \cap D = \emptyset$ for each $\alpha \in \Delta$. Hence $\bigcap\limits_{\alpha\in\Delta} I_\alpha$ is an ideal of L such that $(\bigcap\limits_{\alpha\in\Delta} I_\alpha) \cap D = \emptyset$. By Lemma 3.7(3), we get $\bigcap\limits_{\alpha\in\Delta} \omega(I_\alpha) = \omega(\bigcap\limits_{\alpha\in\Delta} I_\alpha)$. Therefore $\bigcap\limits_{\alpha\in\Delta} F_\alpha$ is an ω-filter of L. □

Note that the class of all ω-filters of a lattice is closed under set-intersection. In general, ω-filters need not be closed under finite joins. However, in the following, we prove that the class $\mathcal{F}_\omega(L)$ of all ω-filters of a lattice L forms a complete lattice.

Theorem 4.2. *Let I, J be two ideals of a lattice L such that $I \cap D = J \cap D = \emptyset$. Then $\omega(I \vee J)$ is the smallest ω-filter containing both $\omega(I)$ and $\omega(J)$.*

Proof. Let I, J be two ideals of L such that $I \cap D = J \cap D = \emptyset$. Clearly $(I \vee J) \cap D = \emptyset$. By Lemma 3.7(2), we get $\omega(I) \subseteq \omega(I \vee J)$ and $\omega(J) \subseteq \omega(I \vee J)$. Suppose $\omega(I) \subseteq \omega(K)$ and

$\omega(J) \subseteq \omega(K)$ for some ideal K of L with $K \cap D = \emptyset$. Then

$$\begin{aligned} x \in \omega(I \vee J) &= x \vee (i \vee j) \in D \quad \text{for some } i \in I \text{ and } j \in J \\ &= x \vee i \in \omega(J) \subseteq \omega(K) \\ &= x \vee i \vee k_1 \in D \quad \text{for some } k_1 \in K \\ &= x \vee k_1 \in \omega(I) \subseteq \omega(K) \\ &= x \vee k_1 \vee k_2 \in D \quad \text{for some } k_2 \in K \\ &= x \in \omega(K) \quad \text{since } k_1 \vee k_2 \in K \end{aligned}$$

Hence $\omega(I \vee J)$ is the supremum of $\omega(I)$ and $\omega(J)$. Denote this supremum by $\omega(I) \sqcup \omega(J)$. Then $(\mathcal{F}_\omega(L), \cap, \sqcup)$ forms a lattice. □

Corollary 4.3. *Let $\{\omega(I_\alpha)\}_{\alpha\in\Delta}$ be a family of ω-filters of a lattice L where $I_\alpha \cap D = \emptyset$ for each $\alpha \in \Delta$. Then $\bigsqcup_{\alpha\in\Delta} \omega(I_\alpha)$ is the smallest ω-filter containing each $\omega(I_\alpha)$.*

It can be easily observed that the class of all ω-filters of a lattice forms a complete lattice with respect to set inclusion $\subseteq$, in which for any $\{\omega(I_\alpha)\}_{\alpha\in\Delta}$ of ω-filters, $\inf \{\omega(I_\alpha)\}_{\alpha\in\Delta} = \omega(\bigcap_{\alpha\in\Delta} I_\alpha)$ and the sup $\{\omega(I_\alpha)\}_{\alpha\in\Delta} = \omega(\bigvee_{\alpha\in\Delta} I_\alpha)$. Since the class of all ideals of a distributive lattice forms a complete distributive lattice, the class $\mathcal{F}_\omega(L)$ of all ω-filters of a distributive lattice L forms a complete distributive lattice. In general, the class $\mathcal{F}_\omega(L)$ of all ω-filters of a lattice L is not a sublattice of the filter lattice $\mathcal{F}(L)$. However, in the following, we derive a set of equivalent conditions for $\mathcal{F}_\omega(L)$ to become a sublattice of $\mathcal{F}(L)$. For this, we first need the following result.

Lemma 4.4. *Every proper ω-filter is contained in a minimal prime D-filter.*

Proof. Let F be a proper ω-filter of L. Then $F = \omega(I)$ for some ideal I of L with $I \cap D = \emptyset$. Hence $D \subseteq \omega(I) = F$. Clearly $F \cap I = \omega(I) \cap I = \emptyset$. Consider, the set

$$\text{Im} = \{J \mid J \text{ is an ideal of } L \text{ such that } I \subseteq J \text{ and } F \cap J = \emptyset\}.$$

Clearly $I \in \text{Im}$ and Im satisfies the Zorn's lemma. Let M be a maximal element of Im. Then M is an ideal of L such that $I \subseteq M$ and $F \cap M = \emptyset$. Since $D \subseteq F$, we get $D \cap M = \emptyset$. Hence M is an ideal which is maximal with respect to the property that $D \cap M = \emptyset$. Therefore $L - M$ is a minimal prime D-filter such that $F \subseteq L - M$. □

Theorem 4.5. *The following conditions are equivalent in a lattice L:*

(1) $\mathcal{F}_\omega(L)$ *is a sublattice of* $\mathcal{F}(L)$*;*

(2) *for* $a, b \in L, a \vee b \in D$ *implies* $(a)^\circ \vee (b)^\circ = L$*;*

(3) *for* $a, b \in L, (a)^\circ \vee (b)^\circ = (a \vee b)^\circ$*;*

(4) *for* $I, J \in \mathcal{I}(L)$*,* $I \vee J = L$ *implies* $\omega(I) \vee \omega(J) = L$*;*

(5) *for* $I, J \in \mathcal{I}(L)$*,* $\omega(I) \vee \omega(J) = \omega(I \vee J)$

Proof. (1) $\Rightarrow$ (2): Assume that $\mathcal{F}_\omega(L)$ is a sublattice of $\mathcal{F}(L)$. Let $a, b \in L$ be such that $a \vee b \in D$. Suppose $(a)^\circ \vee (b)^\circ \neq L$. Since $(a)^\circ$ and $(b)^\circ$ are ω-filters of L, by hypothesis, we get that $(a)^\circ \vee (b)^\circ$ is a proper ω-filter of L. Hence by Lemma 4.4, there exists a minimal prime D-filter P such that $(a)^\circ \vee (b)^\circ \subseteq P$. Hence $(a)^\circ \subseteq P$ and $(b)^\circ \subseteq P$. Since P is a minimal prime D-filter, we get that $a \notin P$ and $b \notin P$. Since P is a prime filter, we get that $a \vee b \notin P$, which is a contradiction to that $a \vee b \in D \subseteq P$. Hence we must have $(x)^\circ \vee (y)^\circ = L$.

(2) $\Rightarrow$ (3): Assume the condition (2). Let $a, b \in L$. Clearly $(a)^\circ \vee (b)^\circ \subseteq (a \vee b)^\circ$. Conversely, let $x \in (a \vee b)^\circ$. Then $(x \vee a) \vee (x \vee b) = x \vee (a \vee b) \in D$. Hence by condition (2), we get $(x \vee a)^\circ \vee (x \vee b)^\circ = L$. Thus $x \in (x \vee a)^\circ \vee (x \vee b)^\circ$. Hence $x = r \wedge s$ for some $r \in (x \vee a)^\circ$ and $s \in (x \vee b)^\circ$. Since $r \in (x \vee a)^\circ$, we get $r \vee x \in (a)^\circ$. Similarly, we can get $s \vee x \in (b)^\circ$. Now, we get

$$\begin{aligned} x &= x \vee x \\ &= x \vee (r \wedge s) \\ &= (x \vee r) \wedge (x \vee s) \in (a)^\circ \vee (b)^\circ. \end{aligned}$$

Hence $(a \vee b)^\circ \subseteq (a)^\circ \vee (b)^\circ$. Therefore $(a)^\circ \vee (b)^\circ = (a \vee b)^\circ$.

(3) $\Rightarrow$ (4): Assume the condition (3). Let I, J be two ideals of L such that $I \vee J = L$. Let d be a dense element of L. Then $d = i \vee j$ for some $i \in I$ and $j \in J$. Hence by (3), we get $L = (d)^\circ = (i \vee j)^\circ = (i)^\circ \vee (j)^\circ \subseteq \omega(I) \vee \omega(J)$. Therefore $\omega(I) \vee \omega(J) = L$.

(4) $\Rightarrow$ (5): Let I, J be two ideals of L. We have always $\omega(I) \vee \omega(J) \subseteq \omega(I \vee J)$. Let $x \in \omega(I \vee J)$. Then $x \vee a \in D$ for some $a \in I \vee J$. Now

$$\begin{aligned} x \in \omega(I \vee J) &\Rightarrow x \vee (i \vee j) \in D \qquad \text{where } i \in I \text{ and } j \in J \\ &\Rightarrow ((x \vee i) \vee (x \vee j)] = (D] \\ &\Rightarrow (x \vee i] \vee (x \vee j] = L \\ &\Rightarrow \omega((x \vee i]) \vee \omega((x \vee j]) = L \\ &\Rightarrow (x \vee i)^\circ \vee (x \vee j)^\circ = L \end{aligned}$$

Hence $x \in (x \vee i)^\circ \vee (x \vee j)^\circ$. Thus $x = a \wedge b$ where $a \in (x \vee i)^\circ$ and $b \in (x \vee j)^\circ$. Since $a \in (x \vee i)^\circ$ and $b \in (x \vee j)^\circ$, we get $x \vee a \in (i)^\circ$ and $x \vee b \in (j)^\circ$. Now

$$\begin{aligned} x &= x \vee x \\ &= x \vee (a \wedge b) \\ &= (x \vee a) \wedge (x \vee b) \\ &\in (i)^\circ \vee (j)^\circ \\ &\subseteq \omega(I) \vee \omega(J) \qquad \text{since } i \in I \text{ and } j \in J \end{aligned}$$

Hence we get $\omega(I \vee J) \subseteq \omega(I) \vee \omega(J)$. Therefore $\omega(I \vee J) = \omega(I) \vee \omega(J)$.

(5) $\Rightarrow$ (1): It is obvious. □

Theorem 4.6. *Let L be a lattice that satisfies the conditions of Theorem 4.5. If $\{F_\alpha\}_{\alpha\in\Delta}$ be an arbitrary family of ω-filters of L, then $\bigvee\limits_{\alpha\in\Delta} F_\alpha$ is again an ω-filter of L.*

Proof. For each $\alpha \in \Delta$, let $F_\alpha = \omega(I_\alpha)$ where I_α is an ideal of L such that $I_\alpha \cap D = \emptyset$. Then $\{I_\alpha\}_{\alpha\in\Delta}$ will be an arbitrary family of ideals in L such that $I_\alpha \cap D = \emptyset$ for each $\alpha \in \Delta$. Clearly $(\bigvee I_\alpha) \cap D = \emptyset$. Since $F_\alpha = \omega(I_\alpha) \subseteq \omega(\bigvee I_\alpha)$ for each $\alpha \in \Delta$, we get $\bigvee F_\alpha \subseteq \omega(\bigvee I_\alpha)$. Conversely, let $x \in \omega(\bigvee I_\alpha)$. Then $x \vee a \in D$ for some $a \in \bigvee I_\alpha$. Then there exists a positive integer n such that $a = a_1 \vee a_2 \vee \cdots \vee a_n$ where $a_i \in I_{\alpha_i}$. By condition (4) of Theorem 4.5, we get

$$\begin{aligned} x \vee a \in D &\Rightarrow x \vee (a_1 \vee a_2 \vee \cdots \vee a_n) \in D \\ &\Rightarrow (x \vee a_1) \vee (x \vee a_2) \vee \cdots \vee (x \vee a_n) \in D \\ &\Rightarrow (x \vee a_1] \vee (x \vee a_2] \vee \cdots \vee (x \vee a_n] = L \\ &\Rightarrow \omega((x \vee a_1]) \vee \omega((x \vee a_2]) \vee \cdots \vee \omega((x \vee a_n]) = L \\ &\Rightarrow (x \vee a_1)^\circ \vee (x \vee a_2)^\circ \vee \cdots \vee (x \vee a_n)^\circ = L \end{aligned}$$

Hence $x \in (x \vee a_1)^\circ \vee (x \vee a_2)^\circ \vee \cdots \vee (x \vee a_n)^\circ$. Thus $x = b_1 \wedge b_2 \wedge \cdots \wedge b_n$ where $b_i \in (x \vee a_i)^\circ$ for $i = 1, 2, \ldots, n$. Now

$$\begin{aligned} x &= x \vee x \\ &= x \vee (b_1 \wedge b_2 \wedge \cdots \wedge b_n) \\ &= (x \vee b_1) \wedge (x \vee b_2) \wedge \cdots \wedge (x \vee b_n) \\ &\in (a_1)^\circ \vee (a_2)^\circ \vee \cdots \vee (a_n)^\circ \\ &\subseteq \omega(I_1) \vee \omega(I_2) \vee \cdots \vee \omega(I_n) \\ &= F_1 \vee F_2 \vee \cdots \vee F_n \\ &\subseteq \bigvee F_\alpha \end{aligned}$$

which concludes that $\omega(\bigvee I_\alpha) \subseteq \bigvee F_\alpha$. Therefore $\bigvee F_\alpha$ is an ω-filter of L. □

Theorem 4.7. *Let L be lattice that satisfies any one of the conditions of Theorem 4.5. For any D-filter F, there exists a unique ω-filter contained in F.*

Proof. Let F be an arbitrary D-filter of L. Consider $\mathrm{Im}_F = \{H \in \mathcal{F}_\omega(L) \mid H \subseteq F\}$. Since D is the ω-filter and $D \subseteq F$, we get $D \in \mathrm{Im}_F$. Clearly Im_F satisfies the hypothesis of Zorn's Lemma. Let M be a maximal element of Im_F. It is enough to show that M is unique. Let M_1 and M_2 be two maximal elements of Im_F. Clearly $M_1 \vee M_2 \subseteq F$. By Theorem 4.5, $M_1 \vee M_2 \in \mathrm{Im}_F$. Thus $M_1 = M_1 \vee M_2 = M_2$. Hence Im_F has a unique maximal element, which is the required ω-filter contained in F. □

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