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Research Paper

SOME ASPECTS OF MARGINAL AUTOMORPHISMS OF A FINITE p-GROUP

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ABSTRACT. Let F be a free group, $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$ and G be a finite $\mathcal{V}$-nilpotent p-group. The automorphism α of G is said to be a marginal automorphism (with respect to V), if for all $x \in G$, $x^{-1}x^{\alpha} \in V^{\star}(G)$, where $V^{\star}(G)$ denotes the marginal subgroup of G. An automorphism α of G is called an IA-automorphism if $x^{-1}x^{\alpha} \in G'$ for each $x \in G$. An automorphism α of G is called a class preserving if for all $x \in G$, there exists an element $g_x \in G$ such that $x^{\alpha} = g_x^{-1}xg_x$. Let $\mathrm{Aut}^{V^{\star}}(G)$, $\mathrm{Aut}^{G'}(G)$ and $\mathrm{Aut}_c(G)$ respectively, denote the group of all marginal automorphisms, IA-automorphisms and class preserving automorphisms of G. In this paper, first we give a necessary and sufficient condition on a finite $\mathcal{V}$-nilpotent p-group G such that each marginal automorphism of G fixes the center of G element-wise. Then we characterize all finite $\mathcal{V}$-nilpotent p-groups G such that $\mathrm{Aut}^{V^{\star}}(G) = \mathrm{Aut}^{G'}(G)$. Finally, we obtain a necessary and sufficient condition for a finite $\mathcal{V}$-nilpotent p-group G such that $\mathrm{Aut}^{V^{\star}}(G) = \mathrm{Aut}_c(G)$.

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1. Introduction

Let F be a free group on the countable set $X = \{x_1, x_2, x_3, ...\}$ and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Then for a group G, two subgroups $V(G)$ and $V^{\star}(G)$ correspond to the variety $\mathcal{V}$, are defined as follows:

$$V(G) = \langle v(g_1, ..., g_r) \mid g_1, ..., g_r \in G, v \in V \rangle,$$

and

$$V^{\star}(G) = \{g \in G \mid v(g_1, ..., g_{i-1}, g_i g, g_{i+1}, ..., g_r) = v(g_1, ..., g_r), \\ \forall v \in V, g_1, ..., g_r \in G, \ and\ i \in \{1, ..., r\}\},$$

which are called the verbal and the marginal subgroups of G, respectively (see [3, 7, 9]). It can be easily seen that $V(G)$ and $V^{\star}(G)$ are fully-invariant and characteristic subgroups of G. If we take $V = \{[x_1, x_2]\}$, where $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, then $\mathcal{V}$ is the variety of abelian groups, $V(G) = G'$ and $V^{\star}(G) = Z(G)$, where G' and $Z(G)$ are the commutator subgroup and the center of G, respectively. We denote by $\Phi(G)$, the Frattini subgroup of G, the intersection of all the maximal subgroups of G and for $x \in G$, x^{α} denotes for the image of x under an automorphism α of G. Let N be a normal subgroup of G and $\alpha \in \mathrm{Aut}(G)$, the group of all automorphisms of G. If $N^{\alpha} = N$ (or $Ng^{\alpha} = Ng$ for all $g \in G$), we shall say α normalizes N (centralizes G/N respectively). Now let M and N be normal subgroups of G. We let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms α of G normalizing N and centralizing G/N (or equivalently, $[g, \alpha] = g^{-1}g^{\alpha} \in N$ for all $g \in G$), and let $C_{\mathrm{Aut}^N(G)}(M)$ denote the group of all automorphisms of $\mathrm{Aut}^N(G)$ centralizing M. If we choose $N = V^{\star}(G)$ or $N = Z(G)$, then $\mathrm{Aut}^N(G)$ is precisely the group of all marginal or central automorphisms of G. For $x \in G$, x^G denote the conjugacy class of all $x^g = g^{-1}xg$, where $g \in G$. An automorphism α of G is called class preserving if $x^{\alpha} \in x^G$ for all $x \in G$. The set of all class preserving automorphisms of G, denoted by $\mathrm{Aut}_c(G)$.

Let p denotes a prime number. Recall an abelian p-group A has invariants or is of type $(a_1, a_2, ..., a_k)$ if it is the direct product of cyclic subgroups of orders $p^{a_1}, p^{a_2}, ..., p^{a_k}$, where $a_1 \geq a_2 \geq ... \geq a_k > 0$.

Next, we introduce the notion of $\mathcal{V}$-nilpotent groups. This gives the usual notion of nilpotent groups if $\mathcal{V}$ is the variety of abelian groups, see also [8].

Definition 1.1. Let G be a group. Then the normal series,

$$1 = G_0 \leq G_1 \leq ... \leq G_c = G,$$

is said to be a $\mathcal{V}$-marginal series, if each factor is marginal, i.e.,

$$G_{i+1}/G_i \leq V^{\star}(G/G_i),\ 0 \leq i \leq c - 1.$$

A group G is said to be $\mathcal{V}$-nilpotent if it has a $\mathcal{V}$-marginal series; the shortest length of such series is called the $\mathcal{V}$-nilpotency class of G.

Let G be a finite p-group and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Assume that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Moreover $G/V(G)$, $G/V(G)Z(G)$ and $V^\star(G)$ are of types $(a_1, a_2, ..., a_k)$, $(b_1, b_2, ..., b_m)$ and $(e_1, e_2, ..., e_n)$. Since $G/V(G)Z(G)$ is a quotient group of $G/V(G)$, by [1, Section 25] we have $m \leq k$ and $b_j \leq a_j$ for all $1 \leq j \leq m$.

Keeping fixed the above terminology, we prove the following theorem:

Theorem A. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Then $\mathrm{Aut}^{V^\star}(G) = C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))$ if and only if $Z(G) \leq V(G)$ or $V(G) < V(G)Z(G)$, $Z(G) \leq \Phi(G)$, $k = m$ and $e_1 \leq b_t$, where t is the largest integer between 1 and m such that $a_t > b_t$.*

Let G be a finite p-group and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Assume that $V^\star(G) \leq Z(G)$. Moreover G/G' and $V^\star(G)$ are of types $(a_1, a_2, ..., a_k)$ and $(e_1, e_2, ..., e_n)$.

The above notation will be used in the following theorem:

Theorem B. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$. Then $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}^{G'}(G)$ if and only if $G' = V^\star(G)$ or $G' < V^\star(G)$, G is purely non-abelian, $m = n$ and $a_1 = b_t$, where $(b_1, b_2, ..., b_m)$ be invariants of G' and t is the largest integer between 1 and m such that $e_t > b_t$.*

In the following theorem, we give a necessary and sufficient condition on a finite $\mathcal{V}$-nilpotent p-group G for which $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}_c(G)$.

Theorem C. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Then $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}_c(G)$ if and only if $G/V^\star(G)$ is abelian, $V^\star(G) \leq V(G)$ and $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G), V^\star(G))$.*

2. Proofs of the theorems

A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. We use C_n for the cyclic group of order n and for a finite group G, $d(G)$ and $\Omega_i(G)$, respectively denote minimal number of generators of G and the subgroup of G generated by its elements of order dividing p^i. Finally, let G and H be any two groups. We denote by $\mathrm{Hom}(G, H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\mathrm{Hom}(G, H)$ forms an abelian group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in \mathrm{Hom}(G, H)$ and $x \in G$.

The following lemma is well-known and will be used in the rest of the paper.

Lemma 2.1. *Let A, B and C be finite abelian groups. Then*

(i) $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$;

(ii) $\mathrm{Hom}(A, B \times C) \cong \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$;

(iii) $\mathrm{Hom}(C_m, C_n) \cong C_d$, *where d is the greatest common divisor of m and n.*

Theorem 2.2. [10, Theorem C] *Let G be a purely non-abelian finite group and $\varnothing \neq V \subseteq F$ such that $V^\star(G) \leq Z(G)$. Then $|\mathrm{Aut}^{V^\star}(G)| = |\mathrm{Hom}(G/V(G), V^\star(G))|$.*

Theorem 2.3. [10, Theorem D] *Let G be a group and $\varnothing \neq V \subseteq F$ such that $V^\star(G) \leq V(G) \cap Z(G)$. Then $\mathrm{Aut}^{V^\star}(G) \cong \mathrm{Hom}(G/V(G), V^\star(G))$.*

Theorem 2.4. [5, Proposition 3.3] *Let G be an arbitrary group and $\varnothing \neq V \subseteq F$ such that $V^\star(G) \leq Z(G)$. Then $C_{\mathrm{Aut}^{V^\star}(G)}(Z(G)) \cong \mathrm{Hom}(G/V(G)Z(G), V^\star(G))$.*

Theorem 2.5. [8, Theorem 2.4] *If G is a $\mathcal{V}$-nilpotent group and N a non-trivial normal subgroup of G, then $N \cap V^\star(G) \neq 1$.*

Proof of the Theorem A.

Assume that $\mathrm{Aut}^{V^\star}(G) = C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))$ and $Z(G) \nleq V(G)$. First, we claim that $Z(G) \leq \Phi(G)$. Suppose, on the contrary, that there exists a maximal subgroup M of G such that $Z(G) \nleq M$. Then $G = M\langle z\rangle$, for some z in $Z(G)\backslash M$. By Theorem 2.5, we choose an element u in $\Omega_1(V^\star(G)) \bigcap M$. Then it is easy to see that the map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, is a marginal automorphism of G. By the given hypothesis $z = z^\alpha = zu$, whence $u = 1$, which is a contradiction. Thus $Z(G) \leq \Phi(G)$ and so $k = d(G/V(G)) = d(G/V(G)Z(G)) = m$, as required. Now since $Z(G) \leq \Phi(G)$, G is purely non-abelian and by Theorem 2.2,

$$|\mathrm{Aut}^{V^\star}(G)| = |\mathrm{Hom}(G/V(G), V^\star(G))|.$$

Also by Theorem 2.4,

$$C_{\mathrm{Aut}^{V^\star}(G)}(Z(G)) \cong \mathrm{Hom}(G/V(G)Z(G), V^\star(G)).$$

It follows that $|\mathrm{Hom}(G/V(G), V^\star(G))| = |\mathrm{Hom}(G/V(G)Z(G), V^\star(G))|$. Therefore

$$\prod_{1\leq i\leq k, 1\leq j\leq n} p^{\min\{a_i, e_j\}} = \prod_{1\leq i\leq m, 1\leq j\leq n} p^{\min\{b_i, e_j\}}.$$

Since $a_i \geq b_i$ for all i such that $1 \leq i \leq m$, we have $\min\{a_i, e_j\} \geq \min\{b_i, e_j\}$, where $1 \leq i \leq m, 1 \leq j \leq n$. Thus $\min\{a_i, e_j\} = \min\{b_i, e_j\}$, for all $1 \leq i \leq m, 1 \leq j \leq n$. Next, since $V(G) < V(G)Z(G)$, there exists some $1 \leq j \leq m$ such that $b_j < a_j$. Let t be the largest integer between 1 and m such that $a_t > b_t$. We show that $e_1 \leq b_t$. Suppose, on the contrary, that $e_1 > b_t$. Then by the above equality, we must have $\min\{a_t, e_1\} = \min\{b_t, e_1\} = b_t$, which is impossible. Hence $e_1 \leq b_t$.

Conversely, if $Z(G) \leq V(G)$, then $\mathrm{Aut}^{V^\star}(G) = C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))$, since $\mathrm{Aut}^{V^\star}(G)$ acts trivially on $V(G)$. Next, assume that $V(G) < V(G)Z(G)$, $Z(G) \leq \Phi(G)$, $k = d(G/V(G)) = d(G/V(G)Z(G)) = m$ and $e_1 \leq b_t$, where t is the largest integer between 1 and m such that $a_t > b_t$. Since $Z(G) \leq \Phi(G)$, G is purely non-abelian and by Lemma 2.1 and Theorem 2.2,

$$|\mathrm{Aut}^{V^\star}(G)| = |\mathrm{Hom}(G/V(G), V^\star(G))| = \prod_{1\leq i\leq k, 1\leq j\leq n} p^{\min\{a_i, e_j\}},$$

and by Theorem 2.4,

$$|C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))| = |\mathrm{Hom}(G/V(G)Z(G), V^\star(G))| = \prod_{1\leq i\leq m, 1\leq j\leq n} p^{\min\{b_i, e_j\}}.$$

Since $e_1 \leq b_t$, we have $b_1 \geq b_2 \geq ... \geq b_t \geq e_1 \geq e_2 \geq ... \geq e_n \geq 1$. Thus $a_i \geq b_i \geq e_j$ for all $1 \leq i \leq t$ and $1 \leq j \leq n$, which shows that $\min\{a_i, e_j\} = e_j = \min\{b_i, e_j\}$ for $1 \leq i \leq t$ and $1 \leq j \leq n$. Since $a_i = b_i$ for all $i \geq t+1$, we have $\min\{a_i, e_j\} = \min\{b_i, e_j\}$ for all $t+1 \leq i \leq k$ and $1 \leq j \leq n$. Thus $\min\{a_i, e_j\} = \min\{b_i, e_j\}$ for all $1 \leq i \leq k$ and $1 \leq j \leq n$. Therefore $|\mathrm{Aut}^{V^\star}(G)| = |C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))|$ and so $\mathrm{Aut}^{V^\star}(G) = C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))$, which completes the proof.□

Proof of Theorem B.

Assume that $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}^{G'}(G)$. For any $x, y \in G$, by the inner automorphism $i_y \in \mathrm{Aut}^{G'}(G)$, induced by $y \in G$, $x^{-1}x^{i_y} = [x, y] \in V^\star(G)$ and thus $G' \leq V^\star(G)$ and $G/V^\star(G)$ is abelian. Let us suppose that $G' \neq V^\star(G)$. We show that G is purely non-abelian. Suppose, on the contrary, that $G = A \times B$, where B is a purely non-abelian and A is non-trivial abelian group. For each non-trivial $f \in \mathrm{Hom}(B, V^\star(G) \cap A)$, the map $\alpha_f : G \to G$ defined by $(a, b)^{\alpha_f} = (af(b), b)$ is a marginal automorphism of G. Since $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}^{G'}(G)$, it follows that $\alpha_f \in \mathrm{Aut}^{G'}(G)$ and so for all $b \in B$, $f(b) \in A \cap G' = 1$, which is a contradiction. Hence by Theorem 2.2 and [10, Proposition 2.1], $|\mathrm{Aut}^{V^\star}(G)| = |\mathrm{Hom}(G, V^\star(G))|$. On the other hand, since G is of class 2, by [13, Lemma 3.1], $\mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G')$. To continue the proof, we claim that $m = n$. Let $(b_1, b_2, ..., b_m)$ be invariants of G'. By [1, Section 25], $m \leq n$ and $b_j \leq e_j$ for all $1 \leq j \leq m$. Suppose for a contradiction, that $m < n$. Since $|\mathrm{Aut}^{G'}(G)| = |\mathrm{Aut}^{V^\star}(G)|$, it follows that $|\mathrm{Hom}(G/G', G')| = |\mathrm{Hom}(G/G', V^\star(G))|$. So by Lemma 2.1, we have

$$|\mathrm{Aut}^{G'}(G)| = |\mathrm{Hom}(G/G', G')| = |\mathrm{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times ... \times C_{p^{b_m}})|$$

$$\leq |\mathrm{Hom}(G/G', C_{p^{e_1}} \times C_{p^{e_2}} \times ... \times C_{p^{e_m}})| < |\mathrm{Hom}(G/G', C_{p^{e_1}} \times C_{p^{e_2}} \times ... \times C_{p^{e_m}})|$$

$$\times |\mathrm{Hom}(G/G', C_{p^{e_{m+1}}} \times ... \times C_{p^{e_n}})| = |\mathrm{Hom}(G/G', V^\star(G))| = |\mathrm{Aut}^{V^\star}(G)|,$$

which is a contradiction. Therefore $d(G') = m = n = d(V^\star(G))$, as required. Next, since $|\operatorname{Aut}^{V^\star}(G)| = |\operatorname{Aut}^{G'}(G)|$, we have

$$\prod_{1\le i\le k, 1\le j\le m} p^{\min\{a_i,e_j\}} = \prod_{1\le i\le k, 1\le j\le m} p^{\min\{a_i,b_j\}}.$$

Since $e_j \ge b_j$ for all j such that $1 \le j \le m$, we have $\min\{a_i, e_j\} \ge \min\{a_i, b_j\}$, where $1 \le i \le k, 1 \le j \le m$. Thus $\min\{a_i, e_j\} = \min\{a_i, b_j\}$, for all $1 \le i \le k, 1 \le j \le m$. Next, since $G' < V^\star(G)$, there exists some $1 \le j \le m$ such that $b_j < e_j$. Let t be the largest integer between 1 and m such that $b_t < e_t$. We show that $a_1 \le b_t$. Suppose, on the contrary, that $a_1 > b_t$. Then by the above equality, we must have $\min\{a_1, e_t\} = \min\{a_1, b_t\} = b_t$, which is impossible. Hence $a_1 \le b_t$. Let $\exp(G/Z(G)) = p^s$, where $s \in \mathbb{N}$. Since $\operatorname{cl}(G) = 2$, by [6, Lemma 0.4], $s = b_1$. But $a_1 \le b_t \le b_{t-1} \le ... \le b_1 = s \le a_1$. Whence $a_1 = b_t$.

Conversely, if $G' = V^\star(G)$, then $\operatorname{Aut}^{V^\star}(G) = \operatorname{Aut}^{G'}(G)$. Assume that $G' < V^\star(G)$, G is purely non-abelian, $m = n$ and $a_1 = b_t$, where $(b_1, b_2, ..., b_m)$, be invariants of G' and t is the largest integer between 1 and m such that $e_t > b_t$. By [13, Lemma 3.1],

$$|\operatorname{Aut}^{G'}(G)| = |\operatorname{Hom}(G/G', G')| = \prod_{1\le i\le k, 1\le j\le m} p^{\min\{a_i,b_j\}}.$$

Also by Theorem 2.2 and [10, Proposition 2.1],

$$|\operatorname{Aut}^{V^\star}(G)| = |\operatorname{Hom}(G/G', V^\star(G))| = \prod_{1\le i\le k, 1\le j\le m} p^{\min\{a_i,e_j\}}.$$

Since $a_1 = b_t$, we have $1 \le a_k \le ... \le a_2 \le a_1 = b_t \le b_{t-1} \le ... \le b_2 \le b_1$. Thus $e_j \ge b_j \ge a_i$ for all $1 \le i \le k$ and $1 \le j \le t$, which shows that $\min\{a_i, e_j\} = a_i = \min\{a_i, b_j\}$ for $1 \le i \le k$ and $1 \le j \le t$. Since $e_j = b_j$ for all $j \ge t + 1$, we have $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $t+1 \le j \le m$. Thus $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $1 \le j \le m$. Therefore $|\operatorname{Aut}^{V^\star}(G)| = |\operatorname{Aut}^{G'}(G)|$, which together with $G' < V^\star(G)$, completes the proof.□

Proof of Theorem C.

First suppose that $\operatorname{Aut}^{V^\star}(G) = \operatorname{Aut}_c(G)$. We prove that $V^\star(G) \le \Phi(G)$. Suppose, on the contrary, that there exists a maximal subgroup M of G such that $V^\star(G) \not\le M$. Then $G = M\langle v\rangle$, for some v in $V^\star(G)\backslash M$. We choose an element u in $\Omega_1(V^\star(G) \cap M)$. Then the map $\alpha : mv^i \mapsto m(vu)^i$, where $m \in M$ and $0 \le i < p$, is a marginal automorphism of G which is not class preserving. Hence $V^\star(G) \le \Phi(G)$. Next, we show that $|\operatorname{Aut}^{V^\star}(G)| = |\operatorname{Hom}(G, V^\star(G))|$. For any $\alpha \in \operatorname{Aut}^{V^\star}(G)$, define $f_\alpha : G \to V^\star(G)$ by $f_\alpha(x) = x^{-1}x^\alpha$. We observe that f_α is a homomorphism from G to $V^\star(G)$, and $\alpha \mapsto f_\alpha$ is an injective map from $\operatorname{Aut}^{V^\star}(G)$ to $\operatorname{Hom}(G, V^\star(G))$. Conversely, if $f \in \operatorname{Hom}(G, V^\star(G))$, then define the map $x^\alpha = xf(x)$, for all $x \in G$, is an endomorphism of G. Since $x^{-1}x^\alpha \in V^\star(G) \le \Phi(G)$, for all $x \in G$, we may write G as the product of the image of α and the Frattini subgroup of G and so the image of α must be G itself. Thus α is an automorphism of G. Consequently, $\alpha = \alpha_f \in \mathrm{Aut}^{V^\star}(G)$, $f_{\alpha_f} = f$ and so $|\mathrm{Aut}^{V^\star}(G)| = |\mathrm{Hom}(G, V^\star(G))|$, as required. Since $V^\star(G)$ is abelian, by [10, Proposition 2.1], $|\mathrm{Hom}(G, V^\star(G))| = |\mathrm{Hom}(G/V(G), V^\star(G))|$. Next, for any $a, b \in G$, by the inner automorphism $i_b \in \mathrm{Aut}^{V^\star}(G)$, induced by $b \in G$, $a^{-1}a^{i_b} = [a, b] \in V^\star(G)$ and thus $G' \leq V^\star(G)$ and so $G/V^\star(G)$ is abelian. We claim that $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G))$. To see this, let $\psi \in \mathrm{Aut}_c(G) = \mathrm{Aut}^{V^\star}(G)$. Then

$$f_\psi : gV(G)V^\star(G) \mapsto g^{-1}g^\psi,$$

defines a homomorphism from $G/V(G)V^\star(G)$ to $V(G) \cap V^\star(G)$ and the map φ sending ψ to f_ψ defines a monomorphism from $\mathrm{Aut}_c(G)$ to the group

$$\mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G)).$$

On the other hand, let $f \in \mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G))$. Then the map $\beta = \beta_f$ defined by $x^\beta = xf(xV(G)V^\star(G))$, for all $x \in G$, is a marginal automorphism of G and $\varphi(\beta) = \varphi(\beta_f) = f$. So φ is onto, and thus

$$\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G)).$$

To continue the proof, we claim that $V^\star(G) \leq V(G)$. Suppose, on the contrary, that $V^\star(G) \not\leq V(G)$. Then $G/V(G)V^\star(G)$ is a proper quotient subgroup of $G/V(G)$ and

$$|G/V(G)/G/V(G)V^\star(G)| = |V(G)V^\star(G)/V(G)| = |V^\star(G)/V(G) \cap V^\star(G)| > 1.$$

Hence by [2, Lemma D], $\mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G))$ is isomorphic to a proper subgroup of $\mathrm{Hom}(G/V(G), V^\star(G))$. Therefore $|\mathrm{Aut}_c(G)| < |\mathrm{Aut}^{V^\star}(G)|$, a contradiction. Thus $V^\star(G) \leq V(G)$ and so $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G), V^\star(G))$.

Conversely, suppose that $G/V^\star(G)$ is abelian, $V^\star(G) \leq V(G)$ and $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G), V^\star(G))$. Since $V^\star(G) \leq V(G) \cap Z(G)$, by Theorem 2.3

$$\mathrm{Aut}^{V^\star}(G) \cong \mathrm{Hom}(G/V(G), V^\star(G)) \cong \mathrm{Aut}_c(G),$$

combining with $\mathrm{Aut}_c(G) \leq \mathrm{Aut}^{V^\star}(G)$, completes the proof.□

3. Application

In this section, we discuss and collect some facts which are consequences of Theorems A, B and C.

First, we may assume that $V = \{[x_1, x_2]\}$. In this situation, $\mathcal{V}$ is the variety of abelian groups, $V(G) = G'$ and $V^\star(G) = Z(G)$. Also $\mathcal{V}$-nilpotent groups gives the usual notion of nilpotent groups.

The next two corollaries are immediate consequences of Theorem A.

Corollary 3.1. [14, Lemma 2.4] *Let G be a finite p-group such that $\mathrm{Aut}^Z(G) = C_{\mathrm{Aut}^Z(G)}(Z(G))$, where $Z = Z(G)$. Then G is purely non-abelian.*

Let G be a finite p-group. Moreover G/G', $G/G'Z(G)$ and $Z(G)$ are of types $(a_1, a_2, ..., a_k)$, $(b_1, b_2, ..., b_m)$ and $(e_1, e_2, ..., e_n)$.

By the above notation, we have the following corollary.

Corollary 3.2. [11, Theorem A] *Let G be a finite p-group. Then $\mathrm{Aut}^Z(G) = C_{\mathrm{Aut}^Z(G)}(Z(G))$ if and only if $Z(G) \leq G'$ or $Z(G) \leq \Phi(G)$, $d(G/G') = d(G/G'Z(G))$ and $e_1 \leq b_t$, where t is the largest integer between 1 and m such that $a_t > b_t$.*

Note that, Corollary 3.2 yields the following result that is the main theorem of Yadav [14].

Let G be a finite p-group of class 2. Then $\exp(G/Z(G)) = \exp(G')$ by [6, Lemma 0.4]. Let $G/Z(G)$ is of type $(b_1, b_2, ..., b_m)$ and t is the largest integer between 1 and m such that $b_1 = b_2 = ... = b_t$. We note that $t \geq 2$. Set $\bar{M} = M/Z(G)$ is of type $(b_1, b_2, ..., b_t)$. Also let G/G' is of type $(a_1, a_2, ..., a_k)$ such that $\bar{M}$ is isomorphic to a subgroup of $\bar{N} = N/G'$ which is of type $(a_1, a_2, ..., a_t)$ and $Z(G)$ has invariants $(e_1, e_2, ..., e_n)$.

By Keeping fixed the above notation, in the following corollary, the shorter proof of [14, main Theorem] is provided.

Corollary 3.3. *Let G be a finite p-group of nilpotency class 2. Then $\mathrm{Aut}^Z(G) = C_{\mathrm{Aut}^Z(G)}(Z(G))$ if and only if $G' = Z(G)$ or $d(G/G') = d(G/Z(G))$, $(G/Z(G))/\bar{M} \cong (G/G')/\bar{N}$, and the exponents of $Z(G)$ and G' are equal.*

Proof. First assume that $\mathrm{Aut}^Z(G) = C_{\mathrm{Aut}^Z(G)}(Z(G))$ and $G' < Z(G)$. By Corollary 3.2, $k = d(G/G') = d(G/Z(G)) = m$ and $\exp(Z(G)) \leq p^{b_1}$. Now since $\exp(G') = \exp(G/Z(G)) = p^{b_1}$, it follows that $\exp(Z(G)) = p^{b_1}$ and so $\exp(G') = \exp(Z(G))$. Next, since $G/Z(G)$ is a proper quotient group of G/G', there exists some $1 \leq j \leq m$ such that $a_j > b_j$. Let l be the largest integer between 1 and m such that $a_l > b_l$. Hence $a_i = b_i$ for all $l+1 \leq i \leq m$ and by Corollary 3.2, $b_1 = e_1 \leq b_l$, which shows that $e_1 = b_1 = b_2 = ... = b_l$ and $l \leq t$. Set $\bar{M} = M/Z(G)$ is of type $(b_1, b_2, ..., b_t)$, $\bar{N} = N/G'$ which is of type $(a_1, a_2, ..., a_t)$ and A and B are of types $(a_{t+1}, a_{t+2}, ..., a_k)$ and $(b_{t+1}, b_{t+2}, ..., b_k)$. Since $a_i = b_i$ for all $t+1 \leq i \leq k$, then $A \cong B$ and therefore $(G/Z(G))/\bar{M} \cong B \cong A \cong (G/G')/\bar{N}$, as required.

Conversely, suppose that $k = d(G/G') = d(G/Z(G)) = m$, $(G/Z(G))/\bar{M} \cong (G/G')/\bar{N}$ and $\exp(Z(G)) = \exp(G')$. If $G' = Z(G)$, it is easy to see that $\mathrm{Aut}^Z(G) = C_{\mathrm{Aut}^Z(G)}(Z(G))$. Next, assume that $G' < Z(G)$. We claim that $Z(G) = G'G^{p^{e_1}}$. Since $\exp(G/Z(G)) = \exp(Z(G)) = p^{e_1}$, we have $G' \leq G'G^{p^{e_1}} \leq Z(G)$. It follows that $G/Z(G)$ is a quotient subgroup of $G/G'G^{p^{e_1}}$ and so $\exp(G/G'G^{p^{e_1}}) = p^{e_1}$. As $d(G/Z(G)) = d(G/G'G^{p^{e_1}})$, we may assume that $G/G'G^{p^{e_1}} = C_{p^{d_1}} \times C_{p^{d_2}} \times ... \times C_{p^{d_m}}$, where $d_1 = e_1 \geq d_2 \geq ... \geq d_m \geq 1$. As $b_i \leq d_i \leq e_1 = b_i$ for $1 \leq i \leq t$, it follows that $b_i = d_i$ for $1 \leq i \leq t$. On the other hand, $b_i \leq d_i \leq a_i = b_i$ for $t+1 \leq i \leq m$ and so $b_i = d_i$ for $t+1 \leq i \leq m$. Hence $G/Z(G) = G/G'G^{p^{e_1}}$ and consequently $Z(G) = G'G^{p^{e_1}}$, which shows that $Z(G) \leq \Phi(G)$. Finally, let l be the largest integer between 1 and m such that $a_l > b_l$. Since $a_i = b_i$ for $t+1 \leq i \leq m$, it follows that $l \leq t$. Now $p^{e_1} = \exp(Z(G)) = \exp(G') = \exp(G/Z(G)) = p^{b_1}$ and so $e_1 = b_1 = b_2 = ... = b_l$, which together with Corollary 3.2, completes the proof. □

Let G be a finite non-abelian p-group. Also let G/G' and $Z(G)$ are of types $(a_1, a_2, ..., a_k)$ and $(e_1, e_2, ..., e_n)$. By this notation and Theorem B, we have the following result.

Corollary 3.4. [12, Theorem B] *Let G be a finite p-group. Then $\mathrm{Aut}^Z(G) = \mathrm{Aut}^{G'}(G)$ if and only if $G' = Z(G)$ or $G' < Z(G)$, G is purely non-abelian, $d(G') = d(Z(G))$ and $a_1 = b_t$, where $(b_1, b_2, ..., b_m)$, be invariants of G' and t is the largest integer between 1 and m such that $e_t > b_t$.*

Remark 3.5. Note that, the above corollary was proved in [12] with extra condition that G must be a p-group of class 2.

As an application of Theorem C, we get the following corollary which is Theorem 3.1 of [4].

Corollary 3.6. *Let G be a finite p-group. Then $\mathrm{Aut}^Z(G) = \mathrm{Aut}_c(G)$ if and only if $G' = Z(G)$ and $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/Z(G), G')$.*

We end the paper by setting $V = \{[x_1, x_2], x_3^p\}$, where p is a prime. In this situation, $\mathcal{V}$ is the variety of elementary abelian p-groups, $V(G) = G'G^p$ and $V^\star(G) = \Omega_1(Z(G))$. We let $\Omega_1(Z) = \Omega_1(Z(G))$.

The following result is an immediate consequence of Theorem A.

Corollary 3.7. *Let G be a finite $\mathcal{V}$-nilpotent p-group. Then*

$$\mathrm{Aut}^{\Omega_1(Z)}(G) = C_{\mathrm{Aut}^{\Omega_1(Z)}(G)}(Z(G))$$

if and only if $Z(G) \leq \Phi(G)$.

Corollary 3.8. *Let G be a finite $\mathcal{V}$-nilpotent p-group. Then $\mathrm{Aut}^{\Omega_1(Z)}(G) = \mathrm{Aut}^{G'}(G)$ if and only if $G' = \Omega_1(Z(G))$.*

Proof. Let G be a finite p-group and $\mathrm{Aut}^{\Omega_1(Z)}(G) = \mathrm{Aut}^{G'}(G)$. By Theorem B, if $G' < \Omega_1(Z(G))$, then $\exp(G') = p$ and so G' is an elementary abelian, which together with $d(G') = d(\Omega_1(Z(G)))$, shows that $G' = \Omega_1(Z(G))$, a contradiction. Now the result has been proved. □

Corollary 3.9. *Let G be a finite $\mathcal{V}$-nilpotent p-group. Then* $\mathrm{Aut}^{\Omega_1(Z)}(G) = \mathrm{Aut}_c(G)$ *if and only if* $G/\Omega_1(Z(G))$ *is abelian,* $\Omega_1(Z(G)) \leq \Phi(G)$ *and*

$$\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/\Phi(G), \Omega_1(Z(G))).$$

Proof. It is readily seen that from Theorem C. □

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