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Research Paper

SOME ASPECTS OF UNITARY ADDITION CAYLEY GRAPH OF EISENSTEIN INTEGERS MODULO n

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Abstract. The unitary addition Cayley graph $G_n[\omega]$ of Eisenstein integers modulo n has the vertex set $\mathbb{E}_n[\omega]$, the set of Eisenstein integers modulo n. Any two vertices $x = a_1 + \omega b_1$, $y = a_2 + \omega b_2$ of $G_n[\omega]$ are adjacent if and only if $gcd(N(x + y), n) = 1$, where N is the norm of any element of $\mathbb{E}_n[\omega]$ given by $N(a + \omega b) = a^2 + b^2 - ab$. In this paper we obtain some basic graph invariants such as degree of the vertices, number of edges, diameter, girth, clique number and chromatic number of unitary addition Cayley graph of Eisenstein integers modulo n. This paper also focuses on determining the independence number of the above mentioned graph.

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1. Introduction

Let $\mathbb{Z}_n$ be the ring of integers modulo n, $n > 1$ and let U_n be the set of all units of the ring $\mathbb{Z}_n$. The unitary Cayley graph of $\mathbb{Z}_n$ introduced by Dejter and Giudici [1] is an undirected graph, whose vertex set is the set $\mathbb{Z}_n$ and any two vertices a and b of $\mathbb{Z}_n$ are adjacent if and only if $a-b \in U_n$, where U_n is the set of all units of $\mathbb{Z}_n$. The various structures and properties of unitary Cayley graphs have been studied extensively by Dejter and Giudici [1], Klotz and Sander [2], Akhtar et. al.[3] and Boggess et. al.[4].

Let Γ be an abelean group. For a subset B of Γ the addition Cayley graph induced by B on G denoted by $G' = Cay^+(\Gamma, B)$ is an undirected graph with the vertex set Γ and the edge set $\{ab \mid a+b \in B,\ a, b \in \Gamma\}$. Various properties of addition Cayley graphs have been discussed by Grynkiewicz et. al. [5, 6].

Sinha et. al. [7] studied the unitary addition Cayley graph by taking $\Gamma = \mathbb{Z}_n$ and $B = U_n$. Several graph theoretic properties of unitary addition Cayley graphs have been studied by them.

Roy and Patra [8] generalised the concept of a unitary addition Cayley graph to the unitary addition Cayley graph of Gaussian integers modulo n, $G_n[i]$ by replacing the vertex set $\mathbb{Z}_n$ to $\mathbb{Z}_n[i]$, the set of Gaussian integers modulo n, and the edge set U_n to $U_n[i]$. The norm of an element $a+ib$ in $\mathbb{Z}_n[i]$ is defined as $N(a+ib) = a^2+b^2$. An element $c+id$ in $\mathbb{Z}_n[i]$ will be a unit if and only if $gcd(N(c+id), n) = 1$ or simply we can say that $c+id$ will be a unit element in $\mathbb{Z}_n$ if and only if $N(c+id)$ is a unit element in $\mathbb{Z}_n$. In $G_n[i]$ any two vertices $a+ib$ and $c+id$ are adjacent whenever $gcd(N((a+c)+i(b+d)), n) = 1$.

In this paper the vertex set $\mathbb{Z}_n[i]$ of the graph $G_n[i]$ is replaced by the set of Eisenstein integers modulo n, $\mathbb{E}_n[\omega]$. The new graph thus obtained is denoted by $G_n[\omega]$ and named as unitary addition Cayley graph of Eisenstein integers modulo n. Any two vertices $x = a_1+\omega b_1$, $y = a_2+\omega b_2$ of $G_n[\omega]$ are adjacent if and only if $gcd(N(x+y), n) = 1$, where N is the norm of any element of $\mathbb{E}_n[\omega]$ given by $N(a+\omega b) = a^2+b^2-ab$. Some basic graph invariants of $G_n[\omega]$ are obtained along with the comparison of different properties between the graphs $G_n[i]$ and $G_n[\omega]$.

2. Preliminaries and Definitions

A *graph* G is a pair $G(V, E)$,where V is a non-empty finite set, and E is a set of unordered pairs of elements of V. The elements of V are called the vertices of G, and the elements of E are the edges of G. The set of vertices and edges of a graph G is denoted by $V(G)$ and $E(G)$ respectively. $|V(G)|$ and $|E(G)|$ denote the cardinality of $V(G)$ and $E(G)$ respectively. The *complement* $\overline{G}$ of a graph G is the graph with $V(G) = V(\overline{G})$ such that uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. The *degree* of a vertex v, denoted by $deg(v)$ in G

is the number of edges incident at v. If the degree of each vertex is equal, say r in G, then G is called an r-regular graph. A *trail* of a *graph* is an alternating sequence of vertices and distinct edges $v_0, e_1, v_1, e_2, ..., e_n, v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. If all the vertices and necessarily all the edges of a trail are distinct then it is called a *path*. The length of a path is the number of edges in it. A closed path(i.e., a path in which $v_0 = v_n$) is called a *cycle*. An *Eulerian trail* is a closed *trail* containing all vertices and edges of a *graph G*. A *graph G* is called an *Eulerian graph* if it contains an *Eulerian trail*. A *bigraph* or *bipartite graph G* is a graph whose vertex set V can be partitioned in to two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If every vertex of V_1 joins every vertex of V_2, then G is called a *complete bipartite graph*. For distinct vertices x and y of a *graph G*, let $d(x, y)$ be the length of a shortest *path* from x to y, the *diameter* of G, denoted by $diam(G) = sup\{d(x, y) : x, y \ are \ vertices \ of \ G\}$. The *girth* of a graph G, denoted by $girth(G)$ is the length of a shortest *cycle* in G, $(girth(G) = \infty \ if \ G \ contains \ no \ cycles)$. A *graph* that can be drawn in the plane so that edges intersect only at vertices is called *planar*. A complete subgraph of a graph G is called a *clique*. A *maximal clique* is a clique which cannot be enlarged by adding additional vertices to it. The cardinality of the maximal clique is called the *clique number* and it is denoted by $\omega(G)$. A subset S of the vertex set of G is *independent* if no two vertices of S are adjacent. The number of vertices in a maximal independent set of S is called the *independence number* of G and is denoted by $\beta(G)$. In another way idependence set is the maximal clique of $\overline{G}$ denoted by $\omega(\overline{G})$. The clique covering number $\theta(G)$ or $\chi(\overline{G})$ of a graph G is the minimum number of cliques in G needed to cover the vertex set of G. Since $\theta(G)$ involves the minimum number of cliques, only maximal cliques need be considered (since non-maximal cliques could not yield a clique cover of smaller size).

The unitary addition Cayley graph G_n=$Cay^+(\mathbb{Z}_n, U_n)$ is an undirected graph with vertex set $\mathbb{Z}_n$ and two distinct vertices a and b are adjacent if and only if $a + b \in U_n$, where $U_n = \{a \in \mathbb{Z}_n | gcd(a, n) = 1\}$ is the set of all units of $\mathbb{Z}_n$.

The set of Eisenstein integers denoted by $\mathbb{E}$ is a subring of the field of complex numbers. The elements of the set are the complex numbers of the form $a + b\omega$, where a, b are integers and ω= $\frac{(-1+i\sqrt{3})}{2}$, a primitive third root of unity. Then $\overline{\omega} = -1 - \omega = \omega^2$. Since $\mathbb{E}$ is a subring of the field of complex numbers, it is an integral domain. For any element $a + b\omega$, a norm is defined on the set $\mathbb{E}$, $N(a + b\omega) = a^2 + b^2 - ab$ which makes it a Euclidean domain. The units of $\mathbb{E}$ are $\pm\omega$, $\pm\overline{\omega}$ and ± 1. It is easily seen that for any positive integer n, the factor ring $\mathbb{E}/n\mathbb{E}$ is canonically isomorphic to the ring $\mathbb{E}_n = \{a + b\omega | a, b \in \mathbb{Z}_n\}$. Thus $\mathbb{E}_n$ is a principal ideal ring. This ring will be called the ring of Eisenstein integers modulo n. Abu Osba and Alkam[9] obtained some interesting results about the ring $\mathbb{E}_n$. In this chapter we have tried to get some basic graph invariants of the unitary addition Cayley graph of Eisenstein integers modulo n, denoted by $G_n[\omega]$ and also tried to compare the properties of both the graphs $G_n[i]$ and $G_n[\omega]$.

3. Basic Invariants

Lemma 3.1. *The total number of elements in $\mathbb{E}_n$ is n^2.*

Proof. The proof is trivial. $\square$

Theorem 3.2. [9] *The total number of unit elements in $\mathbb{E}_n$ is*

$$\begin{cases} 3 \times 2^{2k-2}, \ when \ n = 2^k, \ k \in \mathbb{N} \\ 2 \times 3^{2k-1}, \ when \ n = 3^k, \ k \in \mathbb{N}. \end{cases}$$

Theorem 3.3. [9] *The total number of unit elements in $\mathbb{E}_n$ is*

$$\begin{cases} p^{2k-2}(p^2-1), \ if \ p \equiv 2(mod 3) \\ (p^k - p^{k-1})^2, \ if \ p \equiv 1(mod 3). \end{cases}$$

Example 3.4. The unitary addition Cayley graphs of Eisenstein integers modulo 2 and 3 are shown below.

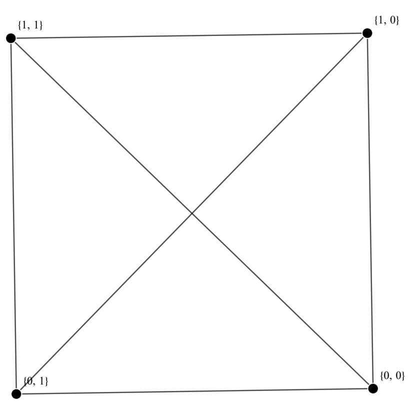

Fig 1: $G_2[\omega]$

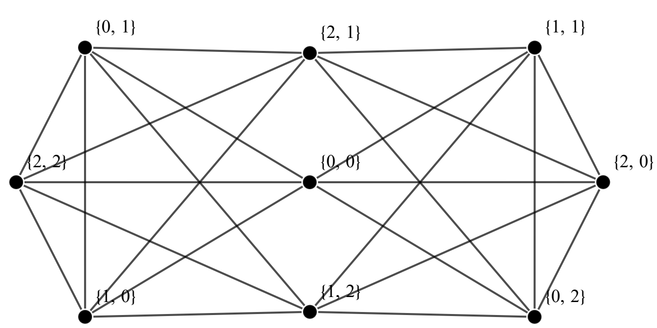

Fig 2: $G_3[\omega]$

Theorem 3.5. *When $n = 2^k$, $k \in \mathbb{N}$, the unitary addition Cayley graph of Eisenstein integers modulo n, $G_n[\omega]$ is a complete 4-partite graph.*

Proof. When $n = 2^k,\ k \in\ \mathbb{N}$, by Theorem 3.2 [9], the number of zero divisors will be 2^{2k-2} which will form one partite set. Next we divide the set of unit elements in three partite sets. Let us take a unit element $a + \omega b$, then

Case 1: when a is multiple of 2 and b is any odd number or its additive inverse in $\mathbb{Z}_n$.

Case 2: when a is an odd number or its additive inverse in $\mathbb{Z}_n$ and b is multiple of 2.

Case 3: when a is an odd number or its additive inverse in $\mathbb{Z}_n$ and b is an odd number or its additive inverse.

But none of the vertices within the partitions are adjacent to each other because the sum of the real parts of the vertices will be an even number and the sum of the imaginary parts of the vertices will also be an even number. But all the vertices of each partition will be adjacent to all the vertices of the other partitions. Thus we get a 4−partite graph. □

Example 3.6. The following figure is the 3-partite structure of the unit elements of the graph $G_4[\omega]$.

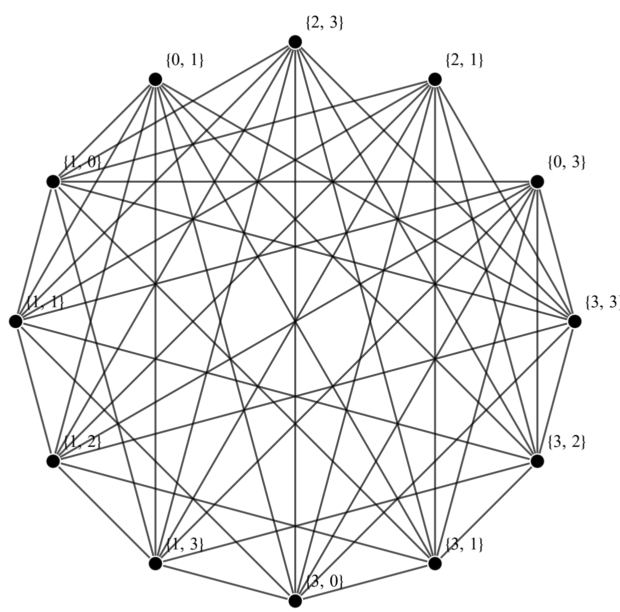

Fig 3: 3-partite structure of the unit elements of $G_4[\omega]$

Theorem 3.7. *When $n = 3^k,\ k \in\ \mathbb{N}$, the unitary addition Cayley graph of Eisenstein integers modulo n, $G_n[\omega]$ has two isomorphic copies of complete graphs $K_{3^{2k-1}}$, whose vertices are all the unit elements of $G_n[\omega]$.*

Proof. Let $n = 3^k,\ k \in\ \mathbb{N}$. Then $0 + 1\omega$ will be a unit element and $0 + 2\omega$ will be its additive inverse. Let $a + b\omega$ be a zero divisor, where none of the real and imaginary parts are multiples of 3. By Theorem 3.2 [9], there are 3^{2k-1} zero divisors and adding $a + b\omega$ to $0 + 1\omega$ 3^{2k-1} times we get 3^{2k-1} distinct unit elements which are adjacent to each other. Similarly, adding $a + b\omega$ to $0 + 2\omega$ 3^{2k-1} times we get 3^{2k-1} distinct unit elements which are adjacent to each other. So all the unit elements are divided into two sets and all the elements within the set are adjacent to each other but not adjacent to any elements of the other set of unit elements as $0 + 2\omega$ is the additive inverse of $0 + 1\omega$. Thus we get two isomorphic copies of complete graphs $K_{3^{2k-1}}$, whose vertices are all the unit elements of $G_n[\omega]$. □

Theorem 3.8. *Let $m = a + b\omega$ be a vertex in $G_n[\omega]$. Then*

$$deg(m) = \begin{cases} 3 \times 2^{2k-2},\ when\ n = 2^k,\ k \in \mathbb{N} \\ 2 \times 3^{2k-1},\ when\ n = 3^k\ , k \in \mathbb{N}\ and\ gcd(N(a + b\omega), n) \neq 1 \\ 2 \times 3^{2k-1} - 1,\ when\ n = 3^k\ , k \in \mathbb{N}\ and\ gcd(N(a + b\omega), n) = 1. \end{cases}$$

Proof. When $n = 2^k$, $\mathbb{Z}_n[\omega]$ has $2^{(2k-2)}$ zero divisors and these zero divisors will be adjacent to all the $3 \times 2^{(2k-2)}$ unit elements. Thus if $a+b\omega$ is a zero divisor then $deg(a+b\omega) = 3 \times 2^{(2k-2)}$. Let $c + d\omega$ and $e + f\omega$ be two unit elements such that both c and e are the multiples of 2 and d and f are the odd numbers or its additive inverse. Then $c + d\omega$ and $e + f\omega$ will never adjacent to each other and there will be $2^{(2k-2)}$ number of such unit elements which are not adjacent to each other. But they will be adjacent to all the elements of the type $g + h\omega$, $j + k\omega$ and $a + b\omega$, where g is an odd number or its additive inverse, h is a multiple of 2,j and k both are either an odd number or their additive inverse. Each type of vertices contains $2^{(2k-2)}$ elements. Thus if $c + d\omega$ is a unit element then $deg(c + d\omega) = 3 \times 2^{(2k-2)}$.
When $n = 3^k$, we consider the following two cases
Case 1: If $gcd(N(a + b\omega), n) \neq 1$ then $a + b\omega$ is a zero divisor. So by Theorem 3.2[9] $deg(a + b\omega) = 2 \times 3^{(2k-1)}$.
Case 2: If $gcd(N(a+b\omega), n) = 1$ then $a+b\omega$ and $2(a+b\omega)$ are both unit elements and $a+b\omega$ will be adjacent to the vertices of type $(c - a) + (d - b)\omega$. But it can not be adjacent to itself. Thus, by Theorem 3.2 [9] $deg(a + b\omega) = 2 \times 3^{(2k-1)} - 1$. □

Theorem 3.9. *If $n = p^k$, p is an odd prime of the form $3q + 1$, $k, q \in \mathbb{N}$. Let $m = a + b\omega$ be a vertex in $G_n[\omega]$. Then*

$$deg(m) = \begin{cases} (p^k - p^{k-1})^2,\ if\ gcd(N(a + b\omega), n) \neq 1 \\ (p^k - p^{k-1})^2 - 1,\ if\ gcd(N(a + b\omega), n) = 1. \end{cases}$$

Proof. When $n = p^k$,$k \in \mathbb{N}$ and p is an odd prime of the form $3q+1$, $q \in \mathbb{N}$. Now using the similar argument as that of the above theorem we can prove that if $a+b\omega$ is a zero divisor and by Theorem 3.3 [9] we have $deg(a+b\omega) = (p^k - p^{k-1})^2$. If $a+b\omega$ is a unit element then $deg(a+b\omega) = (p^k - p^{k-1})^2 - 1$. □

Theorem 3.10. *If $n = p^k$, p is an odd prime of the form $3q+2$, $k, q \in \mathbb{N}$. Let $m = a+b\omega$ be a vertex in $G_n[\omega]$. Then*

$$deg(m) = \begin{cases} p^{2k-2}(p^2-1), \ if \ gcd(N(a+b\omega), n) \neq 1 \\ p^{2k-2}(p^2-1) - 1, \ if \ gcd(N(a+b\omega), n) = 1. \end{cases}$$

Corollary 3.11. *The number of edges in $G_n[\omega]$ is*

$$\begin{cases} 3^{2k-1}(3^{2k}-1) \ when \ n = 3^k \ k \in \mathbb{N} \\ 3 \times 2^{4k-3} \ when \ n = 2^k \ k \in \mathbb{N} \\ \frac{(p^k-p^{k-1})^2(p^{2k}-1)}{2} \ where \ n = p^k, k \in \mathbb{N} \ and \ p \ is \ an \ odd \ prime \ of \ the \ form \ 3q+1, \ q \in \mathbb{N} \\ \frac{p^{4k}-2p^{2k}+p^{(2k-2)}}{2} \ where \ n = p^k, k \in \mathbb{N} \ and \ p \ is \ an \ odd \ prime \ of \ the \ form \ 3q+2, \ q \in \mathbb{N}. \end{cases}$$

Proof. We know that the sum of the degrees of the vertices of a graph is twice the number of lines.

Case 1: when $n = 2^k$, then $2q = \sum_{j=1}^{2^k} deg(x_j)$

$\Rightarrow 2q = 2^{2k} \times (3 \times 2^{(2k-2)})$

$\Rightarrow q = 3 \times 2^{(4k-3)}$.

Case 2: when $n = 3^k$,then $2q = \sum_{j=1}^{2^k} deg(x_j)$

$\Rightarrow 2q = 2 \times 3^{(2k-1)} \times (2 \times 3^{2k-1} - 1) + (2 \times 3^{(2k-1)}) \times (3^{2k} - 2 \times 3^{(2k-1)} + 1)$

$\Rightarrow q = 3^{(2k-1)}(3^{2k} - 1)$.

Case 3: when $n = p^k$, $k \in \mathbb{N}$ *and* p *is an odd prime of the form* $3q+1$, $q \in \mathbb{N}$, then $2q = \sum_{j=1}^{2^k} deg(x_j)$

$\Rightarrow 2q = (p^k - p^{(k-1)})^2 \times \{(p^k - p^{(k-1)})^2 - 1\} + \{p^{2k} - (p^k - p^{(k-1)})^2\}\{(p^k - p^{(k-1)})^2\}$

$\Rightarrow q = \dfrac{(p^k - p^{(k-1)})^2(p^{2k}-1)}{2}$.

Case 4: when $n = n = p^k$, $k \in \mathbb{N}$ *and* p *is an odd prime of the form* $3q+2$, $q \in \mathbb{N}$, then $2q = \sum_{j=1}^{2^k} deg(x_j)$

$\Rightarrow 2q = \{p^{(2k-2)}(p^2-1) - 1\} \times \{(p^{(2k-2)}(p^2-1)\} + \{p^{(2k-2)}(p^2-1)\} \times \{(p^{2k} - p^{(2k-2)}(p^2-1)\}$

$\Rightarrow q = \dfrac{p^{4k} - 2p^{2k} + p^{2k-2}}{2}$. □

Corollary 3.12. $G_n[\omega]$ *is a complete graph when* $n = 2^k$, *and* $k = 1$.

Proof. When $n = 2$, by Theorem 3.5, E_n will have three unit elements and one zero divisor namely $0 + 0\omega$ and thus E_n is a field. So the unitary addition Cayley graph of Eisenstein integers modulo n, $G_n[\omega]$ is a complete graph. □

Lemma 3.13. *When $n = p^k, k \in \mathbb{N}$ and p is an odd prime of the form $3q + 1, q \in \mathbb{N}$, the zero divisors of E_n form an induced complete bipartite subgraph of $G_n[\omega]$.*

Proof. When $n = p^k, k \in \mathbb{N}$ p is an odd prime of the form $3q + 1, q \in \mathbb{N}$, by Theorem 3.3[9], there will be two non associative primes $(a+b\omega)$ and $(a+b\overline{\omega})$ such that $n = (a+b\omega)(a+b\overline{\omega}) = a^2 + b^2 - ab$, Thus $E_n \cong E/ < a + b\omega > \times E/ < a + b\overline{\omega} >$. Hence, $(a + b\omega)$ and $(a + b\overline{\omega})$ are the maximal ideals containing p^{k-1} zero divisors each and they will form a complete bipartite graph. □

Lemma 3.14. *When $n = p^k, k \in \mathbb{N}$ and p is an odd prime of the form $3q + 1, q \in \mathbb{N}$, the unit elements of E_n form an induced regular subgraph of $G_n[\omega]$.*

Proof. If we take the $U_n[\omega]$ as the vertex set of $G_n[\omega]$, then the degree of each vertex will be $(p^k - p^{k-1})^2 - 2p^{k-1}$. Thus, the unit elements of E_n form an induced regular subgraph of $G_n[\omega]$. □

Theorem 3.15. *For $n > 2$, $diam(G_n[\omega]) = 2$.*

Proof. When $n = 2^k$, $k \in \mathbb{N}$, by Theorem 3.5 $G_n[\omega]$ is a complete $4-$partite graph. So, $diam(G_n[\omega]) = 2$. When n is a composite even number, $G_n[\omega]$ will be a $4-$partite graph. Hence $diam(G_n[\omega]) = 2$. Suppose that n is an odd number and let $n + a\omega$ be a unit element. Then $n+a\omega$ and $n+(n-a)\omega$ are not adjacent. But $0+0\omega$ will be adjacent to both $n+a\omega$ and $n+(n-a)\omega$ and there will be a path $n+a\omega - 0+0\omega - n+(n-a)\omega$. Hence, $diam(G_n[\omega]) \leq 2$. But when n is an odd number, $G_n[\omega]$ will never be a complete graph. So, $diam(G_n[\omega]) \geq 2$. Therefore, $diam(G_n[\omega]) = 2$. □

Theorem 3.16. *For $n \geq 2$, $girth(G_n[\omega]) = 3$.*

Proof. When $n = 2^k, k \in \mathbb{N}$, by Theorem 3.5 $G_n[\omega]$ is a $4-$partite graph. So, $girth(G_n[\omega]) = 3$. When n is an odd number, the two unit elements $1 + n\omega$ and $n + \omega$ along with a zero divisor $0 + 0\omega$ are adjacent to each other. Thus $girth(G_n[\omega]) = 3$. □

4. Clique Number, Chromatic Number and Independence Number

Theorem 4.1. *For* $n = 2^k,\ k \in \mathbb{N}$, $\chi(G_n(\omega)) = \omega(G_n(\omega)) = 4$.

Proof. When $n = 2^k, k \in \mathbb{N}$, by Theorem 3.5 $G_n[\omega]$ is a $4-$partite graph. So, there will always be a complete subgraph on four vertices. Thus, $\chi(G_n(\omega)) = \omega(G_n(\omega)) = 4$. □

Theorem 4.2. *For* $n = 3^k,\ k \in \mathbb{N}$, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq 3^{2k-1} + 1$.

Proof. When $n = 3^k,\ k \in\ \mathbb{N}$, by Theorem 3.7 the unitary addition Cayley graph of Eisenstein integers modulo n, $G_n[\omega]$ has two isomorphic copies of complete graphs $K_{3^{2k-1}}$, whose vertices are all the unit elements of $G_n[\omega]$. But all the zero divisors of $G_n[\omega]$ will be adjacent to all the unit elements of $G_n[\omega]$. Thus, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq 3^{2k-1} + 1$. □

Theorem 4.3. *When n is an odd prime of the form* $3q + 1, q \in \mathbb{N}$, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq \frac{(p^k - p^{k-1})^2}{4} + 2$.

Proof. By [9] E_{p^k} is isomorphic to $E/ < (a + b\omega) > \times E/ < (a + b\overline{\omega)} >$. The numbers of unit elements in both $E/ < (a + b\omega) >$ and $E/ < (a + b\overline{\omega)} >$ are $p^k - p^{k-1}$. Removing the additive inverses we have $\frac{(p^k - p^{k-1})}{4}$ unit elements in E_{p^k} and by Lemma 3.13 there will be a maximum subgraph with $\frac{(p^k - p^{k-1})}{4} + 2$ vertices. Thus, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq \frac{(p^k - p^{k-1})^2}{4} + 2$. □

Theorem 4.4. *When n is an odd prime of the form* $3q + 2, q \in \mathbb{N}$, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq \frac{n^2}{2} + 1$.

Proof. When n is an odd prime of the form $3q + 2, q \in \mathbb{N}$, E_n is a field [9]. So there are $\frac{n^2}{2}$ unit elements which are adjacent to each other and along with $0 + 0\omega$ will form a maximal complete subgraph. Thus, $\chi(G_n(\omega)) \geq \omega(G_n(\omega)) \geq \frac{n^2}{2} + 1$. □

We now find the independence number of $G_n[\omega]$, which can be obtained by finding the clique number $\omega(\overline{G_n}[\omega])$ of the complement of the graph $G_n[\omega]$. Along with this we can find the clique covering number $\chi(\overline{G_n}[\omega])$, which is the minimum number of cliques required to cover all the vertices of $G_n[\omega]$.

Theorem 4.5. *When $n > 1, n \in \mathbb{N}$, the independence number of unitary addition Cayley graph of Eisenstein integers modulo n is given by*

$$\chi(\overline{G_n}[\omega]) \geq \omega(\overline{G_n}[\omega]) \geq \begin{cases} 2^{2k-2},\ when\ n = 2^k \\ 3^{2k-1},\ when\ n = 3^k \\ p,\ when\ p\ is\ an\ odd\ prime \\ \quad of\ the\ form\ 3q+1, q \in \mathbb{N} \\ 2,\ when\ p\ is\ an\ odd\ prime \\ \quad of\ the\ form\ 3q+2, q \in \mathbb{N}. \end{cases}$$

Proof. When $n = 2^k,\ k \in \mathbb{N}$, by Theorem 3.5 $G_n[\omega]$ is a complete $4-$ partite graph. But $\overline{G_n}[\omega]$ is a disconnected graph with four components, where each components are the partitions of the vertex set of $G_n[\omega]$. So, each partitions will be a complete graph with 2^{2k-2} vertices. Hence, $\chi(\overline{G_n}[\omega]) \geq \omega(\overline{G_n}[\omega]) \geq 2^{2k-2}$.

When $n = 3^k,\ k \in \mathbb{N}$, by Theorem 3.7 the unitary addition Cayley graph of Eisenstein integers modulo n, $G_n[\omega]$ has two isomorphic copies of complete graphs $K_{3^{2k-1}}$ and all the zero divisors are not adjacent to each other but they are adjacent to all the units elements of $G_n[\omega]$. So, in $\overline{G_n}[\omega]$ all the zero divisors will form a complete graph with 3^{2k-1} elements. $\chi(\overline{G_n}[\omega]) \geq \omega(\overline{G_n}[\omega]) \geq 3^{2k-1}$.

When $n = p,\ p\ is\ an\ odd\ prime\ of\ the\ form\ 3q+1, q \in \mathbb{N}$, by Lemma 3.13 we have the zero divisios of $G_n[\omega]$ which will form a induced complete bipartite graph. In $\overline{G_n}[\omega]$ both the partite sets will form a complete subgraph with p elements. Thus, $\chi(\overline{G_n}[\omega]) \geq \omega(\overline{G_n}[\omega]) \geq p$.

When $n = p,\ p\ is\ an\ odd\ prime\ of\ the\ form\ 3q+2, q \in \mathbb{N}$, by Theorem 3.3 [9], $G_n[\omega]$ is a field. So, $\chi(\overline{G_n}[\omega]) = \omega(\overline{G_n}[\omega]) = 2$. □

Below is a comparison of the graphs $G_n[i]$ and $G_n[\omega]$

Comparison between $G_n[i]$ and $G_n[\omega]$		
Properties	$G_n[i]$	$G_n[\omega]$
Diameter	For $n \geqslant 2, diam(G_n[i]) = \begin{cases} 2, \; if \; n \; is \; even \; or \; odd. \\ 3, \; if \; n = n_1 n_2 \; if \; n_1 \; is \; even \\ \quad and \; n_2 \; is \; an \; odd \; prime. \end{cases}$	For $n > 2$, $diam(G_n[\omega]) = 2$
Girth	For $n \geqslant 2$, $girth(G_n[i]) = \begin{cases} 3, \; if \; n \; is \; odd. \\ 4, \; if \; n \; is \; even. \end{cases}$	For $n \geq 2$, $girth(G_n[\omega]) = 3$
For $n = 2$	$G_2[i]$ *is a complete bipartite graph*	$G_2[\omega]$ *is a complete graph*
For $n = 2^k, \ k \geqslant 2$	$G_n[i]$ *is a complete bipartite graph*	$G_n[\omega]$ *is a complete* $4-$ *partite graph*
Planarity	$G_n[i]$ is planar if $n = 1, 2$	$G_n[\omega]$ is planar if $n = 1, 2$
Traversability	$G_n[i]$ is Eulerian if n is even	$G_n[\omega]$ is Eulerian if n is even
Reducibility	$\mathbb{Z}_n[i]$ is reducible if $n \equiv 1(mod 4)$	$\mathbb{Z}_n[\omega]$ is reducible if $n \equiv 1(mod 3)$
Irreducibility	$\mathbb{Z}_n[i]$ is irreducible if $n \equiv 3(mod 4)$	$\mathbb{Z}_n[\omega]$ is irreducible if $n \equiv 2(mod 3)$

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