Algebraic Structures and Their Applications Vol. 9 No. 1 (2022) pp 81-92.

Research Paper

ON THE COFINITENESS OF LOCAL COHOMOLOGY MODULES

HAJAR ROSHAN-SHEKALGOURABI

Abstract. Let R be a commutative Noetherian ring with identity, I be an ideal of R and M be an R-module such that $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all j. We prove that if $\dim H^i_I(M) \leq 1$ for all i, then for any $i \geq 0$ and for any submodule N of $H^i_I(M)$ that is either I-cofinite or minimax, the R-module $H^i_I(M)/N$ is I-cofinite. This generalizes the main result of Bahmanpour and Naghipour [8, Theorem 2.6]. As a consequence, the Bass numbers and Betti numbers of $H^i_I(M)$ are finite for all $i \geq 0$. Also, among other things, we show that if either $\dim R/I \leq 2$ or $\dim M \leq 2$, then for each finitely generated R-module N, the R-module $\mathrm{Ext}^j_R(N, H^i_I(M))$ is I-weakly cofinite, for all $i \geq 0$ and $j \geq 0$. This generalizes [1, Corollary 2.8].

DOI: 10.29252/as.2021.2382

MSC(2010): Primary:13D45, 13D07, 13C05

Keywords: Local cohomology modules, I-cofinite modules, Minimax modules, Weakly Laskerian modules, Krull dimension, Bass numbers.

Received: 16 November 2019, Accepted: 26 December 2020.

1. Introduction

Let R be a commutative Noetherian ring with identity and I be an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H_I^i(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [11] for more details about the local cohomology.

In [20] Grothendieck conjectured that for any ideal I of R and any finitely generated R-module M, $\operatorname{Hom}_R(R/I, H_I^i(M))$ is a finitely generated R-module for all i, but soon Hartshorne [21] was able to provide a counterexample to Grothendieck's conjecture. He defined an R-module M to be *I-cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all j and he asked:

For which rings R and ideals I are the modules $H_I^i(M)$ I-cofinite for all i and all finitely generated R-modules M?

Concerning this question, Hartshorne [21] and later Chiriacescu [13] showed that if R is a complete regular local ring and I is a prime ideal such that $\dim R/I = 1$, then $H_I^i(M)$ is I-cofinite for any finitely generated R-module M. Huneke and Koh [22, Theorem 4.1] proved that if R is a complete Gorenstein local domain and I is an ideal of R such that $\dim R/I = 1$, then for all non-negative integers i and j, $\operatorname{Ext}_R^j(N, H_I^i(M))$ is finitely generated for any finitely generated R-modules M and N such that $\operatorname{Supp}_R(N) \subseteq V(I)$. Furthermore, Delfino [14] proved that if R is a complete local domain under some mild conditions, then similar results hold. Also, Delfino and Marley [15, Theorem 1] and Yoshida [31, Theorem 1.1] have eliminated the complete hypothesis entirely. Finally, Bahmanpour and Naghipour [8, Theorem 2.6] have removed the local assumption on R.

As we mentioned, these assertions are true when M is finitely generated. As a corollary of the first main theorem of this paper (Corollary 3.7), we improve these results by omitting the finiteness hypothesis on M and replacing it by a more general condition on M. More precisely, we show that the R-module $H_I^i(M)$ is I-cofinite for all i, whenever $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all j and $\dim H_I^i(M) \leq 1$ (e.g., $\dim R/I \leq 1$ or $\dim M \leq 1$), for all i. Our tools for proving this result is the following theorem.

Theorem 1.1. *Let M be an R-module and $t \geq 1$ be a positive integer such that $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all $j \leq t$ and the R-module $H_I^i(M)$ belongs to $(I\text{-cof}, \mathcal{D}_{\leq 1})$ for all $i < t$. Then the following statements hold:*

(i) *The R-module $H_I^i(M)$ is I-cofinite for all $i < t$.*

(ii) *For every submodule N of $H_I^t(M)$ that belongs to $(I\text{-cof}, \mathcal{D}_{\leq 0})$, the R-module $\operatorname{Ext}_R^i(R/I, H_I^t(M)/N)$ is finitely generated for all $i = 0, 1$. In particular, the set $\operatorname{Ass}_R(H_I^t(M)/N)$ is finite.*

Several corollaries of this result are presented. Among these, we extend and improve the main results of Delfino and Marley [15, Theorem 1], Yoshida [31, Theorem 1.1], Bahmanpour and Naghipour [8, Theorem 2.6] and [7, Theorem 2.5], Huneke and Sharp [23, Theorem 2.1], Kawasaki [24, Main Theorem], Brodmann and Lashgari [10, Theorem 2.2], Khashyarmanesh and Salarian [25, Theorem B], Aghapournahr and Bahmanpour [3, Theorem 3.4] and Quy [30, Theorem 3.2].

As the second main result of this paper we characterize the $\mathcal{FD}_{\leq 1}$ local cohomology modules. More precisely, we prove the following theorem:

Theorem 1.2. *Let M be an R-module and $t \geq 1$ be an integer such that $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all $j \leq t$. Then the following statements are equivalent:*

(1) *$H^i_I(M)$ is an $\mathcal{FD}_{\leq 1}$ R-module for all $i < t$;*
(2) *$(H^i_I(M))_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for all $i < t$ and for all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$ with $\dim R/\mathfrak{p} > 1$.*

As the final main result of the present paper, we prove that if M is a non-zero module over a local ring $(R, \mathfrak{m})$ such that $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all j and either $\dim R/I \leq 2$ or $\dim M \leq 2$, then for each finitely generated R-module N, the R-module $\operatorname{Ext}^j_R(N, H^i_I(M))$ is I-weakly cofinite for all i and j. This result is an extension of [1, Corollary 2.9].

Throughout the paper, we assume that R is a commutative Noetherian ring, I is an ideal of R and $V(I)$ is the set of all prime ideals of R containing I. Also, we use $\mathbb{N}_0$ (respectively, $\mathbb{N}$) to denote the set of non-negative (respectively, positive) integers.

2. **Preliminaries**

First, we recall some definitions which are needed in this paper. Let M be an R-module.

(1) M is called *minimax* if there is a finitely generated submodule N of M such that M/N is Artinian (see [33]).
(2) M is called *weakly Laskerian* if $Ass_R(M/N)$ is a finite set for each submodule N of M. Also, M is said to be *I-weakly cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is weakly Laskerian, for all $i \geq 0$ (see [18] and [19]).
(3) M is said to be FSF if there exists a finitely generated submodule N of M such that the support of M/N is a finite set. In the light of [6, Theorem 3.3], over a Noetherian ring R an R-module M is weakly Laskerian if and only if is FSF.
(4) A class of R-modules is a *Serre subcategory* of the category of R-modules when it is closed under taking submodules, quotients and extensions. For example, the classes of Noetherian modules, Artinian modules, minimax modules or weakly Laskerian modules are Serre subcategories. As in standard notation, we let $\mathcal{S}$ stands for a Serre subcategory of the category of R-modules.

The following lemmas are needed in the next section.

Lemma 2.1. *Suppose that M is a finitely generated R-module and $N \in \mathcal{S}$. Then* $\operatorname{Ext}^i_R(M,N) \in \mathcal{S}$ *and* $\operatorname{Tor}^R_i(M,N) \in \mathcal{S}$ *for all* $i \geq 0$.

Proof. The assertion immediately follows from the definition of Ext and Tor functors. □

Lemma 2.2. *Let M be a finitely generated R-module and let N be an arbitrary R-module. Suppose that for some integers $t \geq 0$, $\operatorname{Ext}^i_R(M,N) \in \mathcal{S}$ for all $i \leq t$. Then for any finitely generated R-module L with $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(M)$, $\operatorname{Ext}^i_R(L,N) \in \mathcal{S}$ for all $i \leq t$.*

Proof. See [2, Lemma 2.2]. □

Let $\mathcal{D}_{\leq n}$ denote the class of all R-modules M with $\dim M \leq n$. As a special and interesting example of the class of extension modules introduced by Yoshizawa [32], recently, Aghapournahr and Bahmanpour [3] introduced the class of extension modules of finitely generated modules by the class of all modules T with $\dim T \leq n$ and denoted it by $\mathcal{FD}_{\leq n}$ where $n \geq -1$ is an integer. An R-module M is said to be $\mathcal{FD}_{\leq n}$ if there is a finitely generated submodule N of M such that $\dim M/N \leq n$. Thus, by definition, the class of minimax modules is contained in the class of $\mathcal{FD}_{\leq 0}$ and in the light of [6, Theorem 3.3], the class of weakly Laskerian modules is contained in the class of $\mathcal{FD}_{\leq 1}$. This definition motivates the following.

Definition 2.3. If I-*cof* denotes the class of all I-cofinite modules, then we will define by (I-*cof*, $\mathcal{D}_{\leq n}$) the class of all R-modules M for which there exists an I-cofinite submodule N of M such that $\dim M/N \leq n$.

It is clear that all I-cofinite R-modules, $\mathcal{D}_{\leq n}$ R-modules and I-torsion $\mathcal{FD}_{\leq n}$ R-modules are included in the class of (I-*cof*, $\mathcal{D}_{\leq n}$). We claim that the class of (I-*cof*, $\mathcal{D}_{\leq n}$) is strictly larger than the class of I-*cof* and $\mathcal{D}_{\leq n}$. To see this, consider the following examples.

Examples 2.4.

(i) Suppose that R is a commutative Noetherian ring with $\dim R \geq 2$ and $\mathfrak{p}$ is a two dimensional prime ideal of R. Then there exist $\mathfrak{q} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$ and $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$. Let $M = \operatorname{Hom}_R(R/\mathfrak{p}, R) \oplus E(R/\mathfrak{q})$. It is easy to see that M belongs to ($\mathfrak{p}$-*cof*, $\mathcal{D}_{\leq 1}$) that is neither $\mathfrak{p}$-cofinite nor $\mathcal{D}_{\leq 1}$. Note that $\operatorname{Hom}_R(R/\mathfrak{q}, E(R/\mathfrak{q}))$ is not finitely generated.

(ii) Let $R = k[x,y][[u,v]]$, $\mathfrak{q} = (u,v)$ and $N = R/(xu+yv)$, where k is a field. By Hartshorn's conterexample about cofiniteness of local cohomology modules [21], $\operatorname{Hom}_R(R/\mathfrak{q}, H^2_{\mathfrak{q}}(N))$ is not finitely generated. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$ and

$\dim R/\mathfrak{p} = 3$. Hence, $M = \mathrm{Hom}_R(R/\mathfrak{p}, R) \oplus H^2_{\mathfrak{q}}(N)$ belongs to ($\mathfrak{p}$-*cof*, $\mathcal{D}_{\leq 2}$) that is neither $\mathfrak{p}$-cofinite nor $\mathcal{D}_{\leq 2}$. Note that $\mathrm{Supp}_R(H^2_{\mathfrak{p}}(N)) \subseteq V(\mathfrak{p})$ and so that $\dim H^2_{\mathfrak{p}}(N) \leq 2$.

The following lemma will be quite useful in the proof of one of the main theorems of the paper.

Lemma 2.5. *Let M be a non-zero R-module and $t \in \mathbb{N}_0$. Suppose that the R-module $H^i_I(M)$ is I-cofinite for all $i < t$ and the R-module $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all $j = t, t+1$. Then the R-module $\mathrm{Ext}^j_R(R/I, H^t_I(M))$ is finitely generated for all $j = 0, 1$.*

Proof. See [16, Theorem 2.1] and [17, Theorem A]. □

3. **Main Results**

First, we prove some results which play important roles in what follows.

Proposition 3.1. *If $M \in$ (I-cof, $\mathcal{D}_{\leq 0}$), then the following statements are equivalent:*

(i) $\mathrm{Ext}^i_R(R/I, M)$ *is finitely generated for all $i \geq 0$;*

(ii) *The R-module* $\mathrm{Hom}_R(R/I, M)$ *is finitely generated.*

Proof. $(i) \Rightarrow (ii)$ is clear. For $(ii) \Rightarrow (i)$, considering the exact sequence $0 \to S \to M \to T \to 0$ where S is I-cofinite and $\dim T \leq 0$, we have the long exact sequence

$$0 \to \mathrm{Hom}_R(R/I, S) \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, T) \to \mathrm{Ext}^1_R(R/I, S) \to \cdots$$

which follows that $\mathrm{Hom}_R(R/I, T)$ is finitely generated. Therefore, $\mathrm{Ext}^i_R(R/I, T)$ is finitely generated for all $i \geq 0$ by [4, Lemma 2.5]. This implies that $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. □

Proposition 3.2. *If $M \in$ (I-cof, $\mathcal{D}_{\leq 1}$), then the following statements are equivalent:*

(i) $\mathrm{Ext}^i_R(R/I, M)$ *is finitely generated for all $i \geq 0$;*

(ii) *The R-module* $\mathrm{Ext}^i_R(R/I, M)$ *is finitely generated for all $i = 0, 1$.*

Proof. $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (i)$: By definition, there exists an exact sequence $0 \to S \to M \to T \to 0$ where S is I-cofinite and $\dim T \leq 1$. This induces the long exact sequence

$$0 \to \mathrm{Hom}_R(R/I, S) \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, T) \to \mathrm{Ext}^1_R(R/I, S)$$
$$\to \mathrm{Ext}^1_R(R/I, M) \to \mathrm{Ext}^1_R(R/I, T) \to \mathrm{Ext}^2_R(R/I, S) \to \cdots$$

which implies that $\mathrm{Hom}_R(R/I,T)$ and $\mathrm{Ext}^1_R(R/I,T)$ are finitely generated. Since $\dim T \le 1$, $\mathrm{Ext}^i_R(R/I,T)$ is finitely generated for all $i \ge 0$ by [4, Lemma 2.6]. So, $\mathrm{Ext}^i_R(R/I,M)$ is finitely generated for all $i \ge 0$. $\square$

We are now ready to state and prove the first main theorem of the paper.

Theorem 3.3. *Let M be an R-module and $t \ge 1$ be a positive integer such that $\mathrm{Ext}^j_R(R/I,M)$ is finitely generated for all $j \le t$ and the R-module $H^i_I(M)$ belongs to $(I\text{-cof}, \mathcal{D}_{\le 1})$ for all $i < t$. Then the following statements hold:*

(i) *The R-module $H^i_I(M)$ is I-cofinite for all $i < t$.*

(ii) *For every submodule N of $H^t_I(M)$ that belongs to $(I\text{-cof}, \mathcal{D}_{\le 0})$, the R-module $\mathrm{Ext}^i_R(R/I, H^t_I(M)/N)$ is finitely generated for all $i = 0, 1$. In particular, the set $\mathrm{Ass}_R(H^t_I(M)/N)$ is finite.*

Proof. (i) We proceed by induction on t. Let $t = 1$. By assumption, $\Gamma_I(M)$ belongs to *(I-cof, $\mathcal{D}_{\le 1}$)* and $\mathrm{Hom}_R(R/I,\Gamma_I(M)) = \mathrm{Hom}_R(R/I,M)$ is finitely generated. So, in view of Proposition 3.2, it is enough to show that $\mathrm{Ext}^1_R(R/I,\Gamma_I(M))$ is finitely generated. Considering the exact sequence

$$0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$$

and the fact that $\mathrm{Hom}_R(R/I, M/\Gamma_I(M)) = 0$, we have the exact sequence

$$0 \to \mathrm{Ext}^1_R(R/I,\Gamma_I(M)) \to \mathrm{Ext}^1_R(R/I,M) \to \cdots .$$

Therefore, $\mathrm{Ext}^1_R(R/I,\Gamma_I(M))$ is finitely generated by assumption. Now, assume that $t > 1$ and the result has been proved for all $i < t$. By the inductive hypothesis, $t-1 \in \mathbb{N}_0$ and $H^i_I(M)$ is I-cofinite for all $i < t-1$. Hence, $\mathrm{Ext}^i_R(R/I, H^{t-1}_I(M))$ is finitely generated for all $i = 0, 1$, by Lemma 2.5 and assumption. Since $H^{t-1}_I(M) \in$ *(I-cof, $\mathcal{D}_{\le 1}$)*, $H^{t-1}_I(M)$ is I-cofinite by Proposition 3.2. This completes the inductive step.

(ii) In view of (i) and Lemma 2.5, the R-modules $\mathrm{Hom}_R(R/I, H^t_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^t_I(M))$ are finitely generated. Hence, considering the exact sequence

$$0 \to N \to H^t_I(M) \to H^t_I(M)/N \to 0,$$

implies that N is I-cofinite by Proposition 3.1 and assumption. Also, we obtain the following exact sequence:

$$\cdots \to \mathrm{Hom}_R(R/I, H^t_I(M)) \to \mathrm{Hom}_R(R/I, H^t_I(M)/N) \to \mathrm{Ext}^1_R(R/I,N)$$
$$\to \mathrm{Ext}^1_R(R/I, H^t_I(M)) \to \mathrm{Ext}^1_R(R/I, H^t_I(M)/N) \to \mathrm{Ext}^2_R(R/I,N) \to \cdots .$$

Therefore, $\mathrm{Hom}_R(R/I, H_I^t(M)/N)$ and $\mathrm{Ext}_R^1(R/I, H_I^t(M)/N)$ are finitely generated, as required. □

The next result provides a slight generalization of [3, Theorem 3.4], [5, Theorem 4.1], [10, Theorem 2.2], [25, Theorem B] and [30, Theorem 3.2].

Corollary 3.4. *Let M be an R-module and $t \geq 1$ be a positive integer such that $\mathrm{Ext}_R^j(R/I, M)$ is finitely generated for all $j \leq t$ and the R-module $H_I^i(M)$ is $\mathcal{FD}_{\leq 1}$ (or weakly Laskerian) R-module for all $i < t$. Then the following conditions hold:*

(i) *The R-module $H_I^i(M)$ is I-cofinite for all $i < t$.*

(ii) *For every $\mathcal{FD}_{\leq 0}$ (or minimax) submodule N of $H_I^t(M)$, the R-module $\mathrm{Ext}_R^i(R/I, H_I^t(M)/N)$ is finitely generated for all $i = 0, 1$.*

Proof. The assertion follows from Theorem 3.3 and the fact that any I-torsion $\mathcal{FD}_{\leq n}$ R-module is included in the class of (I-*cof*, $\mathcal{D}_{\leq n}$). □

The following result is a generalization of [8, Proposition 2.2].

Corollary 3.5. *Let M be a non-zero R-module and $t \geq 1$ be a positive integer such that $\mathrm{Ext}_R^j(R/I, M)$ is finitely generated for all $j \leq t$ and $\mathrm{Supp}_R(H_I^i(M)) \subseteq \mathrm{Max}(R)$ for all $i \leq t$. Then the R-module $H_I^i(M)$ is Artinian for all $i \leq t$.*

Proof. By Corollary 3.4, the R-module $\mathrm{Hom}_R(R/I, H_I^i(M))$ is finitely generated for all $i \leq t$. Since $\mathrm{Supp}_R(\mathrm{Hom}_R(R/I, H_I^i(M))) \subseteq \mathrm{Max}(R)$ for all $i \leq t$, it follows that $\mathrm{Hom}_R(R/I, H_I^i(M))$ is Artinian for all $i \leq t$. As $H_I^i(M)$ is an I-torsion R-module, it yields from [28, Theorem 1.3] that $H_I^i(M)$ is Artinian for all $i \leq t$. □

Corollary 3.6. *Let M be an R-module such that $\mathrm{Ext}_R^j(R/I, M)$ is finitely generated for all j and the R-module $H_I^i(M)$ belongs to (I-cof, $\mathcal{D}_{\leq 1}$) for all i. Then*

(i) *The R-module $H_I^i(M)$ is I-cofinite for all i.*

(ii) *For any $i \geq 0$ and for any submodule N of $H_I^i(M)$ that belongs to (I-cof, $\mathcal{D}_{\leq 0}$), the R-module $H_I^i(M)/N$ is I-cofinite for all i.*

Proof. The assertion follows from Proposition 3.2 and Theorem 3.3. □

As an immediate consequence of Corollary 3.6, we derive the following extension of [15, Theorem 1] and [31, Theorem 1.1], [7, Theorem 2.5], [8, Theorem 2.6], [23, Theorem 2.1] and [24, Main Theorem].

Corollary 3.7. *Let M be an R-module such that $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all j and $\dim H^i_I(M) \leq 1$ for all i. Then the R-module $H^i_I(M)$ is I-cofinite for all i. In particular, the Bass numbers and Betti numbers of local cohomology module $H^i_I(M)$ are finite for all $i \geq 0$.*

Proof. It follows from Corollary 3.6, [22, Lemma 4.2] and [29, Theorem 2.1]. □

The following result provides an extension of [1, Corollary 2.8].

Corollary 3.8. *Let M be an R-module such that $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all j and either $\dim R/I \leq 1$ or $\dim M \leq 1$. Then for each finitely generated R-module N, the R-module $\mathrm{Ext}^j_R(N, H^i_I(M))$ is I-cofinite for all $i \geq 0$ and all $j \geq 0$.*

Proof. It follows from Corollary 3.7, and [1, Theorem 2.7] and the fact that

$$\mathrm{Supp}_R(H^i_I(M)) \subseteq \mathrm{Supp}_R(M) \cap V(I).$$

□

The following lemma is needed in the proof of the second main result of this paper.

Lemma 3.9. *Let M be an R-module and i and n be some integers such that $\mathrm{Hom}_R(R/I, H^i_I(M))$ is a finitely generated R-module and $(H^i_I(M))_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Supp}_R(M) \cap V(I)$ with $\dim R/\mathfrak{p} > n$. Then $H^i_I(M)$ is $\mathcal{FD}_{\leq n}$.*

Proof. Let $A_s = (0 :_{H^i_I(M)} I^s)$. Then A_s is a finitely generated R-module by assumption and Lemma 2.2. Also, $\mathrm{Supp}_R(A_{s+2}/A_{s+1}) \subseteq \mathrm{Supp}_R(A_{s+1}/A_s)$ for all $s = 1, 2, \cdots$ by [9, Lemma 2.1]. Therefore, as $\mathrm{Spec}(R)$ is a Noetherian space, it follows that there exists a positive integer k such that for all $s \geq k+1$ we have

$$\mathrm{Supp}_R(A_{s+1}/A_s) = \mathrm{Supp}_R(A_{k+1}/A_k).$$

Now, we show that $\mathrm{Supp}_R(A_{k+1}/A_k) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leq n\}$. To this end, suppose the contrary that there exists $\mathfrak{p} \in \mathrm{Supp}_R(A_{k+1}/A_k)$ such that $\dim R/\mathfrak{p} > n$. Hence, by assumption and the fact that $\mathrm{Supp}_R(A_{k+1}/A_k) \subseteq \mathrm{Supp}_R(M) \cap V(I)$, we conclude that the $R_\mathfrak{p}$-module $(H^i_I(M))_\mathfrak{p}$ is finitely generated. Thus, there exists a finitely generated R-submodule N of $H^i_I(M)$ such that $(H^i_I(M))_\mathfrak{p} = N_\mathfrak{p}$. Since N is I-torsion, there is an integer $s \geq k+1$ such that $N \subseteq (0 :_{H^i_I(M)} I^s) = A_s$. Thus,

$$N_\mathfrak{p} \subseteq (A_s)_\mathfrak{p} \subseteq (A_{s+1})_\mathfrak{p} \subseteq (H^i_I(M))_\mathfrak{p} = N_\mathfrak{p}.$$

This implies that $(A_s)_\mathfrak{p} = (A_{s+1})_\mathfrak{p}$ and so $\mathfrak{p} \notin \mathrm{Supp}_R(A_{k+1}/A_k)$, a contradiction. Since $H^i_I(M) = \bigcup_{s=1}^{\infty} A_s$, it is easy to see that

$$\mathrm{Supp}_R(H^i_I(M)/A_k) = \mathrm{Supp}_R(A_{k+1}/A_k) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leq n\}.$$

Therefore, $\dim H^i_I(M)/A_k \leq n$ and so $H^i_I(M)$ is an $\mathcal{FD}_{\leq n}$ R-module, as required. □

Now we are prepared to prove the second main result of the paper.

Theorem 3.10. *Let M be an R-module and t be a positive integer such that $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all $j \leq t$. Then the following statements are equivalent:*

(i) *$H^i_I(M)$ is an $\mathcal{FD}_{\leq 1}$ R-module for all $i < t$;*

(ii) *$(H^i_I(M))_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for all $i < t$ and for all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$ with $\dim R/\mathfrak{p} > 1$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $H^i_I(M)$ is an $\mathcal{FD}_{\leq 1}$ R-module for all $i < t$. Then, there exists a finitely generated submodule N_i of $H^i_I(M)$ such that $\dim H^i_I(M)/N_i \leq 1$, for all $i < t$. Now, let $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$ such that $\dim R/\mathfrak{p} > 1$. Then $\mathfrak{p} \notin \operatorname{Supp}_R(H^i_I(M)/N_i)$ and so $(H^i_I(M)/N_i)_{\mathfrak{p}} = 0$ for all $i < t$. Hence, $(H^i_I(M))_{\mathfrak{p}} = (N_i)_{\mathfrak{p}}$ for all $i < t$. Therefore, $(H^i_I(M))_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for all $i < t$, as required.

$(ii) \Rightarrow (i)$. The proof is by induction on t. If $t = 1$, then $\operatorname{Hom}_R(R/I, H^0_I(M)) = \operatorname{Hom}_R(R/I, M)$ is finitely generated by assumption. Hence, $H^0_I(M)$ is $\mathcal{FD}_{\leq 1}$ by Lemma 3.9. Now, suppose that $t > 1$ and the assertion holds for $t-1$, we will prove it for t. By the inductive hypothesis, the R-module $H^i_I(M)$ is $\mathcal{FD}_{\leq 1}$ for all $i < t-1$. Thus, in the light of Corollary 3.4 we conclude that $H^i_I(M)$ is I-cofinite for all $i < t-1$ and $\operatorname{Hom}_R(R/I, H^{t-1}_I(M))$ is finitely generated. Therefore, $H^{t-1}_I(M)$ is an $\mathcal{FD}_{\leq 1}$ R-module by Lemma 3.9, as required. □

As an immediate consequence of Theorem 3.10 we have the following result.

Corollary 3.11. *Let M be an R-module such that $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all $j \leq t$ where t is the least integer such that $(H^i_I(M))_{\mathfrak{p}}$ is not finitely generated for some $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$ with $\dim R/\mathfrak{p} \geq 2$. Then the following statements hold:*

(i) *The R-module $H^i_I(M)$ is $\mathcal{FD}_{\leq 1}$ and I-cofinite for all $i < t$;*

(ii) *The R-module $H^t_I(M)$ is not $\mathcal{FD}_{\leq 1}$, whenever t is finite;*

(iii) *For every $\mathcal{FD}_{\leq 0}$ (or minimax) submodule N of $H^t_I(M)$, the R-module $\operatorname{Ext}^i_R(R/I, H^t_I(M)/N)$ is finitely generated for all $i = 0, 1$, whenever t is finite.*

Proof. (i) follows from Theorem 3.10 and Corollary 3.4. (ii) follows from the hypothesis and Proposition 3.10. (iii) follows from (i) and Corollary 3.4. □

Remark 3.12. In Corollary 3.11, if M is a finitely generated R-module, then $t = f_I^2(M)$ introduced in [9]. Therefore, Corollary 3.11 gives extra information rather than [9, Theorem 3.2] about local cohomology modules $H_I^i(M)$ for $i < f_I^2(M)$.

As the final main result of this paper, we prove the following theorem which is a generalization of [1, Corollary 2.8].

Theorem 3.13. *Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero R-module such that $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all j. If either $\dim R/I \leq 2$ or $\dim M \leq 2$, then for each finitely generated R-module N, the R-module $\operatorname{Ext}_R^j(N, H_I^i(M))$ is I-weakly cofinite, for all $i \geq 0$ and $j \geq 0$.*

Proof. Let $\Omega = \{\operatorname{Ext}_R^k(R/I, \operatorname{Ext}_R^j(N, H_I^i(M)))|i \geq 0, j \geq 0, k \geq 0\}$. Suppose that $K \in \Omega$ and K' is a submodule of K. By definition, it suffices to show that $\operatorname{Ass}_R(K/K')$ is finite. For this end, in view of the Flat Base Change Theorem [11, Theorem 4.3.2], [27, Ex. 7.7], [26, Lemma 2.1], without loss of generality, we can assume that R is complete.
Now, suppose that the contrary is true. So, there is a countably infinite subset $\{\mathfrak{p}_r\}_{r=1}^{\infty}$ of non-maximal elements of $\operatorname{Ass}_R(K/K')$. Therefore, $\mathfrak{m} \not\subseteq \bigcup_{r=1}^{\infty} \mathfrak{p}_r$ by [12, Lemma 3]. Let S be the multiplicatively closed subset $R \setminus \bigcup_{r=1}^{\infty} \mathfrak{p}_r$. Then it is easy to see that $\dim S^{-1}R/S^{-1}I \leq 1$ or $\dim S^{-1}M \leq 1$. Hence, it follows from [11, Theorem 4.3.2] and Corollary 3.8 that $\operatorname{Ext}_{S^{-1}R}^j(S^{-1}N, H_{S^{-1}I}^i(S^{-1}M))$ is $S^{-1}I$-cofinite. Therefore, $S^{-1}K/S^{-1}K'$ is a finitely generated $S^{-1}R$-module and so $\operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}K')$ is a finite set. But, $S^{-1}\mathfrak{p}_r \in \operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}K')$ for all $r = 1, 2, \cdots$, a contradiction. □

4. Acknowledgments

The author specially thanks the referee for the helpful suggestions and comments.

References

[1] R. Abazari and K. Bahmanpour, *Cofiniteness of extension functors of cofinite modules*, J. Algebra, **330** No. 1 (2011) 507-516.

[2] A. Abbasi and H. Roshan Shekalgourabi, *Serre subcategory properties of generalized local cohomology modules*, Korean J. Math., **28** No. 1 (2011) 25-37.

[3] M. Aghapournahr and K. Bahmanpour, *Cofiniteness of weakly laskerian local cohomology modules*, Bull. Math. Soc. Sci. Math. Roumanie, **105** No. 4 (2014) 347-356.

[4] ———, *Cofiniteness of general local cohomology modules for small dimensions*, Bull. Korean Math. Soc., **53** No. 5 (2016) 1341-1352.

[5] J. Asadollahi, K. Khashyarmanesh, and Sh. Salarian, *A generalization of the cofiniteness problem in local cohomology modules*, J. Aust. Math. Soc., **75** (2003) 313-324.

[6] K. Bahmanpour, *On the category of weakly laskerian cofinite modules*, Math. Scand., **115** No. 1 (2014) 62-68.

[7] K. Bahmanpour and R. Naghipour, *On the cofiniteness of local cohomology modules*, Proc. Amer. Math. Soc., **136** No. 7 (2008) 2359-2363.

[8] ______, *Cofiniteness of local cohomology modules for ideals of small dimension*, J. Algebra, **321** (2009) 1997-2011.

[9] K. Bahmanpour, R. Naghipour, and M. Sedghi, *Minimaxness and cofiniteness properties of local cohomology modules*, Comm. Algebra, **41** (2013) 2799-2814.

[10] M. P. Brodmann and F. A. Lashgari, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **128** (2000) 2851-2853.

[11] M. P. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, Cambridge, 1998.

[12] L. Burch, *Codimension and analytic spread*, Math. Proc. Camb. Phil. Soc., **72** No. 3 (1972) 369-373.

[13] G. Chiriacescu, *Cofiniteness of local cohomology modules*, Bull. London Math. Soc., **32** (2000) 1-7.

[14] D. Delfino, *On the cofiniteness of local cohomology modules*, Math. Proc. Camb. Phil. Soc., **115** (1994) 79-84.

[15] D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Algebra, **121** No. 1 (1997) 45-52.

[16] M. T. Dibaei and S. Yassemi, *Associated primes and cofiniteness of local cohomology modules*, Manuscripta Math., **117** No. 2 (2005) 199-205.

[17] ______, *Finiteness of extension functors of local cohomology modules*, Comm. Algebra, **34** (2006) 3097-3101.

[18] K. Divaani-Aazar and A. Mafi, *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **133** No. 3 (2005) 655-660.

[19] ______, *Associated primes of local cohomology modules of weakly Laskerian modules*, Comm. Algebra, **34** (2006) 681-690.

[20] A. Grothendieck, *Cohomologie locale des faisceaux et theoremes de lefshetz locaux et globaux (SGA2)*, 1968.

[21] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math., **9** No. 2 (1970) 145-164.

[22] C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Camb. Phil. Soc., **110** No. 3 (1991) 421-429.

[23] C. Huneke and R.Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Soc., **339** No. 17 (1993) 765-779.

[24] K.I. Kawasaki, *On the finiteness of bass numbers of local cohomology modules*, Proc. Amer. Math. Soc., **124** (1996) 3275-3279.

[25] K. Khashyarmanesh and Sh. Salarian, *On the associated primes of local cohomology modules*, Comm. Algebra, **27** No. 12 (1999) 6191-6198.

[26] T. Marley, *The associated primes of local cohomology modules over rings of small dimension*, Manuscripta Math., **104** (2001) 519-525.

[27] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, UK, 1986.

[28] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Camb. Phil. Soc., **107** (1990) 267-271.

[29] ______, *Modules cofinite with respect to an ideal*, J. Algebra, **285** (2005) 649-668.

[30] P. H. Quy, *On the finiteness of associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **6** (2010) 1965-1968.

[31] K. I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J., **147** (1997) 179-191.

[32] T. Yoshizawa, *Subcategories of extension modules by serre subcategories*, Proc. Amer. Math. Soc., **140** (2012) 2293-2305.

[33] H. Zöschinger, *Minimax-moduln*, J. Algebra, **102** (1986) 1-32.

Hajar Roshan-Shekalgourabi

Department of Basic Sciences, Arak University of Technology,

P. O. Box 38135-1177, Arak, Iran.

hrsmath@gmail.com