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Research Paper

A VARIATION OF δ-LIFTING AND δ-SUPPLEMENTED MODULES WITH RESPECT TO AN EQUIVALENCE RELATION

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Abstract. In this paper we introduce Goldie*-δ-supplemented modules as follows. A module M is called Goldie*-δ-supplemented (briefly, G^*_δ-supplemented) if there exists a δ-supplement T of M for every submodule A of M such that $A\beta^*_\delta T$. We say that a module M is called Goldie*-δ-lifting (briefly, G^*_δ-lifting) if there exists a direct summand D of M for every submodule A of M such that $A\beta^*_\delta D$. Note that the last concept given in [4] as a δ-H-supplemented module. We present fundamental properties of these modules. We indicate that these modules lie between δ-lifting and δ-supplemented modules. Also we prove that our modules coincide with some variations of δ-supplemented modules for δ-semiperfect modules.

1. Introduction

Throughout this study, R denotes an associative ring with identity and M denotes a unitary left R-module. The notations $A \leq M$ and $A \leq_\oplus M$ point that A is a *submodule* of M and

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A is a *direct summand* of M, respectively. A submodule A is called *small* in M (denoted by $A \ll M$), if $A + X \neq M$ for any proper submodule X of M. A *supplement* submodule T of A in M provides that $A + T = M$ and $A \cap T \ll T$. A module M is called *supplemented* if every submodule of M has a supplement in M. M is called an *amply supplemented module* if for every submodule A of M with $M = A + B$ there exists a supplement submodule T of A contained in B. A module M is called *lifting* if, for any submodule A of M there exists a decomposition $M = X \oplus Y$ such that $X \leq A$ and $A \cap Y \ll Y$ [3]. And, M is called a $\oplus$-*supplemented module* if every submodule of M has a supplement which is a direct summand in M [10]. Also, in [10], a module M is called H-supplemented if for any $A \leq M$, there exists a submodule $D \leq_{\oplus} M$ such that $M = A + X$ if and only if $M = D + X$ for any $X \leq M$.

An *essential submodule* A of M provides that $\{0\}$ is the only submodule of M whose intersection with A is zero. A module M is called *singular* (*non-singular*) $Z(M) = M$ ($Z(M) = 0$) where $Z(M) = \{m \in M \mid Ann(m) \trianglelefteq R\}$. In [22] and in [9] new generalizations of small submodules, lifting modules and supplemented modules are introduced via singularity as follows. A δ-*small submodule* A of M is a submodule satisfying $A + X \neq M$ for every proper submodule X of M where $\frac{M}{X}$ is singular. We use the notation $\delta(M)$ for the sum of all δ-small submodules of M. Let φ be the class of all singular simple modules. For a module M, $\delta(M) = \cap\{N \leq M \mid \frac{M}{N} \in \varphi\}$. A submodule T of M is called a δ-*supplement* of A in M if $A + T = M$ and $A \cap T \ll_{\delta} T$. M is called δ-supplemented if every submodule of M has a δ-supplement in M. A module M is called δ-*lifting*, if for any submodule A of M there exists a decomposition $M = X + Y$ such that $X \leq A$ and $A \cap Y \ll_{\delta} Y$. If every submodule of M has a δ-supplement which is a direct summand of M, then M is called a $\oplus$-δ-*supplemented module* [18]. In [6], a module M is called δ-*H-supplemented*, if for any $A \leq M$ there exists a submodule $D \leq_{\oplus} M$ such that $M = A + X$ if and only if $M = D + X$ for every $X \leq M$ with $\frac{M}{X}$ is singular.

In [2], the authors defined an equivalence relation β^* and defined G^*-supplemented and G^*-lifting modules via this relation. Therefore, two new algebraic structures are contributed between lifting and supplemented modules. Owing to this fact, the open problem given as 'Is every H-supplemented module supplemented?' in [10] is handled. Thus, the following implications are obtained between some variations of supplemented modules and theirs such that:

$$lifting \Longrightarrow G^* - lifting \Longrightarrow H\text{-}supplemented \Longrightarrow G^*\text{-}supplemented \Longrightarrow supplemented$$

and also we have the ralation

$$lifting \Longrightarrow amply\ supplemented \Longrightarrow G^*\text{-}supplemented$$

In [7], motivated by the equivalence relation β^*, the authors defined the relation β^*_δ as an extended alternative to β^*. For $A, B \leq M$, it is said that A *equivalents to* B *with respect to* β^*_δ if and only if $\frac{A+B}{A} \ll_\delta \frac{M}{A}$ and $\frac{A+B}{B} \ll_\delta \frac{M}{B}$. A module M is called *principally Goldie**-δ-*lifting* (*principally Goldie**-δ-*supplemented*) if for any cyclic submodule Rm of M, there exists a direct summand D (δ-supplement T) in M such that $Rm\beta^*_\delta D$ ($Rm\beta^*_\delta T$). In [6], H-supplemented modules are designed according to the singularity. A module M is called δ-H-*supplemented* if every $A \leq M$ there exists a submodule $D \leq_\oplus M$ such that $M = A + X$ if and only if $M = D + X$ for any $X \leq M$ with $\frac{M}{X}$ is singular. Also we recommend [16] and [17] as a source to get more information about fundamental concepts used in this study.

In this study, inspired from [6] and from the equivalence relation β^*_δ given in [7], we generalize G^*-lifting and G^*-supplemented modules using singularity. We say that a module M is G^*_δ-lifting (G^*_δ-supplemented), if for any submodule A of M there exists a direct summand D (δ-supplement T) in M such that $A\beta^*_\delta D$ ($A\beta^*_\delta T$). By means of these concepts we obtain two new algebraic structures between δ-lifting and δ-supplemented modules. We indicate that δ-H-supplemented modules coincide with G^*_δ-supplemented modules. Also, we prove that our modules coincide with some variations of δ-supplemented modules for δ-semiperfect modules.

2. Preliminaries

Definition 2.1. Given submodules $A \leq B \leq M$, the inclusion $A \leq B$ is called δ-cosmall in M if $\frac{B}{A} \ll_\delta \frac{M}{A}$, denoted by $A \hookrightarrow_{\delta\text{-}cs} B$ [15].

Definition 2.2. Let M be a module and A, $B \leq M$. The submodule A is called β^*_δ equivalent to B (denoted by $A\beta^*_\delta B$) if $\frac{A+B}{A} \ll_\delta \frac{M}{A}$ and $\frac{A+B}{B} \ll_\delta \frac{M}{B}$.

It can be seen from [7, Lemma 3.2] that the relation given above is an equivalence relation.

Theorem 2.3. *Let* A, $B \leq M$. *Then the following statements are equivalent:*

i. $A\beta^*_\delta B$
ii. $A \hookrightarrow_{\delta\text{-}cs} A + B$ and $B \hookrightarrow_{\delta\text{-}cs} A + B$.
iii. For every $X \leq M$ with $\frac{M}{X}$ is singular, if $A + B + X = M$ then $A + X = M$ then $B + X = M$.
iv. If $X \leq M$ with $\frac{M}{X}$ is singular and $A + X = M$ then $B + X = M$ and, if $X \leq M$ with $\frac{M}{X}$ is singular and $B + X = M$ then $A + X = M$.

Proof. $(i) \Longrightarrow (ii)$ Let $A\beta^*_\delta B$. Therefore, we have $\frac{A+B}{A} \ll_\delta \frac{M}{A}$ and $\frac{A+B}{B} \ll_\delta \frac{M}{B}$, that is, $A \hookrightarrow_{\delta\text{-}cs} A + B$ and $B \hookrightarrow_{\delta\text{-}cs} A + B$.

$(ii) \Longrightarrow (iii)$ By assumption, it can be written that $\frac{A+B}{B} + \frac{X+B}{B} = \frac{M}{B}$. As $\frac{\frac{M}{X}}{\frac{B+X}{X}} \cong \frac{M}{B+X}$ is singular, we have $B + X = M$ is obtained. By the same way $A + X = M$ can be verified.

$(iii) \iff (iv)$ Let $A + X = M$ for $X \leq M$ with $\frac{M}{X}$ is singular. By hypothesis, we get $B + X = B$ because $A + B + X = M$. Similarly, $A + X = M$ can be shown whenever $B + X = M$ for $X \leq M$ with $\frac{M}{X}$ is singular. Conversely, let $A + B + X = M$ such that $\frac{M}{X}$ is singular. Since $A + (B + X) = M$. and $\frac{M}{B+X}$ is singular, then $B + (B + X) = B + X = M$ is obtained from the assumption. Similarly, $A + X = M$ is shown. □

Corollary 2.4. *Let A, $B \leq M$ such that $A \leq X + B$ and $B \leq Y + A$, where X, $Y \ll_\delta M$. Then $A\beta^*_\delta B$.*

Proof. Let $A + B + T = M$ for $T \leq M$ with $\frac{M}{T}$ is singular. Since $A \leq X + B$, then we have $(X + B) + B + T = X + B + T = M$. It follows that $B + T = M$ as $X \ll_\delta M$ and $\frac{M}{B+T}$ is singular as a factor module of a singular module $\frac{M}{T}$. Moreover, using the fact $Y \ll_\delta M$, it can be shown that $A + T = M$ and so, $A\beta^*_\delta B$ is obtained from Theorem 2.3. □

Proposition 2.5. *If A, B, $X \leq M$ such that $M = A + X = B + X$, $B \cap X \leq A \cap X$ and $B \hookrightarrow_{\delta\text{-}cs} A + B$, then $A \hookrightarrow_{\delta\text{-}cs} A + B$, so $A\beta^*_\delta B$.*

Proof. It can be proved similar to that of [2, Proposition 2.5] using [22, Lemma 1.2]. □

Proposition 2.6. *Let P, $T \leq M$ where P is maximal such that $\frac{M}{P} \in \varphi$.*

i. *Let A, $B \leq M$ such that $A + B = M$, B is proper in M with $\frac{M}{B}$ is singular and $T\beta^*_\delta A$. Then T is not contained in B.*
ii. *If $T\beta^*_\delta K$ and $T \leq P$, then $K \leq P$.*
iii. *If $T\beta^*_\delta P$, then $T \leq P$. And, if $T\beta^*_\delta K$ then $T \leq \delta(M)$ if and only if $Y \leq \delta(M)$.*

Proof. i. Assume that $T \leq B$. By assumption, we have $A + B + T = M$. Then, $B + T = M$ is obtained from Theorem 2.3 since $\frac{M}{B}$ is singular. Hence, we get the contradiction $B = M$.

ii. Suppose that K is not contained in P. By maximality of P we have $K + P = M$ and so, $T + K + P = M$. As $\frac{M}{P} \in \varphi$ and $T\beta^*_\delta K$, we have $T + P = M$ from Theorem 2.3. Thus, $P = M$ is obtained which is a contradiction.

iii. By (*ii*), taking P instead of K, we get $T \leq P$ as $T\beta^*_\delta P$ and $P \leq P$. □

Proposition 2.7. *Let A, B, C, $D \leq M$ such that $A\beta^*_\delta C$ and $C\beta^*_\delta D$. Then, $(A+C)\beta^*_\delta(B+D)$ and $(A + D)\beta^*_\delta(B + C)$.*

Proof. Let $X \leq M$ with $\frac{M}{X}$ is singular and $(A+C)+(B+D)+X = M$. Then we have $C+B+D+X = M$ and $A+C+D+X = M$ as $\frac{\frac{M}{X}}{\frac{C+D+X}{X}} \cong \frac{M}{C+D+X}$ is singular and $A\beta^*_\delta B$. Following, $B+D+X = M$ and $A+C+X = M$ as $\frac{M}{B+X}$, $\frac{M}{A+X}$ is singular and $C\beta^*_\delta D$. Hence, $(A+C)\beta^*_\delta(B+D)$ is obtained. Similarly, $(A+D)\beta^*_\delta(B+C)$ can be shown from the symmetry of β^*_δ. □

Corollary 2.8. *Let A, B_1, B_2,..., $B_n \leq M$. If $A\beta^*_\delta B_i$ for each $i = 1,.2,...,n$, then $A\beta^*_\delta B_1 + B_2 + ... + B_n$.*

Remark 2.9. The result given in Proposition 2.7 can not be extended to infinite sums. Let us consider that the $\mathbb{Z}$-module Q. It is a known fact that $\delta(\mathbb{Q}) = \mathbb{Q} = \Sigma_{n\in\mathbb{Z}^+}\frac{1}{n}\mathbb{Z}$ where $\frac{1}{n}\mathbb{Z} \ll_\delta \mathbb{Q}$ for each integer n. Clearly, $\frac{1}{n}\mathbb{Z}\ \beta^*_\delta 0$ for each integer n. If the contrast of the claim would be true, then $\Sigma_{n\in\mathbb{Z}^+}\frac{1}{n}\mathbb{Z}\beta^*_\delta 0 = \mathbb{Q}\beta^*_\delta 0$ and so, $\mathbb{Q} \ll_\delta \mathbb{Q}$ is a contradiction.

Definition 2.10. Let $A \leq M$. Then $\beta^*_\delta(A) = \Sigma\{N \leq M \mid A\beta^*_\delta N\}$.

Note that $\beta^*_\delta(0) = \delta(M)$. On the other hand, let $A \leq P$ where $\frac{M}{P} \in \varphi$ which is the set of all singular simple modules. If $A\beta^*_\delta N$, then $N \leq P$ from Proposition 2.6. Hence, $\beta^*_\delta(A) \leq P$. Also, if $A\beta^*_\delta B$, then $\beta^*_\delta(A) = \beta^*_\delta(B)$.

3. Goldie$^*_\delta$-Lifting Modules and Goldie$^*_\delta$-Supplemented Modules

Definition 3.1. A module M is called Goldie$^*_\delta$-lifting (briefly, G^*_δ-lifting) if and only if for each $A \leq M$ there exists a direct summand D of M such that $A\beta^*_\delta D$.

Recall from [6] that a module M is called δ-H-supplemented if for every submodule A of M there exists a direct summand D of M such that $M = A+X$ if and only if $M = D+X$ for any $X \leq M$ with $\frac{M}{X}$ is singular. Let us indicate that this concept is the same with the definition given above. In view of brevity, we will use the term of G^*_δ-lifting for a this type of module. As it is possible to see the other fundamental properties of them in [6], we will omit them and give another ones.

Definition 3.2. A module M is called Goldie$^*_\delta$-supplemented (briefly, G^*_δ-supplemented) if and only if for each $A \leq M$ there exists a δ-supplement D of M such that $A\beta^*_\delta D$.

Note that if M is a singular module or M has no projective submodule, then the concepts of being G^*-supplemented (G^*-lifting) and G^*_δ-supplemented (G^*_δ-lifting) coincide. In particular, a $\mathbb{Z}$-module M is G^*-supplemented (G^*-lifting) if and only if M is G^*_δ-supplemented (G^*_δ-lifting).

Proposition 3.3. *Let M be a δ-hollow module. Then, M is G^*_δ-lifting.*

Proof. Let X be any submodule of M. From the assumption, $X \ll_\delta M$. Therefore, $X\beta_\delta^* 0$ is obtained where $\{0\}$ is a direct summand of M. Hence, M is G_δ^*-lifting. □

Proposition 3.4. *Every semisimple module is G_δ^*-lifting.*

Proof. Let $A \leq M$. As M is semisimple, there exists a submodule B of M such that $M = A \oplus B$. From the symmetry of β_δ^* we have $A\beta_\delta^* B$. Hence, M is G_δ^*-lifting. □

Proposition 3.5. *Let M be a G_δ^*-lifting module and $A \leq M$. If $\frac{A+D}{A} \leq_\oplus \frac{M}{A}$ for any $D \leq_\oplus M$, then $\frac{M}{A}$ is G_δ^*-lifting.*

Proof. Let $\frac{X}{A} \leq \frac{M}{A}$. Since M is G_δ^*-lifting, then there exists a decomposition $M = D \oplus D^{'}$ such that $X\beta_\delta^* D$. Let $\pi : M \longrightarrow \frac{M}{A}$ be the canonical epimorphism. Then, $\pi(X)\beta_\delta^*\pi(D)$ and so, $\frac{X}{A}\beta_\delta^*\frac{D+A}{A}$ is obtained. Hence, M is G_δ^*-lifting from the hypothesis. □

If the sum of any two direct summands of M is a direct summand, then M has the summand sum property.

Proposition 3.6. *Let M be a G_δ^*-lifting module. If M has the summand sum property, then any direct summand of M is G_δ^*-lifting module.*

Proof. Let $D \leq_\oplus M$. Then $M = D \oplus D^{'}$ for some $D^{'} \leq M$. We will verify that $\frac{M}{D^{'}}$ is G_δ^*-lifting. With this aim, we will show that for any $X \leq_\oplus M$, $\frac{X+D^{'}}{D^{'}} \leq_\oplus \frac{M}{D^{'}}$. From the assumption, as $D^{'}$ and X are direct summands of M, then $X + D^{'} \leq_\oplus M$. Therefore, there is a submodule T of M such that $M = (X + D^{'}) \oplus T$. It follows that, $\frac{M}{D^{'}} = \frac{X+D^{'}}{D^{'}} + \frac{T+D^{'}}{D^{'}}$. Moreover, we get $(X + D^{'}) \cap (T + D^{'}) = [(X + D^{'}) \cap T] + D^{'} = 0 + D^{'} = D^{'}$ from modularity. Thus, $\frac{M}{D^{'}} = \frac{X+D^{'}}{D^{'}} \oplus \frac{T+D^{'}}{D^{'}}$. Hence, $D \cong \frac{M}{D^{'}}$ is G_δ^*-lifting from Proposition 3.5. □

Proposition 3.7. *Let M be a π-projective module and let us consider the following statements.*

(1) *M is $\oplus$-δ-supplemented.*

(2) *M is G_δ^*-lifting.*

Then (1) $\Longrightarrow$ (2) holds. In particular, if M is singular, then the converse is provided.

Proof. (1) $\Longrightarrow$ (2) : Let A be any submodule of M. By (1), there exists a δ-supplement T of M which is a direct summand of M such that

$$M = A + T,\ A \cap T \ll_\delta T \text{ and } M = T \oplus T^{'}$$

As M is π-projective, there is a submodule $X \leq A$ provided that $M = X \oplus T$. Clearly, $\frac{X+A}{A} \ll_\delta \frac{M}{A}$. Moreover, as $A = X \oplus (A \cap T)$ from modularity and $A \cap T \ll_\delta T$, we obtain $\frac{A}{X} = \frac{X+A}{X} \ll_\delta \frac{M}{X}$, that is, $A\beta^*_\delta X$ where $X \leq_\oplus M$. Hence, M is G^*_δ-lifting.

Let M be a singular G^*_δ-lifting module and $A \leq M$. Then, there exists a direct summand D of M such that $M = D \oplus D'$ for some $D' \leq M$ and $A\beta^*_\delta D$. Therefore, $\frac{D+A}{D} \ll_\delta \frac{M}{D}$ and $\frac{D+A}{A} \ll_\delta \frac{M}{A}$. As $\frac{M}{A} = \frac{D+A}{A} + \frac{D'+A}{A}$ and $\frac{M}{D+A} \cong \frac{\frac{M}{A}}{\frac{D+A}{A}}$ is singular, then $M = A + D'$ is obtained. Now, it remains to show that $A \cap D' \ll_\delta D'$. Let $(A \cap D') + B = D'$ where $\frac{D'}{B}$ is singular. Thus, $M = (A \cap D') + B + D = A + B + D$ and so $\frac{M}{D} = \frac{A+D}{D} + \frac{B+D}{D}$ is obtained. As $\frac{D+A}{D} \ll_\delta \frac{M}{D}$ and M is singular, we have $M = B + D$ and so $D' = M \cap D' = (B + D) \cap D' = B + (D \cap D') = B$ from modularity. This completes the proof. Hence, M is a $\oplus$-δ-supplemented module. □

Theorem 3.8. *Let M be a singular G^*_δ-lifting module. Then M is δ-supplemented.*

Proof. Let $A \leq M$. By hypothesis, there exists a direct summand D of M such that $M = D \oplus D'$ and $A\beta^*_\delta D$. Then, $\frac{A+D}{D} \ll_\delta \frac{M}{D}$, $\frac{A+D}{A} \ll_\delta \frac{M}{A}$ and we have $\frac{M}{A} = \frac{A+D}{A} + \frac{A+D'}{A}$. As M is singular, $M = A + D'$ is obtained. To complete the proof it must be shown that $A \cap D' \ll_\delta D'$. Let $A \cap D' + B = D'$ with $\frac{D'}{B}$ is singular. Then, $M = D' + D = [A \cap D' + B] + D = A + B + D$ and so, $\frac{M}{D} = \frac{A+D}{D} + \frac{B+D}{D}$ is obtained. By hypothesis, we get $M = B + D$ and so, $D' = B$ by modularity. Hence, M is δ-supplemented. □

Theorem 3.9. *Let M be a π-projective module. If M is G^*_δ-supplemented, then it is G^*_δ-lifting.*

Proof. Let $A \leq M$. Then, there exists a δ-supplement T of M such that $A\beta^*_\delta T$. Assume that T is a δ-supplement of W in M, that is $W + T = M$ and $W \cap T \ll_\delta T$. As M is π-projective, there exists a direct summand T' of M contained in T such that $M = T' \oplus W$. Now, we aim to verify that $A\beta^*_\delta T'$. It is clear that, $\frac{A+T'}{A} \leq \frac{A+T}{A} \ll_\delta \frac{M}{A}$ by [22, Lemma 1.3]. In the remaining part of the proof we will show that $\frac{A+T'}{T'} \ll_\delta \frac{M}{T'}$. Suppose that, $\frac{A+T'}{T'} + \frac{B}{T'} = \frac{M}{T'}$ with $\frac{M}{B}$ is singular. Then $M = A + T' + B = A + T + B$ and so $\frac{M}{T} = \frac{A+T}{T} + \frac{B+T}{T}$. As $A\beta^*_\delta T$ and $\frac{\frac{M}{B}}{\frac{B+T}{B}} \cong \frac{M}{B+T}$ is singular, then $M = B + T$. Moreover, we have $T = T' \oplus (W \cap T)$ by modularity. Thus, $M = B + T = B + [T' + (W \cap T)] = (B + T') + W \cap T$. As $W \cap T \ll_\delta M$ and $\frac{M}{B+T}$ is singular, $M = B + T'$ and so, $M = B$ is obtained due to the fact that $T' \leq B$. Hence the proof is completed. □

Proposition 3.10. *Let M be a Noetherian module which has the summand sum property. Then M is principallly G^*_δ-lifting if and only if M is G^*_δ-lifting.*

Proof. The sufficiency is cleear. For the necessity, let $A \leq M$. As M is Noetherian, A is finitely generated and so $A = Rx_1 + Rx_2 + ... + Rx_n$ for some $x_1, x_2, ..., x_n \in M$. Since M is principally G^*_δ-lifting, there exists direct summands $D_1, D_2, ..., D_n$ of M such that $Rx_1\beta^*_\delta D_1$, $Rx_2\beta^*_\delta D_2, ..., Rx_n\beta^*_\delta D_n$. Then, $A\beta^*_\delta D = D_1 + D_2 + ... + D_n$ where $D \leq_\oplus M$ since M has the summand sum property. Hence, M is G^*_δ-lifting. □

Proposition 3.11. *Let M be a module and $A \leq M$ such that $A = C + S$ where C is cyclic in M and $S \ll_\delta M$ for any $A \leq M$. Then M is principally G^*_δ-lifting and G^*_δ-lifting.*

Proof. Let $A = C + S$ for a cyclic submodule C of M and $S \ll_\delta M$. As M is principally G^*_δ-lifting, then a direct summand D of M corresponds to C such that $C\beta^*_\delta D$. Therefore, $A = (C + S)\beta^*_\delta D$ by [7, Lemma 3.6] which implies M is G^*_δ-lifting. The sufficiency is clear from implications. □

Proposition 3.12. *Let M be a module and A be any submodule of M. If there exists a δ-supplement (direct summand) T and a δ-small submodule S of M such that $A + S = T + S$, then M is a G^*_δ-supplemented (G^*_δ-lifting) module.*

Proof. From assumption, it remains to show that $A\beta^*_\delta T$. Since $A \leq A + S = T + S$, $T \leq T + S = A + S$ and $S \ll_\delta M$, then we have $A\beta^*_\delta T$ from Corollary 2.4. □

Corollary 3.13. *Let M be a module and A be any submodule of M. If there exists a δ-supplement T and a δ-small submodule S of M such that $A = T + S$, then M is a G^*_δ-supplemented module.*

Theorem 3.14. *Let M be a module and consider the statements given below,*

a. *M is δ-lifting.*
b. *M is G^*_δ-lifting.*
c. *M is δ-H-supplemented.*
d. *M is G^*_δ-supplemented.*

Then, $(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longrightarrow (d)$.

Proof. $(a) \Longrightarrow (b)$ Let M be a δ-lifting module. Then, there exists a direct summand D for any submodule A of M satisfying $\frac{A}{D} \ll_\delta \frac{M}{D}$. Therefore, it can be written that $\frac{A+D}{D} \ll_\delta \frac{M}{D}$ and $\frac{A+D}{A} = 0 \ll_\delta \frac{M}{A}$ which implies $A\beta^*_\delta D$. Hence, we obtain the existence of a direct summand D for every submodule A of M such that $A\beta^*_\delta D$, that is, M is G^*_δ-lifting.

$(b) \Longleftrightarrow (c)$ This fact is clear from [6, Lemma 2.2].

$(c) \Longrightarrow (d)$ is clear because every direct summand is a δ-supplement. □

Proposition 3.15. *Let M be a module whose submodules are of δ-supplements which are relatively projective direct summands of M. Then, M is G^*_δ-lifting.*

Proof. Let $A \leq M$. Then, there is a δ-supplement T of M such that $M = A+T$, $A \cap T^{'} \ll_\delta T^{'}$ and $M = T \oplus T^{'}$ where T, $T^{'}$ are relatively projective. It follows that $M = B \oplus T$ for some $B \leq A$ since $T^{'}$ is T-projective from [10, Lemma 4.47]. Therefore, M is δ-lifting. Hence, M is G^*_δ-lifting from Theorem 3.14. □

Proposition 3.16. *Let M be a π-projective and singular module. Then the following statements hold equivalently.*

(1) *M is δ-lifting.*
(2) *M is G^*_δ-lifting.*
(3) *M is $\oplus$-δ-supplemented module.*

Proof. (1) $\Longrightarrow$ (2) : is clear from Theorem 3.14.

(2) $\Longleftrightarrow$ (3) : is clear from Proposition 3.7.

(3) $\Longrightarrow$ (1) : Let $A \leq M$. From assumption, there exists a direct summand D of M such that $M = D \oplus D^{'}$, $A + D = M$ and $A \cap D \ll_\delta D$. On the other hand, as M is π-projective and $D \leq_\oplus M$, then there exists a direct summand $A^{'}$ of M contained A such that $M = A^{'} \oplus D$ from [3, 4.14(1)]. Thus, for every $A \leq M$, there exists a decomposition $M = A^{'} \oplus D$ such that $A^{'} \leq A$ and $A \cap D \ll_\delta D$. Hence M is δ-lifting. □

Proposition 3.17. *Let M be a singular π-projective module. Then, M is G^*_δ-lifting if and only if every submodule of M is a direct sum of a direct summand of M and a δ-small submodule of M.*

Proof. ($\Longrightarrow$) Let M be a G^*_δ-lifting module, then M is a δ-lifting module from Proposition 3.16. Then for any $A \leq M$, there exists a decomposition $M = D \oplus D^{'}$ such that $D \leq A$ and $A \cap D^{'} \ll_\delta M$. It follows that $A = D \oplus (A \cap D^{'})$ where $D \leq_\oplus M$ and $S = A \cap D^{'} \ll_\delta M$.

($\Longleftarrow$) For the necessity, it can be said that M is δ-lifting from [9, Lemma 2.3(b)]. Hence, M is G^*_δ-lifting by Theorem 3.14. □

Proposition 3.18. *LetR be a left non-singular ring, M be a left G^*_δ-supplemented R-module and P be a maximal submodule of M with $\frac{M}{P}$ is singular. If T is a δ-supplement of P with $\frac{M}{T}$ is singular, then $P = S + (P \cap T)$, where S is a δ-supplement of T and T is δ-local.*

Proof. Let M be a G^*_δ-supplemented module. Then, there exists a δ-supplement submodule S corresponding to P satisfying $P\beta^*_\delta S$. By hypothesis, T is a δ-supplement of S. Therefore, we have $P = S+(P\cap T)$ from [7, Theorem 3.7]. Moreover, since T is a δ-supplement submodule of the maximal submodule P, then T is δ-local or semisimple projective from [19, Lemma 2.22]. If T is semisimple projective, then $T \ll_\delta T \leq M$. On the other hand, as T is a δ-supplement of P in M, $P + T = M$ and $P \cap T \ll_\delta T$. Since $T \ll_\delta M$ and $\frac{M}{P}$ is singular, then $P = M$ is got which contradicts with maximality of P in M. Hence, it forces T to be δ-local. □

Example 3.19. Let $R = \frac{\mathbb{Z}}{8\mathbb{Z}}$ and $M = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{8\mathbb{Z}}$. It is a known fact from [9, Example 2.2(2)] that M is not a δ-lifting module. On the other hand, M is a G^*_δ-lifting module as it is G^*-lifting [2, Example 3.9(i)].

Example 3.20. Let $M = \mathbb{F}\oplus\mathbb{F}$ where $\mathbb{F}$ is a quotient field of a DVR R which is not complete. Then it can be seen that clearly M is a δ-supplemented module which is not G^*_δ-supplemented from [3, Example 23.7] and [2, Example 3.9(iii)].

Definition 3.21. A δ-supplemented module M is called *strongly* $\oplus$-δ-supplemented if every δ-supplement submodule of M is a direct summand of M.

Clearly, every δ-lifting module is strongly $\oplus$-δ-supplemented.

Proposition 3.22. *Let M be a module.*

i. *M is δ-lifting if and only if M is amply δ-supplemented and strongly $\oplus$-δ-supplemented.*
ii. *If M is G^*_δ-supplemented and strongly $\oplus$-δ-supplemented, then M is G^*_δ-lifting.*

Proof. *i.*The implication is clear from [1, Proposition 4.2] and [12, Lemma 2.3].

ii. Let A be any submodule of M. By assumption, there is a δ-supplement submodule X of M provided that $A\beta^*_\delta X$. As M is strongly $\oplus$-δ-supplemented X is a direct summand of M. Hence, M is G^*_δ-lifting. □

Proposition 3.23. *Let M be a module.*

i. *M is amply δ-supplemented.*
ii. *For each $A \leq M$ there there is a δ-supplement T and a submodule X of M such that $M = T + X = A + X$, $T + X \leq A + X$ and $T \hookrightarrow_{\delta\text{-}cs} A + T$.*
iii. *M is G^*_δ-supplemented.*

Then, the condition given above implies that $(i) \Longrightarrow (ii) \Longrightarrow (iii)$.

Proof. $(i) \Longrightarrow (ii)$ It is clear from [15, Theorem 3.7].

$(ii) \Longrightarrow (iii)$ By hypothesis, we have $A\beta^*_\delta T$ from Proposition 2.5. Hence, M is G^*_δ-supplemented. □

Proposition 3.24. *Let M be a projective module. Then the following statements are equivalent:*

i. *M is δ-semiperfect.*
ii. *M is δ-lifting.*
iii. *M is $\oplus$-δ-supplemented.*
iv. *M is amply δ-supplemented.*
v. *M is δ-supplemented.*
vi. *M is G^*_δ-supplemented.*
vii. *M is G^*_δ-lifting.*

Proof. It can be seen clearly via Theorem 3.14, Proposition 3.23 and [12, Lemma 2.4]. □

The following hierarchy is valid for given modules below.

$$\delta\text{-}lifting \Longrightarrow G^*_\delta\text{-}lifting \Longrightarrow principally\ G^*_\delta\text{-}lifting$$

Now we will verify the converse implications given above are not provided.

Example 3.25. Let us consider $\mathbb{Z}$-module $\mathbb{Q}$. Since every finitely generated submodule of $\mathbb{Z}$-module $\mathbb{Q}$ is δ-small in $\mathbb{Q}$, then ${}_{\mathbb{Z}}\mathbb{Q}$ is a principally G^*_δ-lifting module. On the other hand, it is not G^*_δ-lifting as it is not δ-supplemented.

Example 3.26. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$. M is a G^*_δ-lifting module which is not δ-lifting [8].

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