Research Paper

WEAKLY PRIMARY SEMI-IDEALS IN POSETS

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ABSTRACT. One of the main goals of science and engineering is to avail human beings cull the maximum propitious decisions. To make these decisions, we need to ken human being's predictions, feasible outcomes of various decisions, and since information is never absolutely precise and accurate, we need to withal information about the degree of certainty. All these types of information will lead to partial orders. A partially ordered set (or poset) theory deals with partial orders and plays a major role in real life. It has wide range of applications in various disciplines such as computer science, engineering, medical field, science, modeling spatial relationship in geographic information systems (GIS), physics and so on. In this paper, we mainly focus on weakly primary semi-ideal of a poset. We introduce the concepts of weakly primary semi-ideal and weakly Q-primary semi-ideal for some prime Q of a poset P and characterize weakly primary semi-ideals of P in terms of prime and primary semi-ideals of P.

1. Preliminaries

A poset is a set together with a partial order relation satisfying reflexive, antisymmetric, and transitive property. Algebraic ordered structures play a great role in mathematics with various applications in many disciplines such as theoretical computer sciences, decision making theory, control engineering, social science, information sciences, coding theory, physics and so on.

In [9], Venkatanarasimhan introduced the notion of semi-ideal of a poset and obtained several important results related to semi-ideals. In [8], Halaš introduced two different kinds of order ideals, namely, k-closed ideals and n-prime ideals of P and shown that an order-ideal is n-closed if and only if it is $(n+1)$-prime. In [4], Elavarasan et al. viewed the structure of P in two ways. Firstly, with respect to the Zariski topology, it was shown that $Spec(P)$ and $Max(P)$ are compact space and compact $T1$ subspace respectively. Secondly, we studied the semi-ideal-based zero-divisor graph $G_I(P)$, for a semi-ideal I of P, and characterized its diameter. In [7], Porselvi et al. defined the notion of z-semi-ideals of P and shown that every minimal prime semi-ideal containing J is z-semi-ideal under certain condition, and characterized union of prime semi-ideals of P. Several researchers pursue their research in algebraic structures (see, [1] and [5]).

Throughout this paper, $(P, \leq)$ denotes a poset with zero element 0. For $T \subseteq P$, let $(T)^l := \{k \in P : k \leq t \text{ for all } t \in T\}$ denotes the lower cone of T in P and $T^* = T \backslash \{0\}$. For $S, T \subseteq P$, we write $(S, T)^l$ instead of $(S \cup T)^l$. Following [9], a non-empty subset K of P is called semi-ideal if $k \in K$ and $a \leq k$, then $a \in K$. A proper semi-ideal K of P is called prime if for any $r, s \in P$, $(r, s)^l \subseteq K$ implies $r \in K$ or $s \in K$. Following [8], let K be a semi-ideal of P. Then $(K : t) = \{x \in P : (x, t)^l \subseteq K\}$ is called the extension of K by $t \in P$. Clearly, for any $t \in P$, $(K : t)$ is a semi-ideal of P.

Following [2], a semi-ideal K of P is called weakly n-prime if for pairwise distinct elements $a_1, a_2, ..., a_n \in P$, if $(a_1, a_2, ..., a_n)^l \subseteq K$, then at least one of n-subsets $(a_2, ..., a_n)^l$, $(a_1, a_3, ..., a_n)^l$, ..., $(a_1, a_2, ..., a_{n-1})^l$ is a subset of K. It is evident that every prime semi-ideal of P is a weakly n-prime semi-ideal of P.

A subset $K \neq \varphi$ of P is known as an ideal if $(s, t)^{ul} \subseteq K$, for all $s, t \in K$. An ideal K of P is called u-ideal if for $s, t \in K$, $(s, t)^u \cap K \neq \varphi$. Dually, we have the concept of filter and l-filter. Following [5], the radical $r(K)$ of a semi-ideal K to be the set of all $s \in P$ such that every prime l-filter containing s has a non-empty intersection with K. It is evident that $r(K)$ is a semi-ideal of P. A proper semi-ideal K of P is said to be primary if for $s, t \in P$, $(s, t)^l \subseteq K$ implies that $s \in K$ or $t \in r(K)$. It is evident that every prime u-ideal of P is primary ideal. A semi-ideal K of P is said to be Q-primary if K is primary and $r(K) = Q$ for some prime u-ideal Q of P.

In [1], Badawi generalized primary ideals and renamed as 2-absorbing primary ideal. A proper ideal K of R is called 2-absorbing primary ideal of a commutative ring R if whenever $x, y, z \in R$ with $xyz \in K$, then $xy \in K$ or $xz \in \sqrt{K}$ or $yz \in \sqrt{K}$. They proved many results concerning 2-absorbing primary ideals of R.

Following [1], we define the notion of weakly primary and weakly Q-primary semi-ideal of P as follows: A proper semi-ideal K of P is called weakly primary if $r_1, r_2, r_3 \in P$, $(r_1, r_2, r_3)^l \subseteq K$, we have $(r_1, r_2)^l \subseteq K$ or $(r_1, r_3)^l \subseteq r(K)$ or $(r_2, r_3)^l \subseteq r(K)$. A semi-ideal K of P is called weakly Q-primary if K is weakly primary and $r(K) = Q$ for some prime u-ideal Q of P. It is evident that every primary semi-ideal is a weakly primary semi-ideal of P, and every weakly prime semi-ideal is a weakly primary semi-ideal of P. We show that if the prime radical $r(K)$ for semi-ideal K is weakly primary, then $r(K)$ is a prime semi-ideal of P.

The given example shows that a weakly primary semi-ideal is not necessary a primary semi-ideal of P.

Example 1.1. Consider the poset $P = \{0, a, b, c, d, e, 1\}$ whose Hasse diagram is given below.
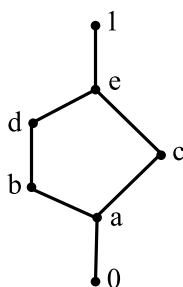
Let $K = \{0, a\}$. Then $r(K) = \{0, a\}$ and K is a weakly primary semi-ideal but not primary as $(b, c)^l \subseteq K$ with $b \notin K$ and $c \notin r(K)$.

Example 1.2. Consider the poset depicted in Example 1.1. Here $K = \{0, a, b, c\}$ is a semi-ideal of P and $r(K) = \{0, a, b, c, d, e\}$. Then K is a weakly primary semi-ideal of P. If we take $Q = r(K)$, then K is weakly Q-primary semi-ideal of P.

In [3], Ebrahimi Atani et al. studied a direct product of commutative rings. Following [3], we studied the direct product of posets in [6]. Recall that a direct product of posets P_1 and P_2 is the poset $P_1 \times P_2 = \{(s_1, t_2) : s_1 \in P_1,\ t_2 \in P_2\}$ such that $(s_1, t_2) \leq (s', t')$ in $P_1 \times P_2$ if $s_1 \leq s'$ in P_1 and $t_2 \leq t'$ in P_2. It is clear that direct product of semi-ideals is semi-ideal. We establish some properties of direct product of weakly primary semi-ideals of P, and if the direct product of semi-ideals is weakly primary, then both the semi-ideals are weakly primary.

Recall that for a semi-ideal K of P and $k \in P$, we define $V(k) = \{K \in Spec(P) : k \in K\}$ and $D(K) = Spec(P) \backslash V(K)$ where $Spec(P)$ is the spectrum of prime semi-ideals of P. Also, $V(K) = \cap_{k \in K} V(k)$ [4]. For basic terminology in poset, we refer to [8].

2. MAIN RESULTS

In this section, we investigate some properties of weakly primary semi-ideals of P. We also discuss intersection of Q-primary semi-ideals for a prime semi-ideal Q of P. Moreover, we show that if the prime radical $r(K)$ for a semi-ideal K is weakly primary, then $r(K)$ is prime semi-ideal of P. We now start this session with a useful lemma.

Lemma 2.1. [5] *Let K_1, K_2 be semi-ideals of P. Then the following assertions hold:*

(i) $K_1 \subseteq r(K_1)$.

(ii) $r(r(K_1)) = r(K_1)$.

(iii) $r(K_1 \cap K_2) = r(K_1) \cap r(K_2)$.

(iv) If K_1 is a prime u-ideal of P, then $r(K_1) = P$.

(v) If K_1 is a prime u-ideal of P such that $K_2 \subseteq K_1$, then $r(K_2) \subseteq K_1$.

Lemma 2.2. *Let K_1, K_2 be subsets of P. Then we have the following:*

(i) $V(K_1) \cup V(K_2) = V(K_1 \cap K_2)$ for any semi-ideals K_1 and K_2 of P.

(ii) If K_1 is a semi-ideal of P, then $V(K_1) = V(r(K_1))$.

(iii) If $V(K_1) \subseteq V(K_2)$, then $K_2 \subseteq r(K_1)$ for every semi-ideals K_1, K_2 of P.

(iv) $V(K_1) = V(K_2)$ if and only if $r(K_1) = r(K_2)$ for every semi-ideals K_1, K_2 of P.

Proof. (i) and (iii) are obvious.

(ii) As $K_1 \subseteq r(K_1)$, we have $V(r(K_1)) \subseteq V(K_1)$. Let $U \in V(K_1)$. Then $K_1 \subseteq U$ and so $r(K_1) \subseteq r(U) = U$. Thus $U \in V(r(K_1))$ and so $V(K_1) \subseteq V(r(K_1))$.

(iv) Let $V(K_1) = V(K_2)$. By (ii), $V(K_1) \subseteq V(r(K_2))$, and so $r(K_2) \subseteq r(K_1)$. Similarly, we can prove that $r(K_1) \subseteq r(K_2)$. So $r(K_1) = r(K_2)$. Conversely, suppose $r(K_1) = r(K_2)$. Then $V(r(K_1)) = V(r(K_2))$ and so $V(K_1) = V(K_2)$. $\square$

Lemma 2.3. *Let K_1 be a semi-ideal of P. If $r(K_1)$ is prime semi-ideal, then K_1 is weakly primary semi-ideal.*

Proof. Let $(u_1, v_2, w_3)^l \subseteq K_1$ and $(u_1, v_2)^l \nsubseteq K_1$ for some $u_1, v_2, w_3 \in P$. Suppose that $(u_1, v_2)^l \nsubseteq r(K_1)$. Since $r(K_1)$ is prime semi-ideal, we have $w_3 \in r(K_1)$. So $(u_1, w_3)^l \subseteq r(K_1)$ and $(v_2, w_3)^l \subseteq r(K_1)$. Suppose that $(u_1, v_2)^l \subseteq r(K_1)$. Since $r(K_1)$ is prime semi-ideal, we have either $u_1 \in r(K_1)$ or $v_2 \in r(K_1)$ which implies $(u_1, w_3)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(K_1)$. $\square$

The below example shows that converse of the above Lemma 2.3 is not true for some semi-ideal of P.

Example 2.4. Consider the set $P = \{1, 2, 4, 5, 10, 20\}$. Then P is a poset under the relation division. Take the semi-ideal $K_1 = \{1\}$. Here $r(K_1) = \{1\}$ is weakly primary semi-ideal, but not prime as $(2, 5)^l \subseteq K_1$ and $2, 5 \notin K_1$.

Theorem 2.5. Let K_1 be a semi-ideal of P. Then $r(K_1)$ is a prime semi-ideal of P if and only if $r(K_1)$ is a primary semi-ideal of P.

Proof. Let $u_1, v_2 \in P$ and $r(K_1)$ be a prime semi-ideal of P. If $(u_1, v_2)^l \subseteq r(K_1)$, then either $u_1 \in r(K_1)$ or $v_2 \in r(K_1)$. Since $r(K_1) = r(r(K_1))$ by Lemma 2.1, we have either $u_1 \in r(K_1)$ or $v_2 \in r(r(K_1))$. So $r(K_1)$ is a primary semi-ideal of P. Conversely, suppose $r(K_1)$ is a primary semi-ideal of P and $u_1, v_2 \in P$. If $(u_1, v_2)^l \subseteq r(K_1)$, then either $u_1 \in r(K_1)$ or $v_2 \in r(r(K_1))$ which imply $u_1 \in r(K_1)$ or $v_2 \in r(K_1)$, by Lemma 2.1. So $r(K_1)$ is a prime semi-ideal of P. $\square$

Theorem 2.6. Let K_1 be a semi-ideal of P. Then $r(K_1)$ is a weakly prime semi-ideal of P if and only if $r(K_1)$ is a weakly primary semi-ideal of P.

Proof. Suppose that $r(K_1)$ is a weakly prime semi-ideal of P and let $u_1, v_2, w_3 \in P$. If $(u_1, v_2, w_3)^l \subseteq r(K_1)$, then either $(u_1, v_2)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(K_1)$ or $(u_1, w_3)^l \subseteq r(K_1)$. Since $r(K_1) = r(r(K_1))$ by Lemma 2.1, we have either $(u_1, v_2)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(r(K_1))$ or $(u_1, w_3)^l \subseteq r(r(K_1))$. So $r(K_1)$ is weakly primary semi-ideal. Conversely, suppose that $r(K_1)$ is weakly primary semi-ideal and let $u_1, v_2, w_3 \in P$. If $(u_1, v_2, w_3)^l \subseteq r(K_1)$, then either $(u_1, v_2)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(r(K_1))$ or $(u_1, w_3)^l \subseteq r(r(K_1))$. By Lemma 2.1, $r(K_1) = r(r(K_1))$. So we have either $(u_1, v_2)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(K_1)$ or $(u_1, w_3)^l \subseteq r(K_1)$ and hence $r(K_1)$ is weakly prime semi-ideal. $\square$

Theorem 2.7. Let K_1 be a weakly primary semi-ideal of P such that $(u_1, v_2, J^*)^l \subseteq K_1$ for some $u_1, v_2 \in P$, and some u-semi-ideal J of P. If $(u_1, v_2)^l \nsubseteq K_1$, then $(u_1, J^*)^l \subseteq r(K_1)$ or $(v_2, J^*)^l \subseteq r(K_1)$.

Proof. Let $(u_1, v_2, J^*)^l \subseteq K_1$ for some $u_1, v_2 \in P$, and some u-semi-ideal J of P, and let $(u_1, v_2)^l \nsubseteq K_1$. Suppose that $(u_1, J^*)^l \nsubseteq r(K_1)$ and $(v_2, J^*)^l \nsubseteq r(K_1)$. Then there exist $i, j \in J\setminus\{0\}$ such that $(u_1, i)^l \nsubseteq r(K_1)$ and $(v_2, j)^l \nsubseteq r(K_1)$. Since $(u_1, v_2, J^*)^l \subseteq K_1$, we have $(u_1, j)^l \subseteq r(K_1)$ and $(v_2, i)^l \subseteq r(K_1)$. As J is a u-semi-ideal of P, we have $(i, j)^u \cap J \neq \varphi$. Let $t \in (i, j)^u \cap J$. Since $(u_1, v_2, t)^l \subseteq K_1$ and $(u_1, v_2)^l \nsubseteq K_1$, we have $(u_1, t)^l \subseteq r(K_1)$ or $(v_2, t)^l \subseteq r(K_1)$. If $(u_1, t)^l \subseteq r(K_1)$, then $(u_1, i)^l \subseteq r(K_1)$, a contradiction. If $(v_2, t)^l \subseteq r(K_1)$, then $(v_2, j)^l \subseteq r(K_1)$, again a contradiction. So $(u_1, J^*)^l \subseteq r(K_1)$ or $(v_2, J^*)^l \subseteq r(K_1)$. $\square$

Lemma 2.8. *Let K_1 be a semi-ideal of P and $s_1 \in P \backslash r(K_1)$. If $r((K_1 : s_1))$ is a prime semi-ideal of P, then $(r(K_1) : s_1) \subseteq r((K_1 : s_1))$.*

Proof. Suppose that $r((K_1 : s_1))$ is prime and $s_1 \notin r(K_1)$. Let $y \in (r(K_1) : s_1)$. Then $(y, s_1)^l \subseteq r(K_1)$. Let $t \in r(K_1)$. Then $M_t \cap K_1 \neq \varphi$, where M_t is a prime l-filter containing t and so $M_t \cap (K_1 : s_1) \neq \varphi$ which implies $t \in r((K_1 : s_1))$. Thus $(y, s_1)^l \subseteq r(K_1) \subseteq r((K_1 : s_1))$. As $r((K_1 : s_1))$ is prime, we have $s_1 \in r((K_1 : s_1))$ or $y \in r((K_1 : s_1))$. If $s_1 \in r((K_1 : s_1))$, then $s_1 \in r(K_1)$, a contradiction. Thus $y \in r((K_1 : s_1))$ and hence $(r(K_1) : s_1) \subseteq r((K_1 : s_1))$. $\square$

Theorem 2.9. *Let K_1 be a weakly primary semi-ideal of P. If $r(K_1)$ is prime, then $(K_1 : s_1)$ is a weakly primary semi-ideal of P with $r((K_1 : s_1)) = r(K_1)$ for all $s_1 \in P \backslash r(K_1)$.*

Proof. Let $s_1 \in P \backslash r(K_1)$ and $y \in (K_1 : s_1)$. Then $(y, s_1)^l \subseteq K_1 \subseteq r(K_1)$. Since $r(K_1)$ is prime, we have $y \in r(K_1)$ and so $(K_1 : s_1) \subseteq r(K_1)$ which implies $r((K_1 : s_1)) \subseteq r(r(K_1)) = r(K_1)$. Clearly, $r(K_1) \subseteq r((K_1 : s_1))$. Therefore $r((K_1 : s_1)) = r(K_1)$.

We now show that $(K_1 : s_1)$ is a weakly primary semi-ideal of P. Let $(u_1, v_2, w_3)^l \subseteq (K_1 : s_1)$ for $u_1, v_2, w_3 \in P$. Then $(u_1, v_2, w_3, s_1)^l \subseteq K_1$. If $(u_1, v_2, w_3)^l \subseteq K_1$, then $(u_1, v_2)^l \subseteq K_1 \subseteq (K_1 : s_1)$ or $(u_1, w_3)^l \subseteq r(K_1) \subseteq (r(K_1) : s_1)$ or $(v_2, w_3)^l \subseteq (r(K_1) : s_1)$. By Lemma 2.8, $(u_1, v_2)^l \subseteq (K_1 : s_1)$ or $(u_1, w_3)^l \subseteq r((K_1 : s_1))$ or $(v_2, w_3)^l \subseteq r((K_1 : s_1))$. If $(u_1, v_2, w_3, s_1)^l \subseteq K_1$, then either $(u_1, v_2, s_1)^l \subseteq K_1$ or $(u_1, w_3, s_1)^l \subseteq r(K_1)$ or $(v_2, w_3, s_1)^l \subseteq r(K_1)$ which implies $(u_1, v_2)^l \subseteq (K_1 : s_1)$ or $(u_1, w_3)^l \subseteq (r(K_1) : s_1) \subseteq (r(K_1 : s_1))$ or $(v_2, w_3)^l \subseteq (r(K_1) : s_1) \subseteq (r(K_1 : s_1))$ and so $(K_1 : s_1)$ is a weakly primary semi-ideal of P. $\square$

Theorem 2.10. *Let K_1 be a semi-ideal of P and $r(K_1)$ is a prime semi-ideal of P. If $(K_1 : s)$ is a prime semi-ideal for each $s \in r(K_1) \backslash K_1$, then K_1 is a weakly prime semi-ideal of P.*

Proof. Let $(u_1, v_2, w_3)^l \subseteq K_1$ and $u_1 \in r(K_1)$ for $u_1, v_2, w_3 \in P$. If $u_1 \in K_1$, then $(u_1, v_2)^l \subseteq K_1$ and so K_1 is a weakly prime semi-ideal of P. Suppose $u_1 \in r(K_1) \backslash K_1$. Then $(v_2, w_3)^l \subseteq (K_1 : u_1)$. Since $(K_1 : u_1)$ is prime, $v_2 \in (K_1 : u_1)$ or $w_3 \in (K_1 : u_1)$ which implies $(u_1, v_2)^l \subseteq K_1$ or $(u_1, w_3)^l \subseteq K_1$ and so K_1 is a weakly prime semi-ideal of P. $\square$

The below example shows that converse of the above Theorem 2.10 is not true in general.

Example 2.11. Consider the poset $P = \{0, x, y, z, u, 1\}$ whose Hasse diagram is given below.
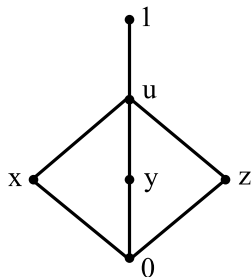
Figure 2

Consider the weakly prime semi-ideal $K_1 = \{0\}$. Here $r(K_1) = \{0, x, y, z, u\}$ is prime semi-ideal but $(K_1 : u) = \{0\}$ is not prime as $(x, y)^l \subseteq (K_1 : u)$ with $x \notin (K_1 : u)$ and $y \notin (K_1 : u)$.

Corollary 2.12. *Let K_1 be a semi-ideal of P and $r(K_1)$ is prime. If $(K_1 : s)$ is a prime semi-ideal for each $s \in r(K_1) \backslash K_1$, then K_1 is a weakly primary semi-ideal of P.*

Lemma 2.13. *Let K_1 be a semi-ideal of P. If K_1 is weakly prime semi-ideal, then $(K_1 : x)$ is a weakly prime semi-ideal of P for each $x \in P \backslash K_1$.*

Proof. Suppose that $(u_1, v_2, w_3)^l \subseteq (K_1 : x)$ for each $x \in P \backslash K_1$ and $u_1, v_2, w_3 \in P$. Then $(u_1, v_2, w_3, x)^l \subseteq K_1$. Since K_1 is weakly prime, we have either $(u_1, v_2, w_3)^l \subseteq K_1$ or $(u_1, v_2, x)^l \subseteq K_1$ or $(u_1, w_3, x)^l \subseteq K_1$ or $(v_2, w_3, x)^l \subseteq K_1$. If $(u_1, v_2, x)^l \subseteq K_1$ or $(u_1, w_3, x)^l \subseteq K_1$ or $(v_2, w_3, x)^l \subseteq K_1$, then $(u_1, v_2)^l \subseteq (K_1 : x)$ or $(u_1, w_3)^l \subseteq (K_1 : x)$ or $(v_2, w_3)^l \subseteq (K_1 : x)$. Suppose $(u_1, v_2, w_3)^l \subseteq K_1$. Then $(u_1, v_2)^l \subseteq K_1$ or $(u_1, w_3)^l \subseteq K_1$ or $(v_2, w_3)^l \subseteq K_1$ and so $(K_1 : x)$ is weakly prime semi-ideal as $K_1 \subseteq (K_1 : x)$. $\square$

Theorem 2.14. *Suppose that I is a P_1-primary and J is a P_2-primary semi-ideal of P for some prime semi-ideals P_1 and P_2 of P. Then $I \cap J$ is a weakly primary semi-ideal of P.*

Proof. Let $K_1 = I \cap J$ where I and J are semi-ideals of P. Then by Lemma 2.1, $r(K_1) = r(I) \cap r(J) = P_1 \cap P_2$. Suppose that $(u_1, v_2, w_3)^l \subseteq K_1$ for some $u_1, v_2, w_3 \in P, (u_1, w_3)^l \nsubseteq r(K_1)$ and $(v_2, w_3)^l \nsubseteq r(K_1)$. Then $u_1, v_2, w_3 \notin r(K_1)$. Since $r(K_1) = P_1 \cap P_2$ is weakly prime semi-ideal of P, we have $(u_1, v_2)^l \subseteq r(K_1)$. Now we claim that $(u_1, v_2)^l \subseteq K_1$. Since $(u_1, v_2)^l \subseteq r(K_1) \subseteq P_1$, we may assume that $u_1 \in P_1$. Since $u_1 \notin r(K_1)$ and $(u_1, v_2)^l \subseteq P_2$, we conclude that $u_1 \notin P_2$ and $v_2 \in P_2$ which implies $v_2 \notin P_1$. If $u_1 \in I$ and $v_2 \in J$, then $(u_1, v_2)^l \subseteq K_1$. Suppose that $u_1 \notin I$. Since I is P_1-primary, we have $(v_2, w_3)^l \subseteq P_1$ which implies $(v_2, w_3)^l \subseteq r(K_1)$, a contradiction. Similarly, assume that $v_2 \notin J$. Since J is P_2-primary, we have $(u_1, w_3)^l \subseteq P_2$ which implies $(u_1, w_3)^l \subseteq r(K_1)$, a contradiction. Hence $(u_1, v_2)^l \subseteq K_1$. $\square$

3. Direct product of weakly primary semi-ideals

In this section, we discuss a direct product of weakly primary semi-ideals of posets. It is clear that direct product of semi-ideals is semi-ideal. We show that if the direct product of semi-ideals is weakly primary, then both the semi-ideals are weakly primary. In general, a direct product of weakly primary semi-ideals of P is not necessary a weakly primary semi-ideal of P.

The below example shows that a direct product of two weakly primary semi-ideals of P need not be a weakly primary semi-ideal of P.

Example 3.1. Let $P_1 = \{1, 2, 3, 6\}$ and $P_2 = \{1, 3, 5\}$ be posets under the relation division. Let $I_1 = \{1\}$ and $I_2 = \{1\}$. Then $r(I_1) = \{1\}$ and $r(I_2) = \{1\}$, and I_1, I_2 are weakly primary semi-ideals of P_1 and P_2 respectively. Here $I_1 \times I_2 = \{(1, 1)\}$ is not a weakly primary semi-ideal of $P_1 \times P_2$ as $((2, 3), (3, 3), (6, 1))^l \subseteq I_1 \times I_2$, but $((2, 3), (3, 3))^l \nsubseteq I_1 \times I_2$ and $((2, 3), (6, 1))^l \nsubseteq r(I_1 \times I_2)$ and $((3, 3), (6, 1))^l \nsubseteq r(I_1 \times I_2)$.

Lemma 3.2. *Let $P = P_1 \times P_2$, where P_1 and P_2 are posets. Let $I_i's$ and $J_j's$ be semi-ideals of P_1 and P_2 respectively. Then $\cap(I_i \times J_j) = \cap I_i \times \cap J_j$.*

Proof. Obvious. □

Proposition 3.3. *Let $P = P_1 \times P_2$, where P_1 and P_2 are posets with greatest elements e_1 and e_2 respectively. Then the below assertions hold:*

(i) If K_1 is a semi-ideal of P_1, then $r(K_1 \times P_2) = r(K_1) \times P_2$.

(ii) If K_2 is a semi-ideal of P_2, then $r(P_1 \times K_2) = P_1 \times r(K_2)$.

Proof. Let $(u_1, v_2) \in r(K_1 \times P_2)$ for any $u_1 \in P_1$, $v_2 \in P_2$. Then $(u_1, v_2) \in \bigcap_i (Q_i \times P_2)$, where $Q_i's$ are all prime u-ideals of P_1 containing K_1. So $u_1 \in \bigcap_i Q_i$ and $v_2 \in P_2$ which imply $u_1 \in r(K_1)$ and $v_2 \in P_2$. So $(u_1, v_2) \in r(K_1) \times P_2$. If $(u_1, v_2) \in r(K_1) \times P_2$, then $u_1 \in r(K_1)$ and $v_2 \in P_2$. So $u_1 \in \bigcap_i Q_i$ and $v_2 \in P_2$ where $Q_i's$ are all prime u-ideals of P_1 which imply $(u_1, v_2) \in \bigcap_i (Q_i \times P_2)$. So $(u_1, v_2) \in r(K_1 \times P_2)$ and hence $r(K_1 \times P_2) = r(K_1) \times P_2$.

(ii) The proof is similar to that of (i). □

The following lemma shows that a direct product of primary semi-ideals is weakly primary semi-ideal.

Lemma 3.4. *Let K_1 and K_2 be primary semi-ideals of P_1 and P_2 respectively. Then $K_1 \times K_2$ is a weakly primary semi-ideal of $P_1 \times P_2$.*

Proof. Suppose that $((u_1, v_2), (s_1, s_2), (t_1, t_2))^l \subseteq K_1 \times K_2$ for $u_1, s_1, t_1 \in P_1$, $v_2, s_2, t_2 \in P_2$. Then $(u_1, s_1, t_1)^l \subseteq K_1$ and $(v_2, s_2, t_2)^l \subseteq K_2$ which imply $((u_1, v_2), (s_1, s_2))^l \subseteq K_1 \times K_2$ or $((u_1, v_2), (t_1, t_2))^l \subseteq r(K_1) \times r(K_2)$ or $((s_1, s_2), (t_1, t_2))^l \subseteq r(K_1) \times r(K_2)$ and so $K_1 \times K_2$ is a weakly primary semi-ideal of $P_1 \times P_2$. $\square$

Theorem 3.5. *Let $P = P_1 \times P_2$, where P_1 and P_2 are posets. Let K_1 and K_2 be proper semi-ideals of P_1 and P_2 respectively. If $K_1 \times K_2$ is a weakly primary semi-ideal of P, then K_1 and K_2 are weakly primary semi-ideals of P_1 and P_2 respectively.*

Proof. Suppose $(u_1, v_2, w_3)^l \subseteq K_1$ for some $u_1, v_2, w_3 \in P_1$. Then $((u_1, v_2, w_3)^l, t) \subseteq K_1 \times K_2$ for some $t \in K_2$. Since $K_1 \times K_2$ is a weakly primary semi-ideal of P, we have either $((u_1, v_2)^l, t) \subseteq K_1 \times K_2$ or $((u_1, w_3)^l, t) \subseteq r(K_1 \times K_2)$ or $((v_2, w_3)^l, t) \subseteq r(K_1 \times K_2)$ which imply either $((u_1, v_2)^l, t) \subseteq K_1 \times K_2$ or $((u_1, w_3)^l, t) \subseteq r(K_1) \times r(K_2)$ or $((v_2, w_3)^l, t) \subseteq r(K_1) \times r(K_2)$, by Proposition 3.3. So $(u_1, v_2)^l \subseteq K_1$ or $(u_1, w_3)^l \subseteq r(K_1)$ or $(v_2, w_3)^l \subseteq r(K_1)$. Hence K_1 is a weakly primary semi-ideal of P_1. Similarly, we can show that K_2 is a weakly primary semi-ideal of P_2. $\square$

Theorem 3.6. *Let $P = P_1 \times P_2$, where P_1 and P_2 are posets with greatest elements $'e_1'$ and $'e_2'$ respectively. Let Q be a proper semi-ideal of P. Then the following statements are equivalent:*

(i) Q is a weakly primary semi-ideal of P.

(ii) Either $Q = K_1 \times P_2$ for some weakly primary semi-ideal K_1 of P_1 or $Q = P_1 \times K_2$ for some weakly primary semi-ideal K_2 of P_2 or $Q = K_1 \times K_2$ for some primary semi-ideals K_1 and K_2 of P_1 and P_2 respectively.

Proof. $(i) \Rightarrow (ii)$: Suppose Q is a weakly primary semi-ideal of P. Then $Q = K_1 \times K_2$ for some semi-ideal K_1 of P_1 and K_2 of P_2.

Case (i): If $K_2 = P_2$, then $K_1 \neq P_1$. So $Q = K_1 \times P_2$. Let $(a_1, b_2, c_3)^l \subseteq K_1$ for some $a_1, b_2, c_3 \in P_1$. Then $((a_1, b_2, c_3)^l, (x_1, y_2, z_3)^l) \subseteq K_1 \times P_2$ where $x_1, y_2, z_3 \in P_2$. Since $K_1 \times P_2$ is weakly primary, we have either $((a_1, b_2)^l, (x_1, y_2)^l) \subseteq K_1 \times P_2$ or $((a_1, c_3)^l, (x_1, z_3)^l) \subseteq r(K_1 \times P_2)$ or $((b_2, c_3)^l, (y_2, z_3)^l) \subseteq r(K_1 \times P_2)$ which imply either $((a_1, b_2)^l, (x_1, y_2)^l) \subseteq K_1 \times P_2$ or $((a_1, c_3)^l, (x_1, z_3)^l) \subseteq r(K_1) \times P_2$ or $((b_2, c_3)^l, (y_2, z_3)^l) \subseteq r(K_1) \times P_2$, by Proposition 3.3. So either $(a_1, b_2)^l \subseteq K_1$ or $(a_1, c_3)^l \subseteq r(K_1)$ or $(b_2, c_3)^l \subseteq r(K_1)$ and hence K_1 is a weakly primary semi-ideal of P_1.

Case (ii): If $K_1 = P_1$, then $K_2 \neq P_2$. By Case (i), K_2 is a weakly primary semi-ideal of P_2.

Case (iii): If $K_1 \neq P_1$ and $K_2 \neq P_2$, then $Q = K_1 \times K_2$. So $r(Q) = r(K_1 \times K_2) = r(K_1) \times r(K_2)$.

Suppose K_1 is not a primary semi-ideal of P_1. Then there exist $x_1, y_2 \in P_1$ such that $(x_1, y_2)^l \subseteq K_1$ with $x_1 \notin K_1$ and $y_2 \notin r(K_1)$. Let $u = (x_1, e_2), v = (e_1, 0)$ and $w = (y_2, e_2)$. Then $(u, v, w)^l \subseteq Q$ as $(x_1, e_2, y_2)^l \subseteq K_1$. Since Q is weakly primary, we have $(u, v)^l \subseteq Q$ or $(u, w)^l \subseteq r(Q)$ or $(v, w)^l \subseteq r(Q)$ which imply $x_1 \in K_1$ or $(x_1, y_2)^l \subseteq K_1$ and $e_2 \in r(K_2)$ or $y_2 \in r(K_1)$, a contradiction. So K_1 is a primary semi-ideal of P_1.

Suppose K_2 is not a primary semi-ideal of P_2. Then there exist $s, t \in P_2$ such that $(s, t)^l \subseteq K_2$ with $s \notin K_2$ and $t \notin r(K_2)$. Let $x_1 = (e_1, s), y_2 = (0, e_2)$ and $z_3 = (e_1, t)$. Then $(x_1, y_2, z_3)^l \subseteq Q$. Since Q is weakly primary, we have $(x_1, y_2)^l \subseteq Q$ or $(x_1, z_3)^l \subseteq r(Q)$ or $(y_2, z_3)^l \subseteq r(Q)$ which imply $s \in K_2$ or $e_1 \in r(K_1)$ or $t \in r(K_2)$, a contradiction. So K_2 is a primary semi-ideal of P_2.

$(ii) \Rightarrow (i)$: Suppose $Q = K_1 \times P_2$ for some weakly primary semi-ideal K_1 of P_1. Let $((y_1, b_1), (y_2, b_2), (y_3, b_3))^l \subseteq K_1 \times P_2$ for some $y_1, y_2, y_3 \in P_1$ and $b_1, b_2, b_3 \in P_2$. Then $(y_1, y_2, y_3)^l \subseteq K_1$. Since K_1 is weakly primary, we have $(y_1, y_2)^l \subseteq K_1$ or $(y_1, y_3)^l \subseteq r(K_1)$ or $(y_2, y_3)^l \subseteq r(K_1)$ which imply $((y_1, b_1), (y_2, b_2))^l \subseteq K_1 \times P_2$ or $((y_1, b_1), (y_3, b_3))^l \subseteq r(K_1) \times P_2$ or $((y_2, b_2), (y_3, b_3))^l \subseteq r(K_1) \times P_2$ and so $((y_1, b_1), (y_2, b_2))^l \subseteq K_1 \times P_2$ or $((y_1, b_1), (y_3, b_3))^l \subseteq r(K_1 \times P_2)$ or $((y_2, b_2), (y_3, b_3))^l \subseteq r(K_1 \times P_2)$, by Proposition 3.3. Thus $Q = K_1 \times P_2$ is a weakly primary semi-ideal of P. Similarly, we can show that $Q = P_1 \times K_2$ is a weakly primary semi-ideal of P.

Suppose $Q = K_1 \times K_2$ for some primary semi-ideals K_1 of P_1 and K_2 of P_2. Then $S = K_1 \times P_2$ and $T = P_1 \times K_2$ are primary semi-ideals of P. So $S \cap T = (K_1 \times P_2) \cap (P_1 \times K_2) = K_1 \times K_2 = Q$. Hence Q is a weakly primary semi-ideal of P. $\square$

4. Conclusion

The concepts of weakly primary semi-ideal and weakly Q-primary semi-ideal for some prime u-ideal Q of P were introduced and investigated their properties. We also found an equivalent assertion for a weakly primary semi-ideal of P. Moreover, we introduced the concept of direct product of weakly primary semi-ideal of a direct product of posets and obtained important results related to these concepts. In future, we will study z-semi-ideal structure of a poset.

5. Acknowledgments

References

[1] A. Badawi, U. Tekir and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., **51** No. 4 (2014) 1163-1173.

[2] J. Catherine Grace John and B. Elavarasan, *Weakly n-prime ideal of posets*, Int. J. Pure Appl. Math., **86** No. 6 (2013) 905-910.

[3] S. Ebrahimi Atani and M. Shajari, *The diameter of a zero-divisor graph for finite direct product of commutative rings*, Sarajevo J. Math., **3** No. 16 (2007) 149-156.

[4] B. Elavarasan and K. Porselvi, *An ideal - based zero-divisor graph of posets*, Commun. Korean Math. Soc., **28** No. 1 (2013) 79-85.

[5] V. Joshi and N. Mundlik, *On primary ideals in posets*, Math. Slovaca, **65** No. 6 (2015) 1237-1250.

[6] K. Porselvi and B. Elavarasan, *Diameters of semi-ideal based zero-divisor graphs for finite direct product of posets*, Ital. J. Pure Appl. Math., **34** (2015) 113-122.

[7] K. Porselvi and B. Elavarasan, *Some properties of prime and z-semi-ideals in posets*, Khayyam J. Math., **6** No. 1 (2020) 46-56.

[8] R. Halaš, *On extensions of ideals in posets*, Discrete Math., **308** (2008) 4972-4977.

[9] P. V. Venkatanarasimhan, *Semi-ideals in posets*, Math. Ann., **185** No. 4 (1970) 338-348.

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