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Research Paper

AN APPROACH TO EXTENDING MODULES VIA HOMOMORPHISMS

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Abstract. The notion of $\mathcal{K}$-extending modules was defined recently as a proper generalization of both extending modules and Rickart modules. Let M be a right R-module and let $S = End_R(M)$. We recall that M is a $\mathcal{K}$-extending module if for every element $\varphi \in S$, $\operatorname{Ker} \varphi$ is essential in a direct summand of M. Since a direct sum of $\mathcal{K}$-extending modules is not a $\mathcal{K}$-extending module in general, an open question is to find necessary and sufficient conditions for such a direct sum to be $\mathcal{K}$-extending. In this paper, we give an answer to this question. We show that if M_i is M_j-injective for all $i, j \in I = \{1, 2, \ldots, n\}$, then $\bigoplus_{i=1}^{n} M_i$ is a $\mathcal{K}$-extending module if and only if M_i is M_j-$\mathcal{K}$-extending for all $i, j \in I$. Other results on $\mathcal{K}$-extending modules and some of their applications are also included.

1. Introduction

Throughout this paper, R denotes an associative ring with identity, M is a unitary right R-module, and $S = End_R(M)$ is the ring of all R-endomorphisms of M. We use the notation

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$N \leq^e M$ to indicate that N is an essential submodule of M (i.e., $L \cap N \neq 0$, for all $0 \neq L \leq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $\Delta(M) = \{f \in S \mid \operatorname{Ker} f \leq^e M\}$ and $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of R $\}$. Recall that the module M is said to be *singular* if $M = Z(M)$ and is called *nonsingular* if $Z(M) = 0$.

Extending modules play important roles in rings and categories of modules, and have been studied extensively by many authors in recent years (see, [6, 13, 16]). A module M is called *extending* (or *CS*) if every submodule of M is essential in a direct summand of M.

In [16], a ring R is called an *ACS-ring* if for every element $a \in R$, it follows that $r_R(a) \leq^e fR$ for some $f^2 = f \in R$. Two papers [1] and [3] generalized the ACS-ring concept to modules in the same ways. In [1], the authors used the terminology " CS-Rickart" for this generalization. Unfortunately, at the time of writing of the paper [3], we were not aware of the paper [1] and we used different terminology "$\mathcal{K}$-extending " for this generalization. As $\mathcal{K}$-extending modules are dual of $\mathcal{I}$-lifting modules which are defined in [2] and $\mathcal{I}$-lifting modules are used in some papers such as [3, 4, 14, 18], we use terminology $\mathcal{K}$-extending modules in this paper.

Recall that a module M is called a $\mathcal{K}$-*extending* module if for every element $\varphi \in S$, $\operatorname{Ker} \varphi$ is essential in a direct summand of M. It is clear that R_R is a $\mathcal{K}$-extending module if and only if R is an ACS-ring. The direct sum of $\mathcal{K}$-extending modules is not a $\mathcal{K}$-extending module, in general (Examples 3.1 and 3.2). In this paper, our main aim is to find necessary and sufficient conditions for a finite direct sum of $\mathcal{K}$-extending modules to be a $\mathcal{K}$-extending module. Among consequences, we show that the finite direct sum of copies of any $\mathcal{K}$-extending module with C_2-condition, is a $\mathcal{K}$-extending module.

2. $\mathcal{K}$-extending modules

We recall that a module M is a $\mathcal{K}$-*extending module* if for every element $\varphi \in S$, it follows that $\operatorname{Ker} \varphi \leq^e eM$ for some $e^2 = e \in S$. It is clear that for $M = R_R$, the notion of a $\mathcal{K}$-extending module coincides with that of an ACS-ring.

Examples 2.1. (1) Every extending module is a $\mathcal{K}$-extending module.

(2) Lee, Rizvi, and Roman [11] introduced the notion of Rickart modules as follow:. A module M is said to be *Rickart* if $\operatorname{Ker} \varphi \leq^{\oplus} M$ for every $\varphi \in End_R(M)$. Rickart modules are precisely nonsingular $\mathcal{K}$-extending modules.

(3) $\mathbb{Z}^{(\mathbb{N})}$ is a Rickart $\mathbb{Z}$-module by [11, Example 2.3]. Hence it is a $\mathcal{K}$-extending module, but $\mathbb{Z}^{(\mathbb{N})}$ is not extending, since if it were, then we would have an epimorphism $f : \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$ with nonessential kernel. Then by the extending property, $\operatorname{Ker}(f)$ is essential in some direct summand K of $\mathbb{Z}^{(\mathbb{N})}$. Hence $\mathbb{Q} \cong K/\operatorname{Ker}(f) \oplus T$ for some direct summand T of $\mathbb{Z}^{(\mathbb{N})}$. Since $\mathbb{Q}$ is nonsingular, $K = \operatorname{Ker}(f)$. It follows that $\mathbb{Q}$ embeds in $\mathbb{Z}$, which is a contradiction.

(4) The $\mathbb{Z}$-module $\mathbb{Z}_4$ is extending and so it is a $\mathcal{K}$-extending $\mathbb{Z}$-module but it is not a Rickart $\mathbb{Z}$-module by [11, Example 2.6].

In the following example, we introduce a $\mathcal{K}$-extending module that is neither Rickart nor extending.

Example 2.2. Denote $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ where p is a prime number in $\mathbb{N}$. Then it is clear that $\mathbb{Z}$ and $\mathbb{Z}_p$ are both Rickart and extending $\mathbb{Z}$-modules and so they are $\mathcal{K}$-extending. However, by [12, Example 1.1], the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_p$ is not Rickart and, by [1, Example 1], M is not extending but it is a $\mathcal{K}$-extending module.

Let M and N be R-modules. We recall that M is *N-$\mathcal{K}$-extending* if for every homomorphism $\varphi : M \to N$, there exists $L \leq^{\oplus} M$ such that $\operatorname{Ker}\varphi \leq^{e} L$; see [3]. It is clear that a module M is $\mathcal{K}$-extending if and only if M is M-$\mathcal{K}$-extending.

Example 2.3. Let M be a semisimple or uniform R-module. Then M is N-$\mathcal{K}$-extending for any right R-module N. Thus the simple $\mathbb{Z}$-module $\mathbb{Z}_p$ is $\mathbb{Z}$-$\mathcal{K}$-extending and also $\mathbb{Z}$ is $\mathbb{Z}_p$-$\mathcal{K}$-extending.

Proposition 2.4. [3, Proposition 3.4] The following conditions are equivalent for a module M:

(1) M is a $\mathcal{K}$-extending module;

(2) For any submodule N of M, every direct summand L of M is N-$\mathcal{K}$-extending;

(3) For every pair of summands L and N of M and any $\varphi \in Hom_R(M, N)$, the kernel of the restricted map $\varphi|_L$ is essential in a direct summand of L.

Corollary 2.5. [3, Corollary 3.5] Every direct summand of a $\mathcal{K}$-extending module is $\mathcal{K}$-extending.

A module M_R has the (*strong*) *summand intersection property*, abbreviated (SSIP) SIP, if the intersection of (any family) every pair of direct summands of M_R is a direct summand of M_R; see [7]. In [8], a module M_R is called an *SIP-extending module* provided that the intersection of every pair of direct summands of M is essential in a direct summand of M. A module M_R is said to be an *SSIP-extending module* if the intersection of any family of direct summands of M is essential in a direct summand of M.

Note that every $\mathcal{K}$-extending module is SIP-extending by [1, Proposition 1]. The following results on relatively $\mathcal{K}$-extending modules, will be useful in this study on direct sums.

Proposition 2.6. Let $\{M_i\}_{i\in\mathcal{I}}$ and N be right R-modules. Then the following statements hold:

(1) If N is an SIP-extending module, then N is $\oplus_{i\in\mathcal{I}}M_i$-$\mathcal{K}$-extending if and only if N is M_i-$\mathcal{K}$-extending for all $i\in\mathcal{I}$, $\mathcal{I}=\{1,2,\dots,n\}$.

(2) If N is an SSIP-extending module, then N is $\oplus_{i\in\mathcal{I}}M_i$-$\mathcal{K}$-extending if and only if N is M_i-$\mathcal{K}$-extending for all $i\in\mathcal{I}$, $\mathcal{I}$ is an arbitrary index set.

Proof. (i) Assume that N is $\oplus_{i\in\mathcal{I}}M_i$-$\mathcal{K}$-extending, where $\mathcal{I}=\{1,2,\dots,n\}$. By Proposition 2.4, N is M_i-$\mathcal{K}$-extending for all $i\in\mathcal{I}$. Conversely, assume that N is M_i-$\mathcal{K}$-extending for all $i\in\mathcal{I}$. Let f be a homomorphism from N to $\oplus_{i\in\mathcal{I}}M_i$ and let $\pi_i:\oplus_{i\in\mathcal{I}}M_i\to M_i$ be the natural projection map for each $i\in\mathcal{I}$. Since $f=(\pi_1 f,\pi_2 f,\dots,\pi_n f)$, $\operatorname{Ker} f=\cap_{i=1}^{n}\operatorname{Ker}(\pi_i f)$ is essential in a direct summand of N because $\operatorname{Ker}(\pi_i f)$ is essential in a direct summand of N and N is SIP-extending.

(ii) It is similar to the proof of (i). □

Corollary 2.7. For every $i\in\mathcal{I}=\{1,2,\dots,n\}$, it follows that M_i is $\oplus_{j\in\mathcal{I}}M_j$-$\mathcal{K}$-extending if and only if M_i is M_j-$\mathcal{K}$-extending for all $j\in\mathcal{I}$.

Proof. It is easily proved from [1, Proposition 1 (4)] and Proposition 2.6. □

3. Direct sums of $\mathcal{K}$-extending modules

The following examples show that a direct sum of $\mathcal{K}$-extending modules is not $\mathcal{K}$-extending in general.

Example 3.1. Let $R=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$, $e_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $e_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Then the modules $e_1R=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & 0\end{pmatrix}$ and $e_2R=\begin{pmatrix}0 & 0\\ 0 & \mathbb{Z}\end{pmatrix}$ are Rickart R-modules because $End(e_1R)\cong\mathbb{Z}\cong End(e_2R)$. Thus they are $\mathcal{K}$-extending. But $M=R_R$ is not a $\mathcal{K}$-extending module. As $End_R(M)\cong R$, the only direct summands of M are as follows: $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$, $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & n\\ 0 & 1\end{pmatrix}\mathbb{Z}$, where $n\in\mathbb{Z}$. Consider $\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}\in End_R(M)$. Then $r_M(\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix})=\begin{pmatrix}0 & -1\\ 0 & 2\end{pmatrix}\mathbb{Z}$ is not essential in a direct summand of M.

Example 3.2. (See [1, Example 2]) Let K be a field and let

$$R=\{\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & \alpha_4 & 0\\ 0 & 0 & \alpha_5\end{pmatrix}\mid \alpha_1,\alpha_2,\alpha_3\in K\}.$$

Consider $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $S_1 = \{ \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha \in K\}$,

and $S_2 = \{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha \in K\}$.

Then S_1 and S_2 are simple right R-modules and are nonisomorphic. It is easy to see that eR and eR/S_1 are $\mathcal{K}$-extending modules. By [1, Example 2], $M = eR \oplus eR/S_1$ is not $\mathcal{K}$-extending.

In the following, we recall some notions which are useful for proving Theorem 3.4.

A ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$; see [15]. Let M and N be R-modules. Nicholson and Zhou [17] extended the notion of the Jacobson radical of a ring to $Hom_R(M,N)$ as defined below:

$J(Hom_R(M,N)) = \{\alpha \in Hom_R(M,N) \mid 1_M - \alpha\beta \in Aut_R(M) \text{ for all } \beta \in Hom_R(N,M)\} = \{\alpha \in Hom_R(M,N) \mid 1_N - \beta\alpha \in Aut_R(N) \text{ for all } \beta \in Hom_R(N,M)\}$.

They called $J(Hom_R(M,N))$ the *Jacobson radical* of $Hom_R(M,N)$ (see also [9]). They also called a morphism $\alpha \in Hom_R(M,N)$ *semiregular* if there exists $\beta \in Hom_R(N,M)$ such that

$$\beta = \beta\alpha\beta \quad \text{and} \quad \alpha - \alpha\beta\alpha \in J(Hom_R(M,N)).$$

We recall that $Hom_R(M,N)$ is *semiregular* if every $\alpha \in Hom_R(M,N)$ is semiregular. Hence $End_R(M)$ is a semiregular ring if and only if $Hom_R(M,M)$ is semiregular.

Let $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ be modules. Then, by using the canonical injections and projections, $Hom_R(M,N)$ has a natural matrix representation as

$$Hom_R(M,N) = \begin{pmatrix} Hom_R(M_1,N_1) & Hom_R(M_1,N_2) & \cdots & Hom_R(M_1,N_t) \\ Hom_R(M_2,N_1) & Hom_R(M_2,N_2) & \cdots & Hom_R(M_2,N_t) \\ \vdots & \vdots & \ddots & \vdots \\ Hom_R(M_s,N_1) & Hom_R(M_s,N_2) & \cdots & Hom_R(M_s,N_t) \end{pmatrix}$$
$$= [Hom_R(M_i,N_j)],$$

where the elements of M and N are written as rows, and the matrix $[Hom_R(M_i,N_j)]$ acts by the right matrix multiplication.

Theorem 3.3. [17, Theorem 10] Let $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ be modules. Then $J(Hom_R(M,N)) = [J(Hom_R(M_i,N_j))]$.

Beidar and Kasch [5] defined the *singular ideal* $\Delta[M,N]$ of $Hom_R(M,N)$ by

$$\Delta[M,N] = \{\alpha \in Hom_R(M,N) \mid \operatorname{Ker}\alpha \leq^e M\}.$$

An R-module M has (C_2) if any submodule of M isomorphic to a summand of M is itself a summand. A ring R is called a *right C_2-ring* if R_R has (C_2). A module M is called N-C_2 (or *relatively C_2 to N*) if any submodule $N' \leq N$ with $N' \cong M' \leq^{\oplus} M$ implies $N' \leq^{\oplus} N$. Note that M has the C_2 condition if and only if M is M-C_2. Let N be a semisimple module and let M be any right R-module. Then M is N-C_2.

We now prove the following main theorem.

Theorem 3.4. Let $\{M_i\}_{i\in\mathcal{I}}$ be a family of right R-modules, where $\mathcal{I} = \{1, 2, \ldots, n\}$. Suppose that M_i is M_j-C_2 for all $i, j \in \mathcal{I}$. Then $\oplus_{i=1}^{n} M_i$ is a $\mathcal{K}$-extending module if and only if M_i is M_j-$\mathcal{K}$-extending for all $i, j \in \mathcal{I}$.

Proof. Let M_i be M_j-$\mathcal{K}$-extending and let M_i be M_j-C_2 for all $i, j \in \mathcal{I} = \{1, 2, \ldots, n\}$. Then, by [17, Theorem 33], $Hom_R(M_i, M_j)$ is semiregular and $\Delta(Hom_R(M_i, M_j)) = J(Hom_R(M_i, M_j))$ for all $i, j \in I$. By [17, Corollary 23], $S = End_R(M)$ is semiregular. Using Theorem 3.3, we have the following equalities:

$$\begin{aligned} J(S) = J(Hom_R(M, M)) &= J[Hom_R(M_i, N_j)] = [J(Hom(M_i, M_j))] \\ &= [\Delta(Hom(M_i, M_j))] = \Delta[Hom(M_i, M_j)] = \Delta(S). \end{aligned}$$

Thus, by [19, 41.22], M is $\mathcal{K}$-extending. □

Corollary 3.5. Let M be a $\mathcal{K}$-extending module with C_2 condition. Then any finite direct sum of copies of M is a $\mathcal{K}$-extending module.

The following example follows from Corollary 3.5.

Example 3.6. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Consider $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ as a right R-module. Then M is a $\mathcal{K}$-extending module with C_2 condition because M is a nonsingular quasi-injective R-module. Thus, by Corollary 3.5, $M^{(n)}$ is $\mathcal{K}$-extending for any $n \in \mathbb{N}$.

Corollary 3.7. Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of right R-modules, where $\mathcal{I} = \{1, 2, \ldots, n\}$. Assume that M_i is M_j-injective for all $i, j \in \mathcal{I}$. Then $\oplus_{i=1}^{n} M_i$ is a $\mathcal{K}$-extending module if and only if M_i is M_j-$\mathcal{K}$-extending for all $i, j \in \mathcal{I}$.

Proof. It follows from Theorem 3.4 and [12, Proposition 2.27]. □

In the reset of this note, our focus is on an application of Corollary 3.7 (Theorem 3.9). First, we state the following trivial lemma.

Lemma 3.8. Let M and N be R-modules and let $T \leq N$. If M is an N-$\mathcal{K}$-extending module, then M is T-$\mathcal{K}$-extending.

We recall that the module M is $\mathcal{K}$-*nonsingular* if, for all $\varphi \in S$, $\operatorname{Ker}\varphi \leq^e M$ implies $\varphi = 0$. Note that, by [1, Lemma 6], a module M is $\mathcal{K}$-nonsingular $\mathcal{K}$-extending if and only if M is Rickart. Thus injective hull $E(M)$ of a $\mathcal{K}$-nonsingular module M is Rickart. It was shown in [12] that for every nonsingular extending module M, the modules M and $E(M)$ are relatively Rickart to each other and $E(M) \oplus M$ is a Rickart module. In the following result, we prove this note with another conditions.

Theorem 3.9. Let M be a nonsingular $E(M)$-$\mathcal{K}$-extending module. Then M and $E(M)$ are relatively Rickart to each other and $E(M) \oplus M$ is a Rickart module.

Proof. Assume that M is a nonsingular $E(M)$-$\mathcal{K}$-extending module. Since $M \subseteq E(M)$, the module M is $\mathcal{K}$-extending by Lemma 3.8. Then, by [1, Lemma 6], M is a Rickart module. We know that the injective hull $E(M)$ of a $\mathcal{K}$-nonsingular module M is Rickart. As $M \subseteq E(M)$, $E(M)$ is M-Rickart by [12, Theorem 2.6].

Now we show that M is $E(M)$-Rickart. Let $f \in Hom_R(M, E(M))$ be arbitrary and let ι be the natural inclusion map from M to $E(M)$. As $E(M)$ is injective, there exists $g \in End_R(E(M))$ such that $f = g\iota$, so $\operatorname{Ker} f = \operatorname{Ker} g \cap M$. Then $\operatorname{Ker} g$ is a direct summand of $E(M)$. Since M is $E(M)$-$\mathcal{K}$-extending , there exists a direct summand H of M such that $\operatorname{Ker} f$ is essential in H. Thus $\operatorname{Ker} f$ is essential in $E(H)$ and $E(H)$ is a direct summand of $E(M)$. It is clear that $\operatorname{Ker} f$ is essential in $\operatorname{Ker} g$ and $\operatorname{Ker} g$ is a direct summand of $E(M)$. Because $E(M)$ is nonsingular, $E(H) = \operatorname{Ker} g$. Thus $H \leq \operatorname{Ker} g \cap M = \operatorname{Ker} f$. Hence $\operatorname{Ker} f$ is a direct summand of M. Therefore M is $E(M)$-Rickart. Now by [12, Corollary 2.13], $E(M) \oplus M$ is a Rickart module. □

Corollary 3.10. ([12, Theorem 2.16]) Let M be a nonsingular extending module. Then M and $E(M)$ are relatively Rickart to each other and $E(M) \oplus M$ is a Rickart module.

Proof. It is easy to see that if M is extending, then M is $E(M)$-$\mathcal{K}$-extending. Thus, by Theorem 3.9, the result holds. □

The condition "M is $E(M)$-$\mathcal{K}$-extending" in Theorem 3.9 is not superfluous as we can see in the following example.

Example 3.11. Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. It is clear that A is a commutative ring and von Neumann regular. Consider $R = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant }\}$. Then R is a subring of A and, by [10, Example 7.54], R is a von Neumann regular ring. Let $M = R_R$. Then M is a nonsingular Rickart, and so M is $\mathcal{K}$-extending. On the other hand, the injective hull, $E(M) = A$, is an injective Rickart R-module. In this case, $E(M)$ is M-injective and M-Rickart, but M is not $E(M)$-$\mathcal{K}$-extending and so M is not $E(M)$-Rickart. For $\varphi = (1, 0, 1, 0, \dots, 1, 0, \dots) \in Hom_R(M, E(M))$, $\operatorname{Ker}\varphi$ is not essential in a direct summand of M. Hence, $E(M) \oplus M$ is not a Rickart module by [12, Theorem 2.6].

In Theorem 3.9, the nonsingular condition is not superfluous as shown in the next example.

Example 3.12. (See [12, Example 2.19]) The $\mathbb{Z}$-module $M = \mathbb{Z}_p$, where p is a prime number in $\mathbb{N}$, is not nonsingular but is $\mathcal{K}$-extending. Note that $E(M) = \mathbb{Z}_{p^\infty}$ is not a Rickart $\mathbb{Z}$-module. Hence $E(M) \oplus M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is not a Rickart $\mathbb{Z}$-module.

The following example states an application of Theorem 3.9.

Example 3.13. It is well known that $E(\mathbb{Z}) = \mathbb{Q}$. Since all nonzero homomorphisms from $\mathbb{Z}$ to $\mathbb{Q}$ are monomorphism, the $\mathbb{Z}$-module $\mathbb{Z}$ is a $\mathbb{Q}$-$\mathcal{K}$-extending module. Hence, by Theorem 3.9, $\mathbb{Z} \oplus \mathbb{Q}$ is a Rickart $\mathbb{Z}$-module.

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