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Research Paper

2-DOMINATION IN VAGUE GRAPHS

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Abstract. A vague graph is a generalized structure of a fuzzy graph that gives more precision, flexibility and compatibility to a system when compared with systems that are designed using fuzzy graphs. In this paper, the notions of (perfect-total) 2-dominating set and (perfect-total) 2-domination numbers on vague graphs are introduced and some properties are investigated. Especially, it is proven that in any strong vague graph on a Petersen graph, any minimal 2-dominating set is a minimal perfect 2-dominating set and minimal dominating set. Then, the concepts of (total) 2-cobondage set and (total) 2-cobondage number in vague graphs are expressed and related results obtained. Finally, an application related to Fire Stations and Emergency Medical centers is provided.

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1. Introduction

Euler first introduced the concept of graph theory in 1736. The theory of graph is regarded as an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science. Cockayne and Hedetniemi [9] introduced the domination number and the independence number. Further, Janakiram and Kulli [12] suggested the concept of the cobondage number in graphs. In addition Zadeh [22] first proposed the theory of fuzzy sets. Then, Rosenfeld [20] introduced the concept of fuzzy graph theory as a generalization of Eulers graph. Somasundaram [21] discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graphs. Gau and Buehrer [10] proposed the concept of the vague set by replacing the value of an element in a set with a subinterval of $[0,1]$. Namely, a true-membership function $t_v(x)$ and a false membership function $f_v(x)$ are used to describe the boundaries of the membership degree. Accordingly, Ramakrishna [17] introduced the concept of vague graphs, along with some of their properties. Nagoor Gani and Chandrasekaran[13] discussed domination in the fuzzy graph using strong arcs. Nagoor Gani and Prasanna Devi[14] discussed 2-domination in fuzzy graphs. Borzooei and Rashmanlou [6, 7, 8, 18, 19] studied different types of dominating set in vague graphs. Borzooei and Banitalebi [5] introduced concepts of additions of an arc, cobondage sets, and cobondage numbers in vague graphs.

The purpose of this paper is to discuss the concepts of (total) 2-dominating set, (total) 2-domination number, (total) 2-cobondage set and (total) 2-cobondage number in vague graphs and it's application.

2. Preliminaries

In this section, we review some definitions and results from[3, 4, 5, 6, 11, 12, 17], which we need in what follows.

A *fuzzy graph* $G = (\sigma, \mu)$ on simple graph $G^* = (V, E)$ is a pair of functions $\sigma : V \to [0,1]$ and $\mu : E \to [0,1]$ such that, for any $uv \in E$, $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$, where $\wedge$ denote minimum. A *vague set* A on a finite non-empty set X, is a pair (t_A, f_A), where $t_A : X \to [0,1]$ and $f_A : X \to [0,1]$ are true and false membership functions, respectively such that for all $x \in X$, $0 \leq t_A(x) + f_A(x) \leq 1$. Note that $t_A(x)$ is considered as the lower bound for degree of membership of x in A and $f_A(x)$ is the lower bound for negative of membership of x in A. So, the degree of membership of x in the vague set A, is characterized by the interval $[t_A(x), 1 - f_A(x)]$.

A *vague graph* on a non-empty set V is a pair $G = (A, B)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague relation on V such that

$$t_B(xy) \leq \min\{t_A(x), t_A(y)\} \quad , \quad f_B(xy) \geq \max\{f_A(x), f_A(y)\},$$

for all $x, y \in V$. Note that B is called vague relation on A.
The underlying *crisp graph* of a vague graph $G = (A, B)$, is the graph $H = (V_1, E_1)$, where

$$V_1 = \{v \in V : t_A(v) > 0, f_A(v) > 0\} \subseteq V \quad , \quad E_1 = \big\{uv \in E : t_B(uv) > 0, f_B(uv) > 0\big\} \subseteq E$$

A vague graph G is called *strong* if for any $v_iv_j \in E$,

$$t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\} \ , \ f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\}.$$

A vague graph G is called *complete* if for any $v_i, v_j \in V$,

$$t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\} \ , \ f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\}.$$

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. Then,
(i) the *vertex cardinality* of G is defined by,

$$|V| = \sum_{v_i \in V} \big(\frac{t_A(v_i) + (1 - f_A(v_i))}{2}\big),$$

(ii) the *edge cardinality* of G is defined by,

$$|E| = \sum_{v_iv_j \in E} \big(\frac{t_B(v_iv_j) + (1 - f_B(v_iv_j))}{2}\big),$$

(iii) the *cardinality* of G is defined by,

$$|G| = |V| + |E|$$

(iv) for any $U \subseteq V$, the vertex cardinality of U is denoted by $O(U)$ and defined by,

$$O(U) = \sum_{v_i \in U} \big(\frac{t_A(v_i) + (1 - f_A(v_i))}{2}\big),$$

(v) for any $F \subseteq E$, the *edge cardinality* of F is denoted by $S(F)$ and defined by,

$$S(F) = \sum_{v_iv_j \in F} \big(\frac{t_B(v_iv_j) + (1 - f_B(v_iv_j))}{2}\big).$$

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. Then
(i) the *complement* of G is a vague graph $\overline{G} = (\overline{A}, \overline{B})$, where $\overline{A} = A$, and for any $v_iv_j \in E$,

$$\overline{t_B}(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\} - t_B(v_iv_j)$$

and

$$\overline{f_B}(v_iv_j) = f_B(v_iv_j) - \max\{f_A(v_i), f_A(v_j)\}.$$

(ii) an edge $e = uv \in E$ is called an *strong edge* of G, if

$$t_B(uv) \geq (t_B)^{\infty}(uv) \quad , \quad f_B(uv) \leq (f_B)^{\infty}(uv)$$

where

$$(t_B)^{\infty}(uv) = \max\{(t_B)^k(uv) | \ k = 1, 2, \cdots, n\} \ , \ (f_B)^{\infty}(uv) = \min\{(f_B)^k(uv) | \ k = 1, 2, \cdots, n\}$$

and

$$t_B^k(uv) = \min\{t_B(ux_1), t_B(x_1x_2), \cdots, t_B(x_{k-1}v) \big| u, x_1, \cdots, x_{k-1}, v \in V, k = 1, 2, \cdots, n\},$$

$$f_B^k(uv) = \max\{f_B(ux_1), f_B(x_1x_2), \cdots, f_B(x_{k-1}v) \big| u, x_1, \cdots, x_{k-1}, v \in V, k = 1, 2, \cdots, n\}.$$

(iii) the *neighborhood* of $u \in V$ is denoted by $N(u)$ and is defined as follows:

$$N(u) = \{v \in V \mid uv \text{ is a strong edge in } G\}.$$

(iv) we say that u *dominate* v in G if there exists a strong edge between u and v.
(v) $S \subset V$ is called a *dominating set* in G, if for any $v \in V - S$, there exists $u \in S$ such that u dominates v.
(vi) a dominating set S in G is called a *minimal dominating set*, if no proper subset of S is a dominating set.
(vii) the *lower domination number* of G denoted by $d_V(G)$ and is defined by,

$$d_V(G) = \min\{O(D) \big| D \text{ is a minimal domination set of } G\},$$

$(viii)$ the *upper domination number* of G denoted by $D_V(G)$ and is defined by,

$$D_V(G) = \max\{O(D) \big| D \text{ is a minimal domination set of } G\}.$$

(ix) the *domination number* of G denoted by $\Delta_V(G)$ and is defined by,

$$\Delta_V(G) = \frac{d_V(G) + D_V(G)}{2}.$$

(x) two vertices $u, v \in V$ are called *independent,* if there is no any strong edge between them.
(xi) $S \subseteq V$ is called an *independent set* in G, if for any $u, v \in S$, $t_B(uv) < (t_B)^{\infty}(uv)$ and $f_B(uv) > (f_B)^{\infty}(uv)$.

Let $G^* = (V, E)$ be a simple graph which is not complete graph and $e = xy \notin E$, where $x, y \in V$. Suppose that $G = (A, B)$ be a vague graph on G^* and $G_e = (A_e, B_e)$ be a vague graph on $G_e^* = (V, E_e)$, as an extension of G on G^*, where $E_e = E \cup \{e\}$. Then e is called an *additional strong edge*, if e is a strong edge in G_e.

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. Then
(i) the *cobondage set* of a vague graph G is the set C of additional strong edges of G_e, that reduces the domination number of G. i.e.,.

$$\Delta_V(G_C) < \Delta_V(G),$$

(ii) a cobondage set C of G is said to be *minimal cobondage set* if no proper subset of X is a cobondage set,
(iii) the *lower cobondage number* of G denoted by $b_E(G)$ and is defined by,

$$b_E(G) = \min\{S(C)\big| C \text{ } is \text{ } a \text{ } minimal \text{ } cobondage \text{ } set \text{ } of \text{ } G\},$$

(iv) the *upper cobondage number* of G denoted by $B_E(G)$ and is defined by,

$$B_E(G) = \max\{S(C)\big| C \text{ } is \text{ } a \text{ } minimal \text{ } cobondage \text{ } set \text{ } of \text{ } G\}.$$

Notation. From now one, in this paper, we let $G = (A, B)$ be a vague graph on a finite simple graph $G^* = (V, E)$.

3. 2-dominating set in vague graphs

Dominating sets in the graph theory have some applications in electrical networks, neural networks, operations research and monitoring communication. There are some phenomenal in the above mentioned fields in which dominating sets appear as 2-dominating sets. Therefore, considering the 2-dominating sets and the related results is worthwhile. In this section, we define the notions of 2-dominating set, perfect 2-dominating set, total 2-dominating set, 2-domination numbers, perfect 2-domination numbers and total 2-domination numbers on vague graphs and investigated some related results.

Definition 3.1. $D_2 \subset V$ is called a 2-dominating set of G, if for any node $v \in V - D_2$, then v dominate by at least two vertices in D_2.

Example 3.2. Consider the vague graph G as Figure 1. It is clear that, $D_2 = \{u_2, u_3, u_4, u_5\}$ is a 2-dominating set in G.

Definition 3.3. (i) A 2-dominating set D_2 of G is called a minimal 2-dominating set, if no proper subset of D_2 is a 2-dominating set of G.
(ii) Minimum vertex cardinality among all minimal 2-dominating sets of G, is called lower 2-domination number of G and is denoted by $d_V^2(G)$.
(iii) Maximum vertex cardinality among all minimal 2-dominating sets of G is called upper 2-domination number of G and is denoted by $D_V^2(G)$.

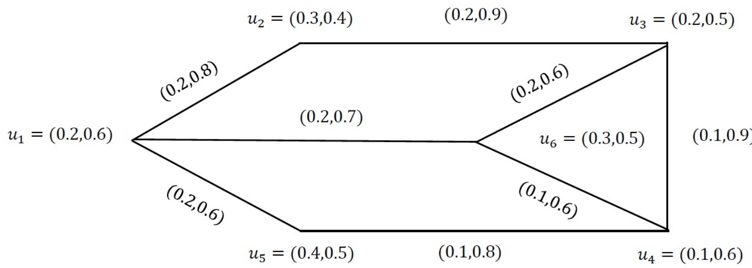

Figure 1. Vague graph G.

(iv) The 2-domination number of G is denoted by $\Delta_V^2(G)$ and is defined by:

$$\Delta_V^2(G) = \frac{d_V^2(G) + D_V^2(G)}{2}.$$

Example 3.4. Consider the vague graph G as Figure 2. It is easy to see that two sets

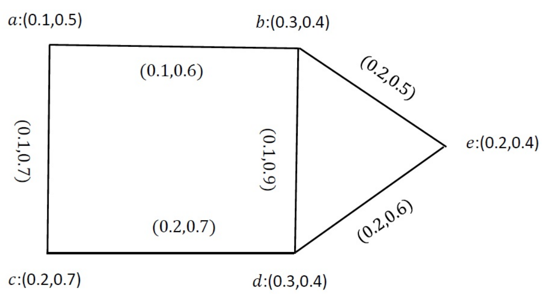

Figure 2. Vague graph G.

$M_1 = \{a, c, e\}$ and $M_2 = \{b, c, d\}$ are minimal 2-dominating sets of G. By routine calculations, $D_V^2(G) = 1.2$, $d_V^2(G) = 1.05$ and $\Delta_V^2(G) = 1.125$.

Proposition 3.5. (i) *If* $|N(v)| \leq 1$, *for* $v \in V$, *then* v *belongs to the every 2-dominating sets of* G.
(ii) *Every 2-dominating set of* G *is a dominating set.*
(iii) $d_V^2(G) \geq d_V(G)$, $D_V^2(G) \geq D_V(G)$ *and so* $\Delta_V^2(G) \geq \Delta_V(G)$.
(iv) $d_V^2(G) + d_V^2(\overline{G}) \leq 2O(G)$, *where* $d_V^2(\overline{G})$ *is the lower 2-domination number of* $\overline{G}$.
(v) *If* G *is a complete vague graph with* $n \geq 2$ *nodes, then*

$$d_V^2(G) = \min\{O(S) | S \text{ is a 2-element subset of } V\}$$

and

$$D_V^2(G) = \max\{O(S)|S \textit{ is a 2-element subset of } V\}.$$

Proof. The proofs are straightforward. □

Theorem 3.6. *A 2-dominating set D_2 of a vague graph G is a minimal 2-dominating set if and only if for any vertex $d \in D_2$, one of the following conditions holds.*
(i) d is not a strong neighbor of any vertex in D_2,
(ii) d is dominated by a unique vertex in D_2 (that is, d is one of nodes of an isolated strong edge),
(iii) there are vertices $v \in V - D_2$ and $u \in D_2$ such that $N(v) \cap (D_2 - \{d\}) = \{u\}$.

Proof. Assume that D_2 is a minimal 2-dominating set of G. Then, for any vertex $d \in D_2$, $D_2 - \{d\}$ is not a 2-dominating set. Thus there exists $v \in V - (D_2 - \{d\})$ which is not dominated by two vertices in $D_2 - \{d\}$. If $v = d$, then v is not a strong neighbor of any vertex in D_2 or v is dominated by unique vertex in $D_2 - \{d\}$. If $v \neq d$, then v is not dominated by two vertices in $D_2 - \{d\}$, but is dominated by two vertices in D_2. Hence there is a vertex $u \in D_2$ such that $N(v) \cap (D_2 - \{d\}) = \{u\}$.

Conversely, assume that D_2 is not a minimal 2-dominating set. Then there exists a vertex $d \in D_2$ such that $D_2 - \{d\}$ is a 2-dominating set. Thus d is a strong neighbor of at least two vertices in $D_2 - \{d\}$, and so (i) and (ii) do not hold. Also, every vertex in $V - D_2$ is a strong neighbor of at least two vertices in $D_2 - \{d\}$, and so (iii) does not hold, which is a contradiction. Since at least one of the conditions should be hold. Therefore, D_2 is a minimal 2-dominating set. □

Remark 3.7. Conditions (i) and (ii) of Theorem 3.6 equivalent by $\left|N(d)\right| \leq 1$.

Note. If D is a minimal dominating set of a vague graph $G = (A, B)$ with out isolated vertex, then $V - D$ is a dominating set of G. The next example shows that if D_2 is a minimal 2-dominating set, then $V - D_2$ is not necessarily 2-dominating set.

Example 3.8. Consider the vague graph G as Figure 3. It is clear that, $D_2 = \{u_3, u_4, u_5\}$ is a minimal 2-dominating set of G but $M = \{u_1, u_2\}$ is not a 2-dominating set of G.

Theorem 3.9. *Let D_2 be a minimal 2-dominating set of G. Then $V - D_2$ is not a 2-dominating set of G.*

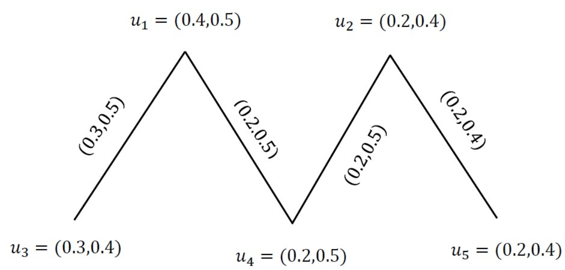

Figure 3. Vague graph G.

Proof. Assume $v \in V$ and $|N(v)| \leq 1$. Then v belongs to every 2-dominating set of G. Thus $v \in D_2$ and so v does not belong to $V - D_2$. Therefore, $V - D_2$ is not a 2-dominating set of G. □

Theorem 3.10. *Let G be a vague graph and D_2 be an independent 2-dominating set of G. Then D_2 is a minimal dominating set of G.*

Proof. If D_2 is an independent 2-dominating set of vague graph G, then D_2 is dominating set of G and $D_2 - \{d\}$ is not a dominating set for any $d \in D_2$. Hence D_2 is a minimal dominating set of G. □

Theorem 3.11. *Let G^* be a Petersen graph and G be a strong vague graph on G^*. Then any minimal 2-dominating set D_2 of G is an independent set of G.*

Proof. Let there are $x, y \in D_2$ such that $xy \in E$, by the contrary. Then by Theorem 3.6, there are $w \in V - D_2$ and $u \in D_2$ such that $N(w) \cap (D_2 - \{x\}) = \{u\}$ and so $wx, wu \in E$. Now, we consider the following cases:

Case 1. If $u = y$, then (wxy) is a cycle in G^* with length 3 , which is a contradiction.

Case 2. If $u \neq y$, then by Theorem 3.6, there are $t \in V - D_2$ and $z \in D_2$ such that $tz, tu \in E$. Now, in this case, if $z = x$, then $(txwu)$ is a cycle with length 4 in G^*, which is a contradiction. If $z = y$, since $d_{G^*}(y) = 3$, then there exists $k \in V$ such that $ky \in E$. If $kx \in E$, then (kxy) is a cycle in G^* with length 3, which is a contradiction. If $ku \in E$, then $(kytu)$ is a cycle in G^* with length 4, which is a contradiction. Thus there exist $v \in D_2$, such that $kv \in E$ and by Theorem 3.6 there exists $f \in V - D_2$ such that $fv \in E$. Now, if $fx \in E$, then $(fxwutykv)$ is a cycle in G^* with length 8, which is a contradiction. If $fy \in E$, then $(kvfy)$ is a cycle in G^* with length 4, which is a contradiction. If $fu \in E$, then $(fuwxykv)$ is a cycle in G^* with length 7, which is a contradiction. Therefore, xy does not belong to E. □

Corollary 3.12. *Let G^* be a Petersen graph and G be a strong vague graph on G^*. Then any minimal 2-dominating set of G is a minimal dominating set of G and so $D_V^2(G) = D_V(G)$.*

Theorem 3.13. *Let G^* be a star graph and G be a vague graph on G^*. Then*

$$d_V^2(G) = D_V^2(G) = \Delta_V^2(G).$$

Proof. The nodes except the center node, will have only one strong neighbor and it will be in every 2-dominating set of G and the center node will have all other nodes as its strong neighbors. Hence the nodes except the center node, form an unique minimal 2-dominating set of G and

$$d_V^2(G) = D_V^2(G) = \sum_{i=2}^{n} \frac{1 + t_A(v_i) - f_A(v_i)}{2},$$

where $v_i \in V$, for $i = 1, 2, ..., n$ and v_1 is a center node. Therefore, $d_V^2(G) = D_V^2(G) = \Delta_V^2(G)$.

□

Note. If e is an additional strong arc in G_e^*, then

$$d_V^2(G_e) \leq d_V^2(G) \ , \ D_V^2(G_e) \leq D_V^2(G),$$

and so, $\Delta_V^2(G_e) \leq \Delta_V^2(G)$.

Definition 3.14. $D_2^p \subset V$ is called a perfect 2-dominating set of G, if for any node $v \in V - D_2^p$, then v dominate by exactly two vertices in D_2^p.

Definition 3.15. (i) A perfect 2-dominating set D_2^p of G is called a minimal perfect 2-dominating set, if no proper subset of D_2^p is a perfect 2-dominating set of G.
(ii) Minimum vertex cardinality among all minimal perfect 2-dominating sets of G, is called lower perfect 2-domination number of G and is denoted by $d_p^2(G)$.
(iii) Maximum vertex cardinality among all minimal perfect 2-dominating sets of G is called upper perfect 2-domination number of G and is denoted by $D_p^2(G)$.
(iv) The perfect 2-domination number of G is denoted by $\Delta_P^2(G)$ and is defined by:

$$\Delta_p^2(G) = \frac{d_p^2(G) + D_p^2(G)}{2}.$$

Example 3.16. Consider the vague graph G as Figure 4. It is easy to see that two sets $M_1 = \{a, c, e\}$ and $M_2 = \{b, f, d\}$ are minimal perfect 2-dominating sets of G. By routine calculations, $D_p^2(G) = 1.35$, $d_p^2(G) = 0.95$ and $\Delta_p^2(G) = 1.15$.

Remark 3.17. If D_2^p be a minimal perfect 2-dominating set of G. Then $V - D_2^p$ is not a perfect 2-dominating set of G.

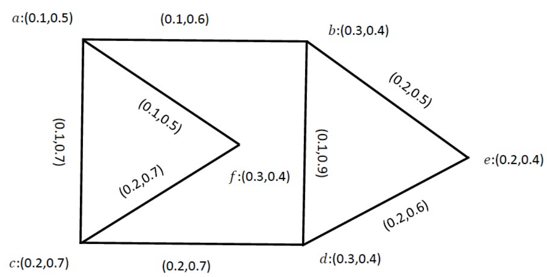

FIGURE 4. Vague graph G.

Remark 3.18. If D_2^p be an independent perfect 2-dominating set of G. Then D_2^p is a minimal dominating set of G.

Corollary 3.19. *A perfect 2-dominating set D_2^p of a vague graph G is a minimal perfect 2-dominating set if and only if for any vertex $d \in D_2^p$, one of the following conditions holds.*
(i) d is not a strong neighbor of any vertex in D_2^p,
(ii) d is dominated by a unique vertex in D_2^p (that is, d is one of nodes of an isolated strong edge),
(iii) there are vertices $v \in V - D_2^p$ and $u \in D_2^p$ such that $N(v) \cap (D_2^p - \{d\}) = \{u\}$.

Note. The next example shows that if D_2^p is a minimal 2-dominating set, then D_2^p is not necessarily minimal perfect 2-dominating set.

Example 3.20. Consider the vague graph G as Figure 5. It is clear that, $D_2 = \{a, c, e\}$ is a

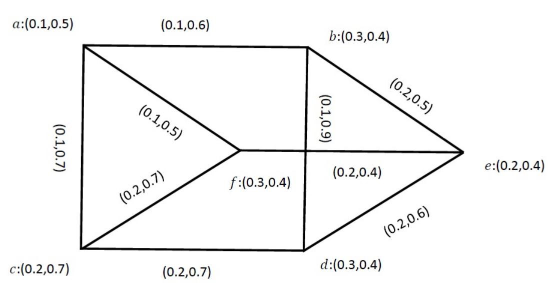

FIGURE 5. Vague graph G.

minimal 2-dominating set of G but is not a minimal perfect 2-dominating set of G.

Theorem 3.21. *Let G^* be a Petersen graph and G be a strong vague graph on G^*. Then any minimal 2-dominating set D_2 of G is a minimal perfect 2-dominating set of G.*

Proof. Assume that D_2 is not a minimal perfect 2-dominating set by the contrary. Then there is $u \in V - D_2$ such that $N(u) \cap D_2 = \{x, y, z\}$. Now by Theorem 3.6, there are $w \in V - D_2$ and $v \in D_2$ such that $N(w) \cap (D_2 - \{v\}) = \{x\}$ and also by Theorem 3.11, D_2 is an independent set of G, which is contradiction. Therefore, D_2 is a minimal perfect 2-dominating set of G. □

Corollary 3.22. *Let G^* be a Petersen graph and G be a strong vague graph on G^*. Then*

$$D_V^2(G) = D_V(G) = D_p^2(G).$$

Definition 3.23. $D_2^t \subset V$ is called a total 2-dominating set of G, if for any node $v \in V$, v dominated by at least two vertices in D_2^t.

Definition 3.24. (i) A total 2-dominating set D_2^t of G is called minimal total 2-dominating set if no proper subset of D_2^t is a total 2-dominating set of G.
(ii) Minimum vertex cardinality among all minimal total 2-dominating sets of G is called lower total 2-domination number of G and is denoted by $td_V^2(G)$.
(iii) Maximum vertex cardinality among all minimal total 2-dominating sets of G is called upper total 2-domination number of G and is denoted by $TD_V^2(G)$.
(iv) The total 2-domination number of G is denoted by $T\Delta_V^2(G)$ and is defined as follows,

$$T\Delta_V^2(G) = \frac{td_V^2(G) + TD_V^2(G)}{2}.$$

Example 3.25. Consider the vague graph G as Figure 6. Clear that, $D_2^t = \{b, c, e\}$ is a

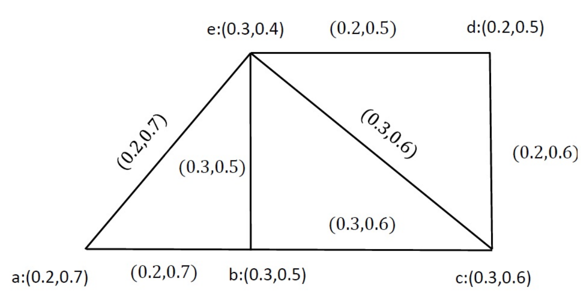

FIGURE 6. Vague graph G.

minimal total 2-dominating set in G and $td_V^2(G) = TD_V^2(G) = T\Delta_V^2(G) = 1.2$.

Theorem 3.26. *A total 2-dominating set D_2^t of a vague graph G is a minimal total 2-dominating set if and only if for any vertex $v \in D_2^t$ one of the following conditions holds.*
(i) There are vertices $w \in V - D_2^t$ and $x \in D_2^t$ such that $N(w) \cap (D_2^t - \{x\}) = \{v\}$.

(ii) There are vertices $y,\ z \in D_2^t$ *such that* $N(y) \cap (D_2^t - \{z\}) = \{v\}$.

Proof. Assume that D_2^t is a minimal total 2-dominating set of G. Then for any vertex $v \in D_2^t$, $D_2^t - \{v\}$ is not a total 2-dominating set. Hence there exists $w \in V$, which is not dominated by at least two vertices in $D_2^t - \{v\}$. If $w \in V - D_2^t$, then $|N(w) \cap (D_2^t - \{v\})| = 1$, but w is dominated by two vertices in D_2^t. So there is a vertex $x \in D_2^t$ such that $N(w) \cap (D_2^t - \{x\}) = \{v\}$. If $w \in D_2^t$, then $|N(w) \cap (D_2^t - \{v\})| = 1$, but w is dominated by two vertices in D_2^t. Hence there is a vertex $z \in D_2^t$ such that $N(w) \cap (D_2^t - \{z\}) = \{v\}$.

Conversely, assume that D_2^t is not a minimal total 2-dominating set by the contrary. Then there exists a vertex $v \in D_2^t$ such that $D_2^t - \{v\}$ is a total 2-dominating set. Thus for any vertex $w \in V - (D_2^t - \{v\}$, there exists at least two neighbors in $D_2^t - \{v\}$ and so (i) does not holds. Also, for any vertex $y \in D_2^t - \{v\}$ there exists at least two neighbors in $D_2^t - \{v\}$ and so (ii) does not holds, which is a contradiction. Therefore, D_2^t is a minimal total 2-dominating set of G. □

Note. If G has a vertex u such that $|N(u)| < 2$, then G has not any total 2-dominating set and $td_V^2(G) = TD_V^2(G) = 0$.

Theorem 3.27. $td_V^2(G) = TD_V^2(G) = |V|$ *if and only if there exists exactly two strong neighbors, for any vertex of* G.

Proof. If for any vertex of G, there exist exactly two neighbors, then $D_2^t = V$ is the only total 2-dominating set in G and so $td_V^2(G) = TD_V^2(G) = |V|$. Conversely, assume that $td_V^2(G) = TD_V^2(G) = |V|$. If there exists a vertex u such that $|N(u)| \geq 3$ and v, w and z be neighbors of u, then $V - \{z\}$ is a total 2-dominating set of G. So that $td_V^2(G) < |V|$, which is a contradiction. Therefore, for any vertex of G there exists exactly two neighbors. □

Corollary 3.28. (i) *Every total 2-dominating set of vague graph* G *is a 2-dominating set of* G,
(ii) *if* G *has a total 2-dominating set, then* G *consists of cycle or cycles made of strong edges.*

Theorem 3.29. *Let* G *be a vague graph and* D_2 *be a minimal 2-dominating set of* G. *Then* D_2 *is a minimal total 2-dominating set of* G *if and only if all vertices of* D_2 *belong of cycles made of strong edges of* G *and its vertices.*

Proof. If D_2 is a minimal total 2-dominating set of G, then any vertex of D_2 dominated by at least two vertices of D_2. So that D_2 is on cycles made of strong edges of G. Conversely, assume that all vertices of D_2 are on cycles made of strong edges of G and its vertices, then for any vertex of D_2, there exists at least two neighbors in D_2, also D_2 is a minimal 2-dominating set of G. Therefore, D_2 is a minimal total 2-dominating set of G. □

Corollary 3.30. *(i)* $d_V^2(G) \leq td_V^2(G)$, $D_V^2(G) \leq TD_V^2(G)$ *and so* $\Delta_V^2(G) \leq T\Delta_V^2(G)$,
(ii) if minimal 2-dominating set D_2 of vague graph G is a minimal total 2-dominating set, then the number of vertices of D_2 is at least three.

4. 2-cobondage number of a vague graph

In this section, we define the notion of 2-cobondage sets and 2-cobondage numbers in vague graphs and we investigate some related results.

Definition 4.1. (i) The 2-cobondage set of G is the set C_2 of additional strong arcs to G, such that reduces the 2-domination number of G, i.e.,.

$$\Delta_V^2(G_{C_2}) \leq \Delta_V^2(G).$$

(ii) A 2-cobondage set C_2 of G is said to be minimal 2-cobondage set if no proper subset of C_2 is a 2-cobondage set of G.

Remark 4.2. The cobondage set is not necessarily 2-cobondage set.

Definition 4.3. (i) Minimum edge cardinality among all minimal 2-cobondage sets of G is called lower 2-cobondage number of G, and denoted by $b_E^2(G)$.
(ii) Maximum edge cardinality among all minimal 2-cobondage sets of G is called upper 2-cobondage number of G, and denoted by $B_E^2(G)$.

Note. $b_E^2(G) \geq b_E(G)$ and $B_E^2(G) \geq B_E(G)$.

Example 4.4. Consider the vague graph G as Figure 7. Then two sets $\{a, c, e\}$ and $\{b, d, e\}$

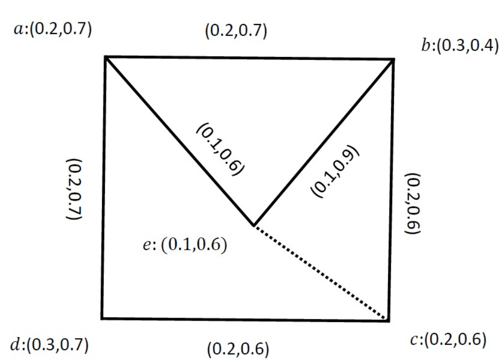

Figure 7. Vague graph G.

are minimal 2-dominating sets of vague graph G. Then by routine calculations, it is clear that $D^2_V(G) = 1$, $d^2_V(G) = 0.8$ and $\Delta^2_V(G) = 0.9$. Now by adding strong arc $ec = (0.1, 0.6)$ to G, $\{a, c\}$ is a minimal 2-dominating set of $(G)_{C_2}$, where $C_2 = \{ec\}$. Hence, C_2 is a minimal 2-cobondage set of G and $b^2_E(G) = 0.25$

Theorem 4.5. *If G has an isolated vertex $u \in V$, then*

$$b^2_E(G) \leq 2\big|\{u\}\big|.$$

Proof. Let $u \in V$ be an isolated vertex in G. Then u belongs to any minimal 2-dominating set D_2 of G. Suppose $w, z \in D_2$ and $w \neq z \neq u$. By adding two strong arcs $e_1 = uw$ and $e_2 = uz$ to G, then $D_2 - \{u\}$ is a minimal 2-dominating set of G_{C_2}, where $C_2 = \big\{e_1, e_2\big\}$. Hence, C_2 is a 2-cobondage set of G. Also,

$$t_B(e_1) \leq t_A(u),\ t_B(e_2) \leq t_A(u) \quad , \quad f_B(e_1) \geq f_A(u),\ f_B(e_2) \geq f_A(u)$$

Therefore,

$$b^2_E(G) \leq S(C_2) \leq \frac{1 + t_A(u) - f_A(u)}{2} + \frac{1 + t_A(u) - f_A(u)}{2} \leq 1 + t_A(u) - f_A(u) = 2\big|\{u\}\big|.$$

□

Theorem 4.6. *If G has an isolated strong edge $e = uv$, then*

$$b^2_E(G) \leq \min\{\big|\{u\}\big|, \big|\{v\}\big|\}.$$

Proof. Let $e = uv$ be an isolated strong edge in G. Then any one of u and v belongs to any minimal 2-dominating set D_2 of G. Suppose $w \in D_2$, $w \neq u$ and $w \neq v$. By adding strong arc $e_1 = uw$ or $e_2 = vw$ to G, then $D_2 - \{u\}$ or $D_2 - \{v\}$ is a minimal 2-dominating set of G_{e_1} or G_{e_2}. So, if $K_1 = \{e_1\}$ and $K_2 = \{e_2\}$, then $d^2_V(G_{e_1}) < d^2_V(G)$ and $d^2_V(G_{e_2}) < d^2_V(G)$. Hence, K_1 and K_2 are 2-cobondage sets of G. Also,

$$t_B(e_1) \leq t_A(u),\ t_B(e_2) \leq t_A(v) \quad , \quad f_B(e_1) \geq f_A(u),\ f_B(e_2) \geq f_A(v).$$

Thus,

$$b^2_E(G) = \min\big\{S(K_1), S(K_2)\big\} \leq \min\big\{\frac{1 + t_A(u) - f_A(u)}{2}, \frac{1 + t_A(v) - f_A(v)}{2}\big\} = \min\{\big|\{u\}\big|, \big|\{v\}\big|\}.$$

□

Theorem 4.7. *If G has a vertex u such that $N(u) = \{v\}$, then*

$$b^2_E(G) \leq 2\big|\{u\}\big| + \big|\{v\}\big|.$$

Proof. Let $u \in V$ such that $N(u) = \{v\}$. Then u belongs to any minimal 2-dominating set D_2 of G. If v belongs to D_2, then for any $w \in D_2$ and $w \neq u \neq v$, by adding strong arc uw to G, then $D_2 - \{u\}$ is a minimal 2-dominating set of G_{uw} and $d_V^2(G_{uw}) < d_V^2(G)$. So, if $C_2 = \{uw\}$, then C_2 is a 2-cobondage set of G. Also,

$$b_E^2(G) \leq S(C_2) \leq \big|\{u\}\big|$$

If v is not belong to D_2, then there is $t \in D_2$ such that $t \neq u$ and tv is a strong arc in G. By adding three strong arcs ut, uq and vq to G, for any $q \in D_2$ and $t \neq u \neq q$, then $D_2 - \{u\}$ is a minimal 2-dominating set of G_{C_2}, where $C_2 = \{ut, uq, vq\}$ and $d_V^2(G_{C_2}) < d_V^2(G)$. Hence C_2 is a 2-cobondage set of G and

$$b_E^2(G) \leq S(C_2) \leq 2(\frac{1 + t_A(u) - f_A(u)}{2}) + \frac{1 + t_A(v) - f_A(v)}{2}.$$

Therefore

$$b_E^2(G) \leq 2\big|\{u\}\big| + \big|\{v\}\big|.$$

□

Theorem 4.8. *Let D_2 is a minimal 2-dominating set of G. Then*

$$b_E^2(G) \leq 2\big|\{u\}\big| + 1,$$

where $u \in D_2$ and $\big|\{u\}\big| = \max\{\big|\{v\}\big| \; \big|v \in D_2\}$.

Proof. Let D_2 be a minimal 2-dominating set of G and $v \in D_2$ such that $\big|N(v)\big| > 1$. Then by Theorem 3.6, there are vertices $x \in V - D_2$ and $y \in D_2$ such that $N(x) \cap (D_2 - \{v\}) = \{y\}$. Then for any $w \in D_2$ and $w \neq y \neq v$, by adding strong arcs vy, vw and xw to G, $D_2 - \{v\}$ is a minimal 2-dominating set of G_{C_2} where $C_2 = \{vy, vw, xw\}$ and $d_V^2(G_{C_2}) < d_V^2(G)$. Hence, C_2 is a 2-cobondage set of G. Also

$$b_E^2(G) \leq S(C_2) \leq 2(\frac{1 + t_A(v) - f_A(v)}{2}) + \frac{1 + t_A(x) - f_A(x)}{2} \leq 2\big|\{u\}\big| + 1.$$

□

Example 4.9. Consider the vague graph G as Figure 8. Then $D_2 = \{a, c, d, e\}$ is a minimal 2-dominating set of G and by routine calculations, $b_E^2(G) \leq 0/45$.

Definition 4.10. (i) The total 2-cobondage set of a vague graph G is the set C_2^t of additional strong arcs to G, such that reduces the total 2-domination number of G, i.e.,.

$$T\Delta_V^2(G_{C_2^t}) \leq T\Delta_V^2(G),$$

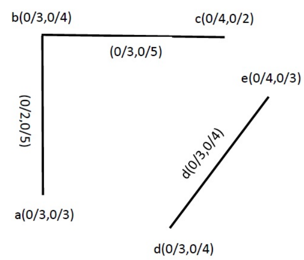

FIGURE 8. Vague graph G.

(ii) a total 2-cobondage set C_2^t of G is said to be minimal total 2-cobondage set if no proper subset of C_2^t is a total 2-cobondage set of G.

Remark 4.11. In the Example 4.4 show that 2-cobondage set and total 2-cobondage set are not necessarily equivalent.

Definition 4.12. (i) Minimum edge cardinality among all minimal total 2-cobondage sets of G is called lower total 2-cobondage number of G, and denoted by $tb_E^2(G)$.
(ii) Maximum edge cardinality among all minimal total 2-cobondage sets of G is called upper total 2-cobondage number of G, and denoted by $TB_E^2(G)$.

Example 4.13. Consider the vague graph G as Figure 9. Then, $D_2^t = \{a, b, c, f\}$ is a minimal

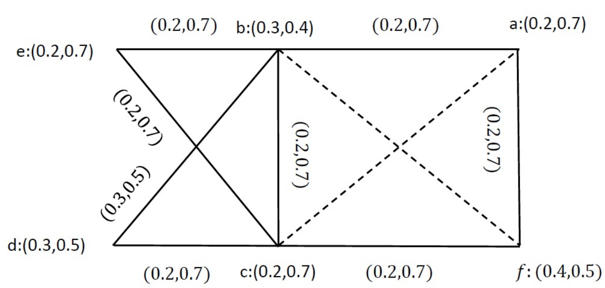

FIGURE 9. Vague graph G.

total 2-dominating set in G. By routine calculations, it is clear that $td_V^2(G) = TD_V^2(G) = 1/4$. Now by adding strong arc $bf = (0.3, 0.5)$ or $ac = (0.2, 0.7)$ to G, then $D_2^t - \{a\}$ or $D_2^t - \{b\}$ is a minimal total 2-cobondage set of G_{bf} or G_{ac}, respectively. Hence $C_2^t = \{bf\}$ and $C_2'^t = \{ac\}$ are minimal total 2-cobondage sets of G. Moreover,

$$TB_E^2(G) = 0/4\ ,\ tb_E^2(G) = 0/25$$

Proposition 4.14. $tb^2_E(G) \leq b^2_E(G)$ *and* $TB^2_E(G) \leq B^2_E(G)$.

Proof. The proof is straightforward □

Proposition 4.15. *Let* G^* *be a cycle and* G *be a vague graph on* G^* *with* $n \geq 4$ *nodes. Then*

$$T\Delta^2_V(G) = |V| \quad , \quad tb^2_E(G) \leq \min\{|u| \big| u \in V\}.$$

Proof. The proof is straightforward. □

5. Application

Domination and dominating set in graphs have applications to several fields, especially in the fields of operations research, neural networks and monitoring communication. And in some cases, it is essential and worthwhile to address domination and dominating sets in larger dimensions.

5.1. **Placement of fire stations and emergency medical centers.** A city map can be considered as a vague graph. In this vague graph, the vertices indicate the regions and edges indicate their communication paths in the city. We define the values of f-strength and t-strength for any $v \in V$ and $e \in E$, as follows:
$t_A(v)$: The minimum assurance of non-incidentalism in region v.
$f_A(v)$: The minimum assurance of incidentalism in region v.
$t_B(e)$: The minimum assurance of the timely presence at the incident scene through the e path.
$f_B(e)$: The minimum assurance of the timely absence at the incident scene through the e path.
Thus, the size of each vertex stands for $|v| = \dfrac{1 + t_A(v) - f_A(v)}{2}$, for any $v \in V$, represents the optimal level of assurance of the non-incidentalism of that region. Also, the size of each edge stands for $|e| = \dfrac{1 + t_B(e) - f_B(e)}{2}$, represents the optimal level of assurance of timely presence in the incident scene through that edge(path). It should be noted that such things as urban texture, the driving culture and the extent to which drivers and pedestrians follow driving rules, the type of industry and the presence of high-risk industries in one area are factors that contribute to the estimation of the incidentalism or non-incidentalism in that area. Also, the factors such as the volume of traffic, the number of traffic lights, the squares, the overpasses and pedestrian underpasses and the maximum and minimum speed of vehicles per path, etc., are affected by the route in the estimation of the assurance of timely presence or absence in that path. According to the above comparison, the 2-dominating sets could be considered as the location of fire stations and medical emergency centers in the city.
For example, suppose graph G represents a city map. In this case, $D_2 = \{u_2, u_3, u_5\}$ Set

represents the location of the Fire Stations and the Emergency Medical Center throughout the city. We note that $D2$ is the minimal 2-dominating set of graph G with vertex cardinality

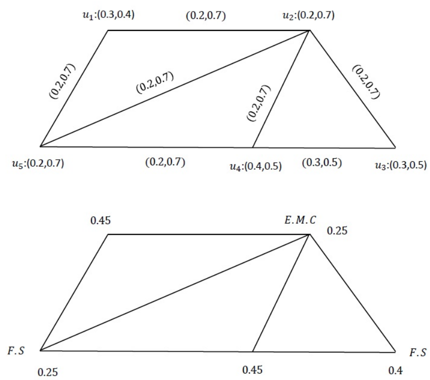

FIGURE 10.

of 0.95.

Purpose: How can we increase the optimum reliability of the timely presence of relief teams at the incident scene in the city, while not increasing the number of fire stations and emergency medical centers?

Answer: By reopening the path $e = u_3u_5$ with coordinates $(0.2, 0.7)$ between u_3 and u_5 areas, one can replace the location of the Emergency Medical Centers and Fire Stations D_2 (with vertex cardinality of 0.9) by set $D_2^* = \{u_2, u_5\}$(with vertex cardinality of 0.5), and also increase the optimum reliability of the timely presence of relief teams in the city by 0.25.

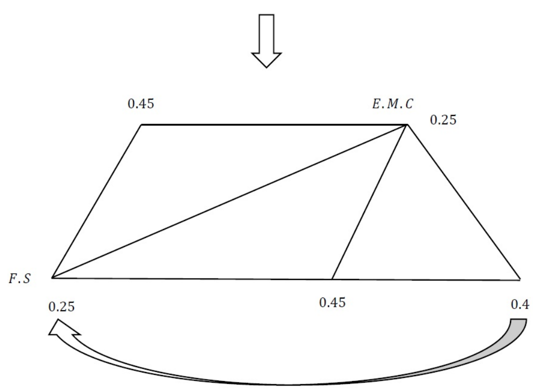

FIGURE 11. Vague graph G_e.

6. Conclusion

The theory of vague graphs has many applications in new science and technology. Since the vague models compare the classical and fuzzy models to the system, they give more accuracy, flexibility and compatibility. The concept of domination in graph is very important in both theoretical developments and applications. Nowadays, the concepts of domination and dominating set and numbers are considered as the fundamental concepts in the theory of vague graphs. In this paper, we introduced (perfect-total)2-dominating set, (perfect-total)2-domination numbers and some relations between 2-domination number and additions of arc in vague graphs and furthermore, we discussed concepts of 2-cobondage number in vague graphs. Especially, it was proved that in any strong vague graph on a Petersen graph, any minimal 2-dominating set is a minimal perfect 2-dominating set and minimal dominating set. Finally, by using the concept of 2-dominating set and the reduction effect of an additional strong arc on the 2-domination number parameter, a model for optimizing the 2-domination parameter was presented.

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