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Research Paper

ON GRADED J_{gr}-CLASSICAL PRIME SUBMODULES

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ABSTRACT. Let G be a group with identity e. Let R be a G-graded commutative ring with identity 1 and M a graded R-module. A proper graded submodule C of M is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in C$, then either $rm \in C$ or $sm \in C$. In this paper, we introduce the concept of graded J_{gr}-classical prime submodule as a new generalization of graded classical submodule and we give some results concerning such graded modules. We say that a proper graded submodule N of M is *a graded J_{gr}-classical prime submodule of* M if whenever $rsm \in N$ where $r, s \in h(R)$ and $m \in h(M)$, then either $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$, where $J_{gr}(M)$ is the graded Jacobson radical.

1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

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The notion of graded classical prime submodules as a generalization of graded prime submodules was introduced in [9] and studied in [1, 3, 4, 6, 7]. The purpose of this paper is to introduce the concept of graded J_{gr}-classical prime submodules as a new generalization of graded classical prime submodules and give a number of its properties.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [10] and [12–14] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group and let e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_α of R indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha\in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all α, $\beta \in G$. The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{\alpha\in G} R_\alpha$. If $r \in R$, then r can be written uniquely as $\sum_{\alpha\in G} r_\alpha$, where r_α is called a homogeneous component of r in R_α. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \bigoplus_{\alpha\in G}(I \cap R_\alpha) := \bigoplus_{\alpha\in G} I_\alpha$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A left R-module M is said to be *a graded R-module* (or *G-graded R-module*) if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Here, $R_\alpha M_\beta$ denotes the additive subgroup of M consisting of all finite sums of elements $r_\alpha m_\beta$ with $r_\alpha \in R_\alpha$ and $m_\beta \in M_\beta$. Also if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called a homogeneous element. Note that M_α is an R_e-module for every $\alpha \in G$. So, if $I = \bigoplus_{\alpha\in G} I_\alpha$ is a graded ideal of R, then I_α is an R_e-module for every $\alpha \in G$. A submodule N of M is said to be *a graded submodule of* M if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$.

Let R be a G-graded ring and M a graded R-module. A proper graded ideal P of R is said to be *a graded prime ideal* if whenever $rs \in P$, we have $r \in P$ or $s \in P$, where $r, s \in h(R)$ (see [15].) It is shown in [8, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rN \subseteq M\}$ is a graded ideal of R. A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$ (see [8].)

A proper graded submodule N of M is called *a graded classical prime* submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [9].)

A graded submodule N of M is said to be *a graded maximal submodule* if $N \neq M$ and if there is a graded submodule L of M such that $N \subseteq L \subseteq M$, then $N = L$ or $L = M$ (see [14].)

The graded Jacobson radical of a graded module M, denoted by $J_{gr}(M)$, is defined to be the intersection of all graded maximal submodules of M (if M has no graded maximal submodule then we shall take, by definition, $J_{gr}(M) = M$) (see [14].)

2. Results

The following lemma is known, (see [11, Lemma 1.2], but we write it here for the sake of references.

Lemma 2.1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(1) If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.

(2) If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M.

Definition 2.2. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be *a graded J_{gr}-classical prime submodule of M* if whenever $rsm \in N$ where $r, s \in h(R)$ and $m \in h(M)$, then either $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$.

Example 2.3. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$. Then M is a G-graded R-module with $M_0 = \mathbb{Z} \times \mathbb{Z}$ and $M_1 = \{(0, 0)\}$. Now, consider the graded submodule $K = 6\mathbb{Z} \times (0)$ of M. Then it is not a graded J_{gr}-classical prime submodule of M since $2 \cdot 3 \cdot (1, 0) \in K$ but $2 \cdot (1, 0) \notin K + J_{gr}(M)$ and $3 \cdot (1, 0) \notin K + J_{gr}(M)$.

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded classical prime submodule of M, then N is a graded J_{gr}-classical prime submodule of M.*

Proof. Suppose that N is a graded classical prime submodule of M. Let $r, s \in h(R)$, $m \in h(M)$ such that $rsm \in N$, then by our assumption we get either $rm \in N$ or $sm \in N$, hence $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$. □

The next example shows that a graded J_{gr}-classical prime submodule is not necessarily a graded classical prime submodule.

Example 2.5. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}_8$. Then M is a G-graded R-module with $M_0 = \mathbb{Z} \times \mathbb{Z}_8$ and $M_1 = \{(0, 0)\}$. Now, consider the graded submodule $N = \mathbb{Z} \times (4)$ of M. Then N is a graded J_{gr}-classical prime since for each $m \in h(M)$, $(\mathbb{Z} \times (4) + J_{gr}(\mathbb{Z} \times \mathbb{Z}_8) :_R (m)) = (\mathbb{Z} \times (4) + \{0\} \times (2) :_R (m)) = (\mathbb{Z} \times (2) :_R (m)) = 2\mathbb{Z}$, which is a graded prime ideal of $\mathbb{Z}$. However the graded submodule N is not a graded classical prime submodule of M, since $2 \cdot 2 \cdot (0, 1) = (0, 4) \in N$, but $2 \cdot (0, 1) \notin N$.

It is clear that every graded prime submodule is a graded J_{gr}-classical prime submodule since every graded prime submodule is graded classical prime (see [6].) But the converse is not true in general see Example 2.5.

Remark 2.6. Let R be a G-graded ring and M a graded R-module.

(i) If $J_{gr}(M) = 0$, then every graded J_{gr}-classical prime submodule of M is a graded classical prime submodule of M.

(ii) If $J_{gr}(M)$ is contained in every graded submodule of M, then every graded J_{gr}-classical prime submodule of M is a graded classical prime submodule of M.

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and K a graded submodule of M with $J_{gr}(M) \subseteq J_{gr}(K)$. If N is a graded J_{gr}-classical prime submodule of M with $N \subseteq K$, then N is a graded J_{gr}-classical prime submodule of K.*

Proof. Let $r, s \in h(R)$ and $m \in K \cap h(M)$ with $rsm \in N$. Then either $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$ as N is a graded J_{gr}-classical prime submodule of M. Since $J_{gr}(M) \subseteq J_{gr}(K)$, we get $rm \in N + J_{gr}(K)$ or $sm \in N + J_{gr}(K)$. Therefore, N is a graded J_{gr}-classical prime submodule of K. □

The following results give us a characterization of a graded J_{gr}-classical prime submodule of a graded module.

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded J_{gr}-classical prime submodule of M.*

(ii) *If $K = \oplus_{g\in G} K_g$ is a graded submodule of M, $r, s \in h(R)$ and $g \in G$ with $rsK_g \subseteq N$, then either $rK_g \subseteq N + J_{gr}(M)$ or $sK_g \subseteq N + J_{gr}(M)$.*

Proof. $(i) \Rightarrow (ii)$ Let $K = \oplus_{g\in G} K_g$ be a graded submodule of M and let $r, s \in h(R)$, $g \in G$ such that $rsK_g \subseteq N$. Assume that $rK_g \nsubseteq N + J_{gr}(M)$ and $sK_g \nsubseteq N + J_{gr}(M)$. This implies there are elements $k_{g_1}, k_{g_2} \in K_g$ such that $rk_{g_1} \notin N + J_{gr}(M)$ and $sk_{g_2} \notin N + J_{gr}(M)$. $rsk_{g_1} \in N$ and $rk_{g_1} \notin N + J_{gr}(M)$, we get $sk_{g_1} \in N + J_{gr}(M)$ since N is a graded J_{gr}-classical prime submodule of M. Similarly, by $rsk_{g_2} \in N$ and $sk_{g_2} \notin N + J_{gr}(M)$, so that we get $rk_{g_2} \in N + J_{gr}(M)$. Since k_{g_1}, $k_{g_2} \in K_g$, we have $k_{g_1} + k_{g_2} \in K_g$. $rs(k_{g_1} + k_{g_2}) \in rsK_g \subseteq N$. By (i) either $r(k_{g_1} + k_{g_2}) \in N + J_{gr}(M)$ or $s(k_{g_1} + k_{g_2}) \in N + J_{gr}(M)$ as N is a graded J_{gr}-classical prime. If $rk_{g_1} + rk_{g_2} = r(k_{g_1} + k_{g_2}) \in N + J_{gr}(M)$, we get $rk_{g_1} \in N + J_{gr}(M)$ since $rk_{g_2} \in N + J_{gr}(M)$, a contradiction. Similarly, if $sk_{g_1} + sk_{g_2} = s(k_{g_1} + k_{g_2}) \in N + J_{gr}(M)$, then we get a contradiction. Therefore either $rK_g \subseteq N + J_{gr}(M)$ or $sK_g \subseteq N + J_{gr}(M)$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$. Let K be a submodule of M generated by $m \in M$, that is, $K = Rm$. Then by Lemma 2.1 (1), K is a graded submodule of M and $K = \bigoplus_{g\in G} R_g m$. Moreover, for every $g \in G$,

$K_g = R_g m$, in particular, $K_e = R_e m$. Since $rsK_e \subseteq N$, by our assumption, we get either $rK_e \subseteq N + J_{gr}(M)$ or $sK_e \subseteq N + J_{gr}(M)$, it follows that $rm = r1m \in rR_e m = rK_e \subseteq N + J_{gr}(M)$ or $sm = s1m \in rR_e m = sK_e \subseteq N + J_{gr}(M)$. Thus N is a graded J_{gr}-classical prime submodule of M. □

Theorem 2.9. *Let R be a G-graded ring and M a graded R-module. If N is a graded J_{gr}-classical prime submodule of M and K is any proper graded submodule of M such that $J_{gr}(K) = J_{gr}(M)$ and $K \nsubseteq N$, then $N \cap K$ is a graded J_{gr}-classical prime submodule of K.*

Proof. By Lemma 2.1(2) and the hypothesis that $K \nsubseteq N$, $N \cap K$ is a proper graded submodule of K. Let $r, s \in h(R)$ and $m \in K \cap h(M)$ such that $rsm \in N \cap K$. Then either $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$ as N is a graded J_{gr}-classical prime submodule of M. This yields that either $rm \in (N + J_{gr}(K)) \cap K$ or $sm \in (N + J_{gr}(K)) \cap K$. By the modular law, we get $rm \in (N \cap K) + J_{gr}(K)$ or $sm \in (N \cap K) + J_{gr}(K)$. Therefore, $N \cap K$ is a graded J_{gr}-classical prime submodule of K. □

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$ (see [14].)

Recall that a proper graded submodule N of a graded R-module M is said to be *a gr-small submodule of* M (for short $N <<_g M$) if for every proper graded submodule L of M, we have $N + L \neq M$ (see [2].)

Theorem 2.10. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ be a graded epimorphism.*

(i) *If N is a graded J_{gr}-classical prime submodule of M such that $ker(f) \subseteq N$, then $f(N)$ is a graded J_{gr}-classical prime submodule of M'.*
(ii) *If N' is a graded J_{gr}-classical prime submodule of M' with $ker(f) <<_g M$, then $f^{-1}(N')$ is a graded J_{gr}-classical prime submodule of M.*

Proof. (i) Suppose that N is a graded J_{gr}-classical prime submodule of M. It is easy to see that $f(N)$ is a proper graded submodule of M'. Now, let $r, s \in h(R)$ and $m' \in h(M')$ such that $rsm' \in f(N)$. Since f is a graded epimorphism, there exists $m \in h(M)$ such that $f(m) = m'$. Hence $f(rsm) = rsf(m) = rsm' \in f(N)$, so there exists $n \in N \cap h(M)$ such that $f(rsm) = f(n)$. Thus $rsm - n \in ker(f) \subseteq N$, and $rsm \in N$. Hence $rm \in N + J_{gr}(M)$ or $sm \in N + J_{gr}(M)$ as N is a graded J_{gr}-classical prime submodule of M. This yields that $rm' \in f(N) + f(J_{gr}(M)) \subseteq f(N) + J_{gr}(M')$ or $sm' \in f(N) + f(J_{gr}(M)) \subseteq f(N) + J_{gr}(M')$

by [5, Theorem 2.12(i)]. Therefore $f(N)$ is a graded J_{gr}-classical prime submodule of M'.

(ii) Suppose that N' is a graded J_{gr}-classical prime submodule of M'. It is easy to see that $f^{-1}(N')$ is a proper graded submodule of M. Now, let $r, s \in h(R)$ and $m \in h(M)$ such that $rsm \in f^{-1}(N')$, hence $rsf(m) \in N'$. Then either $rf(m) \in N' + J_{gr}(M')$ or $sf(m) \in N' + J_{gr}(M')$ as N' is a graded J_{gr}-classical prime submodule of M'. Since $ker(f) <<_g M$, by [5, Theorem 2.12(ii)], we get $f(J_{gr}(M)) = J_{gr}(M')$. This yields that either $f(rm) \in N' + f(J_{gr}(M))$ or $f(sm) \in N' + f(J_{gr}(M))$. So $rm \in f^{-1}(N') + J_{gr}(M)$ or $sm \in f^{-1}(N') + J_{gr}(M)$. Therefore, $f^{-1}(N')$ is a graded J_{gr}-classical prime submodule of M. □

Recall that a graded R-module M is called a graded semisimple if and only if every graded submodule of M is a direct summand of M (see [14].)

Theorem 2.11. *Let R be a G-graded ring and M a graded semisimple R-module. If N is a graded J_{gr}-classical prime submodule of M, then N is a graded classical prime submodule of M.*

Proof. Suppose that N is a graded J_{gr}-classical prime submodule of M. Now, since M is a graded semisimple, M has no graded small nonzero submodules. By [5, Theorem 2.10], we have $J_{gr}(M) = \sum\{N : N <<_g M\}$, it follows that $J_{gr}(M) = 0$. Thus N is a graded classical prime submodule of M by Remark 2.6. □

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