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Research Paper

HOMODERIVATIONS AND SEMIGROUP IDEALS IN 3-PRIME NEAR-RINGS

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Abstract. This paper studies homoderivations satisfying certain conditions on semigroup ideals of near-rings. In addition, we include some examples of the necessity of the hypotheses used in our results.

1. Introduction

An additively written group $(\mathcal{N}, +)$ equipped with a binary operation $\cdot : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, $(x, y) \longmapsto xy$, such that $(xy)z = x(yz)$ and $x(y+z) = xy + xz$ for all $z, y, z \in \mathcal{N}$ is called a left near-ring. The results obtained in near-rings can be used in various fields inside and outside of pure mathematics. The best known is to balanced incomplete block designs using planar near-rings. Precisely, we can construct efficient codes and block designs with the help of finite near-rings. Also, there are other applications in cryptography, digital computing, automata theory, sequential mechanics, and combinatorics. For the basic results of near-ring theory and

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its applications, we recommend the references of Clay [7], Meldrum [11], Pilz [12], and Lakehal [9].

Throughout this paper, by a near-ring we mean that left near-ring $\mathcal{N}$ with center $Z(\mathcal{N})$. A non empty subset U of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. Recall that $\mathcal{N}$ is 3-prime, that is, for all $a, b \in \mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $x0 = 0$). As usual for all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. For $S \subseteq \mathcal{N}$, a mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called zero-power valued on S if for each $x \in S$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$. A mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ preserves S if $f(S) \subseteq S$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [13], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. According to [6], an additive mapping h from $\mathcal{N}$ into itself is said to be a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{N}$.

Many results on commutativity in prime and semi-prime rings admitting suitably constrained derivations, generalized derivations, and homoderivations have been published in the literature (see [1], [2], [3], [4], [5], [8], and [10]). Recently, A. Boua has proved comparable results on 3-prime near-rings in [6].

Our aim in this paper is to investigate 3-prime near-rings admitting homoderivations satisfying certain identities, in the case where the constraints are initially assumed to hold on semigroup ideal of near-rings.

2. Preliminaries

In the following, we give some well-known results of near-rings in the literature, which will be used extensively in the proof of our results.

Lemma 2.1. [3, Lemmas 1.2 (i), 1.2 (iii), and 1.3 (iii)] *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) If $z \in Z(\mathcal{N}) \setminus \{0\}$, then z is not a zero divisor.

(ii) If $Z(\mathcal{N})$ contains a nonzero element z for which $z + z \in Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.

(iii) If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.

Lemma 2.2. [3, Lemmas 1.3 (i), 1.4 (i), and 1.3 (iii)] *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) If U is a nonzero semigroup right (resp. semigroup left) ideal of $\mathcal{N}$ and $x \in \mathcal{N}$ such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.

(ii) If U is a nonzero semigroup ideal of $\mathcal{N}$ and $x, y \in \mathcal{N}$ such that $xUy = \{0\}$, then $x = 0$ or $y = 0$.

(iii) If U is a nonzero semigroup right ideal of $\mathcal{N}$ and $x \in \mathcal{N}$ which centralizes U, then $x \in Z(\mathcal{N})$.

Lemma 2.3. [3, Lemmas 1.5] *If $\mathcal{N}$ is a 3-prime near-ring and $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.4. [6, Lemma 2.4 (ii)] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a homoderivation h such that $h^2(\mathcal{N}) = \{0\}$, then $h = 0$.*

Lemma 2.5. [6, Lemma 2.4] *Let $\mathcal{N}$ be a prime 3-near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then for all $x, y, a \in \mathcal{N}$ we have*

$$h(xy)(h(a)+a) = h(x)h(y)(h(a)+a) + h(x)y(h(a)+a) + xh(y)(h(a)+a).$$

3. Some results for homoderivation and semigroup ideal in 3-prime near-ring

We begin this paragraph with a crucial result, which is necessary for developing the proof of our main results.

Lemma 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero additive map f on $\mathcal{N}$ which is zero-power valued on $\mathcal{N}$. Then the following assertions are equivalent:*
(i) $f(x) + x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.
(ii) $x + f(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.
(iii) $\mathcal{N}$ is a commutative ring.

Proof. It is clear that the implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are trivial.
$(i) \Rightarrow (iii)$ Suppose that

$$f(x) + x \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}. \tag{1}$$

If $f(x) \neq 0$ for all $x \in \mathcal{N} \setminus \{0\}$. By recurrence we have $f^n(x) \neq 0$ for all $x \in \mathcal{N} \setminus \{0\}$ and $n \in \mathbb{N}^*$. Since f is zero-power valued on $\mathcal{N}$, for each $x \in \mathcal{N}$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$, it follows that for $z = f^{k(x)-1}(x) \neq 0$, $f(z) = f^{k(x)}(x) = 0$ which is a contradiction. Hence there exists $y \in \mathcal{N} \setminus \{0\}$ such that $f(y) = 0$, so we get $y = f(y) + y \in Z(\mathcal{N}) \setminus \{0\}$ and $y + y = f(y+y) + y + y \in Z(\mathcal{N})$, which forces that $\mathcal{N}$ is abelian.

Now by replacing x by $x - f(x) + f^2(x) + ... + (-1)^{k(x)-1} f^{k(x)-1}(x)$ in (1) and using $\mathcal{N}$ is abelian we get $x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$, thus $\mathcal{N} \subseteq Z(\mathcal{N})$. Hence $\mathcal{N}$ is commutative ring by Lemma 2.3. □

Lemma 3.2. *Let $\mathcal{N}$ be a 2-torsion free near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$, then $\mathcal{N}$ is zero symmetric near-ring.*

Proof. We have for all $z \in \mathcal{N}$

$$\begin{aligned} h(0z) &= h(0)h(z) + h(0)z + 0h(z) \\ &= 0h(z) + 0z + 0h(z). \end{aligned}$$

On the other hand

$$\begin{aligned} h(0z) &= h(0(0z)) \\ &= h(0)h(0z) + h(0)0z + 0h(0z) \\ &= 0h(0z) + 0z + 0h(0z) \\ &= 0h(z) + 0z + 0h(z) + 0z + 0h(z) + 0z + 0h(z). \end{aligned}$$

Comparing the last two expressions, we find $2(0h(z) + 0z) = 0$ for all $z \in \mathcal{N}$. Using 2-torsion freeness of $\mathcal{N}$ we obtain $0h(z) + 0z = 0$ for all $z \in \mathcal{N}$. By recurrence, we obtain

$$0h^n(z) + (-1)^{n+1}0z = 0 \text{ for all } z \in \mathcal{N} \quad \text{and} \quad n \in \mathbb{N}^*. \tag{2}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(z) > 1$ such that $h^{k(z)}(z) = 0$. Replacing n by $k(z)$ in (2), we get $(-1)^{k(z)+1}0z = 0$ for all $z \in \mathcal{N}$. Thus $\mathcal{N}$ is zero symmetric near-ring. □

Lemma 3.3. *Let $\mathcal{N}$ be a 3-prime near-ring and h be a nonzero homoderivation of $\mathcal{N}$.*

(i) If $\mathcal{N}$ is zero symmetric and U is a nonzero semigroup right ideal of $\mathcal{N}$, then $h(U) \neq \{0\}$.

(ii) If U is a nonzero semigroup left ideal of $\mathcal{N}$, then $h(U) \neq \{0\}$.

Proof. (i) Let U be a nonzero semigroup right ideal. Suppose that $h(U) = \{0\}$. Then for all $u \in U$ and $x \in \mathcal{N}$ we have $0 = h(ux) = h(u)h(x) + h(u)x + uh(x) = uh(x)$, that is $uh(x) = 0$ for all $u \in U, x \in \mathcal{N}$. This implies that $Uh(x) = \{0\}$ for all $x \in \mathcal{N}$. Hence $h = 0$ by Lemma 2.2 (i).

(ii) The argument for semigroup left ideal is similar. □

Lemma 3.4. *Let $\mathcal{N}$ be a 3-prime near-ring admiting a nonzero homoderivation h and U a nonzero semigroup right ideal of $\mathcal{N}$.*

(i) If $x \in \mathcal{N}$ and $h(U)(h(x) + x) = \{0\}$, then $h(x) + x = 0$.

(ii) If $x \in \mathcal{N}$ and $xh(U) = \{0\}$, then $x = 0$.

(iii) If $\mathcal{N}$ is a 2-torsion free, then $h^2(U) \neq \{0\}$.

Proof. (i) Let $x \in \mathcal{N}$ and $h(U)(h(x)+x) = \{0\}$. We have

$$\begin{aligned} 0 &= h(yu)(h(x)+x) \\ &= (h(y)h(u)+h(y)u+yh(u))(h(x)+x) \\ &= h(y)u(h(x)+x) \text{ for all } u \in U, y \in \mathcal{N}. \end{aligned}$$

Then $h(y)U(h(x)+x) = \{0\}$ for all $y \in \mathcal{N}$ and by Lemma 2.2 (ii) and Lemma 3.3, we conclude that $h(x)+x = 0$.

For (ii), suppose $xh(U) = \{0\}$. For all $u \in U$ and $y \in \mathcal{N}$, we have

$$\begin{aligned} 0 &= xh(uy) - x(h(u)h(y)+h(u)y+uh(y)) \\ &= xuh(y). \end{aligned}$$

Hence $xUh(y) = \{0\}$ for all $y \in \mathcal{N}$ and $x = 0$ by Lemma 2.2 (ii) and Lemma 3.3.

For (iii), assume that $h^2(U) = \{0\}$, then $0 = h^2(uv) = 2h(u)h(v)$ for all $u, v \in U$, since $\mathcal{N}$ is a 2-torsion free, we get $h(u)h(v) = 0$, thus $h(U)(h^2(v)+h(v)) = \{0\}$ for all $v \in U$, in view of Lemma 3.4 (i), we obtain $h^2(v)+h(v) = h(v) = 0$. Thus, part (ii) $h(U) = \{0\}$ which is a contradiction by Lemma 3.3. □

Lemma 3.5. *Let $\mathcal{N}$ be a 3-prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$ and h a nonzero homoderivation on $\mathcal{N}$ which preserves U. If $a \in \mathcal{N}$ and $[h(a)+a, h(U)] = \{0\}$, then $h(a)+a \in Z(\mathcal{N})$.*

Proof. Let $a \in \mathcal{N}$ and $[h(a)+a, h(U)] = \{0\}$.

We set $C(a) = \{x \in \mathcal{N} \mid [h(a)+a, x] = 0\}$. Note that $h(U) \subset C(a) \cap U$. Thus, if $y \in C(a) \cap U$ and $u \in U$, then both $h(yu)$, $h(u)$, $h(y)$ and $yh(u)$ are in $C(a)$. Therefore, $h(y)u \in C(a)$ for all $u \in U, y \in C(a) \cap U$. Hence, $h(y)uv \in C(a)$ for all $u, v \in U, y \in C(a) \cap U$ and so, $0 = [h(a)+a, h(y)uv] = h(y)u[(h(a)+a), v]$. Thus, $h(y)U[(h(a)+a), v] = \{0\}$ for all $v \in U, y \in C(a) \cap U$. Since $h(U) \subset C(a) \cap U$, then $h^2(y)U[(h(a)+a), v] = \{0\}$ for all $y, v \in U$. Since, by Lemma 3.4 (iii), $h^2(U) \neq \{0\}$, by Lemma 2.2 (ii) we get $[h(a)+a, U] = \{0\}$, and $h(a)+a \in Z(\mathcal{N})$ by Lemma 2.2 (iii). □

Theorem 3.6. *Let $\mathcal{N}$ be a 3-prime near-ring and let U be a nonzero semigroup left ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$. Then the following assertions are equivalent:*

(i) $h(u)+u \in Z(\mathcal{N})$ for all $u \in U$.

(ii) $-u+h(-u) \in Z(\mathcal{N})$ for all $u \in U$.

(iii) $\mathcal{N}$ is a commutative ring.

Proof. It is clear that the implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are trivial.

$(i) \Rightarrow (iii)$ Let $U \neq \{0\}$ a semigroup left ideal such that $h(u) + u \in Z(\mathcal{N})$ for all $u \in U$. Since $xu \in U$, we get $h(xu) + xu \in Z(\mathcal{N})$. Thus

$$\begin{aligned} h(xu) + xu &= h(x)h(u) + h(x)u + xh(u) + xu \\ &= (h(u) + u)(h(x) + x) \in Z(\mathcal{N}) \text{ for all } \ u \in U, x \in \mathcal{N}. \end{aligned}$$

Since $h(u) + u \in Z(\mathcal{N})$, it follows that $h(u) + u = 0$ for all $u \in U$ or $h(x) + x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.

Suppose that $h(u) + u = 0$ for all $u \in U$. By recurrence, it follows that

$$h^n(u) + (-1)^{n+1}u = 0 \text{ for all } u \in U \quad \text{and} \quad n \in \mathbb{N}^*. \tag{3}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(u) > 1$ such that $h^{k(u)}(u) = 0$. Replacing n by $k(u)$ in (3), we get $(-1)^{k(u)+1}u = 0$ for all $u \in U$, so $U = \{0\}$ which is a contradiction. Hence

$$h(x) + x \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}.$$

By Lemma 3.1, we conclude that $\mathcal{N}$ is commutative ring.

$(ii) \Rightarrow (iii)$ Let $U \neq \{0\}$ a semigroup left ideal such that $-u + h(-u) \in Z(\mathcal{N})$ for all $u \in U$. Since $xu \in U$ it follows that $-(h(xu) + xu) \in Z(\mathcal{N})$. Thus

$$\begin{aligned} -(h(xu) + xu) &= -(h(x)h(u) + h(x)u + xh(u) + xu) \\ &= -xu - xh(u) - h(x)u - h(x)h(u) \\ &= x(-u) + xh(-u) + h(x)(-u) + h(x)h(-u) \\ &= x(-u - h(u)) + h(x)(-u - h(u)) \\ &= (-u - h(u))(x + h(x)) \in Z(\mathcal{N}) \quad \text{for all } \ u \in U, x \in \mathcal{N}. \end{aligned}$$

Since $-u - h(u) \in Z(\mathcal{N})$, it follows that $-u - h(u) = 0$ for all $u \in U$ or $x + h(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.

First suppose that $-u - h(u) = 0$ for all $u \in U$. Thus $h(u) + u = 0$ for all $u \in U$. As above, it follows that $U = \{0\}$ which is a contradiction. So

$$x + h(x) \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}. \tag{4}$$

From Lemma 3.1, we find that $\mathcal{N}$ is commutative ring. □

Remark 3.7. Using Theorem 3.6 and Lemma 3.5, we can easly find the following Theorem.

Theorem 3.8. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$, that preserves U and satisfies $[h(U), h(U)+U] = \{0\}$, then $\mathcal{N}$ is a commutative ring.*

Theorem 3.9. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves U, then the following assertions are equivalent:*

(i) $h([x,y]) + [x,y] = [h(x)+x, y]$ for all $x, y \in U$.

(ii) $\mathcal{N}$ is a commutative ring.

Proof. It is clear that $(ii) \Longrightarrow (i)$.

$(i) \Rightarrow (ii)$ Assume that

$$h([x,y]) + [x,y] = [h(x)+x, y] \quad \text{for all} \quad x, y \in U. \tag{5}$$

Replacing y by xy in (5), and using the fact that $[h(x)+x, x] = 0$ for all $x \in U$, we get

$$h(x)h([x,y]) + h(x)[x,y] + x(h([x,y]) + [x,y]) = x[h(x)+x, y] \quad \text{for all} \quad x, y \in U.$$

By using (5), we finds

$$h(x)[h(x)+x, y] = 0 \quad \text{for all} \quad x, y \in U. \tag{6}$$

Hence

$$h(x)y(h(x)+x) = h(x)(h(x)+x)y \quad \text{for all} \quad x, y \in U. \tag{7}$$

Putting yt instead of y in (7), we arrive at

$$h(x)y[t, h(x)+x] = 0 \quad \text{for all} \quad x, y \in U, t \in \mathcal{N},$$

which leads to

$$h(x)U[t, h(x)+x] = \{0\} \quad \text{for all} \quad x \in U, t \in \mathcal{N}.$$

By Lemma 2.2 (ii), we obtain

$$h(x) = 0 \quad \text{or} \quad h(x)+x \in Z(\mathcal{N}) \quad \text{for all} \quad x \in U. \tag{8}$$

If there exists $x_0 \in U$ such that $h(x_0) = 0$, using (5) we get $h([x_0, y]) = 0$ for all $y \in U$, thus

$$x_0 h(y) = h(y) x_0 \quad \text{for all} \quad y \in U. \tag{9}$$

Which means that $(h(x_0)+x_0)h(y) = h(y)(h(x_0)+x_0)$ for all $y \in U$. Taking $h(y)t$ instead of y, then by Lemma 2.5, we have

$$x_0 h^2(y)h(t) + x_0 h^2(y)t + x_0 h(y)h(t) = h^2(y)h(t)x_0 + h^2(y)t x_0 + h(y)h(t)x_0 \tag{10}$$

for all $y, t \in U$. Using (9), (10) becomes

$$x_0 h^2(y)t = h^2(y)tx_0 \quad \text{for all} \quad y, t \in U. \tag{11}$$

Replacing t by tm in (11) and using it again, we get $h^2(y)t[x_0, m] = 0$ for all $y, t \in U, m \in \mathcal{N}$, ie. $h^2(y)U[x_0, m] = \{0\}$ for all $y \in U, m \in \mathcal{N}$. By Lemma 3.4 (iii) and Lemma 2.2 (ii), $x_0 \in Z(\mathcal{N})$. In this case, (8) becomes $h(x) + x \in Z(\mathcal{N})$ for all $x \in U$ which forces that $\mathcal{N}$ is a commutative ring by Theorem 3.6. □

Theorem 3.10. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves U, then the following assertions are equivalent:*

(i) $h([x, y]) = [x, y]$ for all $x, y \in U$.

(ii) $\mathcal{N}$ is a commutative ring.

Proof. It is obvious that (ii) implies (i).

$(i) \Rightarrow (ii)$ Assume that

$$h([x, y]) = [x, y] \quad \text{for all} \quad x, y \in U \tag{12}$$

Putting xy in place of y in (12), and using the fact that $[x, xy] = x[x, y]$, we get

$$\begin{aligned} x[x, y] &= h(x[x, y]) \\ &= h(x)h([x, y]) + h(x)[x, y] + xh([x, y]) \\ &= 2h(x)[x, y] + x[x, y] \quad \text{for all} \quad x, y \in U. \end{aligned}$$

Which implies that $2h(x)[x, y] = 0$ for all $x, y \in U$. By 2-torsion freeness of $\mathcal{N}$, we finds $h(x)[x, y] = 0$ for all $x, y \in U$, which implies that

$$h(x)xy = h(x)yx \quad \text{for all} \quad x, y \in U. \tag{13}$$

Substituting yt for y in (13) and using it again, we obtain $h(x)y[x, t] = 0$ for all $x, y \in U, t \in \mathcal{N}$, ie. $h(x)U[x, t] = \{0\}$ for all $x \in U, t \in \mathcal{N}$. By Lemma 2.2 (ii), we arrive at

$$h(x) = 0 \quad \text{or} \quad x \in Z(\mathcal{N}) \quad \text{for all} \quad x \in U. \tag{14}$$

Suppose there is an element x_0 of U such that $h(x_0) = 0$, by (12) we can easly see that $[x_0, h(y)] = [x_0, y]$ for all $y \in U$ and invoking the definition of h. By recurrence we arrive at

$$[x_0, h^k(y)] = [x_0, y] \quad \text{for all} \quad y \in U, k \in \mathbb{N}^*. \tag{15}$$

Using the fact that h is zero-power valued on $\mathcal{N}$, there exists an integer $k(y) > 1$ such that $h^{k(y)}(y) = 0$. Replacing k by $k(y)$ in (15), we obviously get $x_0 \in Z(\mathcal{N})$. In this case, (14) becomes $x \in Z(\mathcal{N})$ for all $x \in U$ which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.3. □

Theorem 3.11. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a non zero homoderivation h which is zero-power valued on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $(h(x)+x)\circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$.

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is obvious that (ii) implies (i).

$(i) \Rightarrow (ii)$ Assume that

$$(h(x)+x)\circ y \in Z(\mathcal{N}) \text{ for all } x, y \in U. \tag{16}$$

Replacing y by $(h(x)+x)y$ in (16), we get

$$(h(x)+x)((h(x)+x)\circ y) \in Z(\mathcal{N}) \text{ for all } x, y \in U.$$

By Lemma 2.1, it follows that

$$(h(x)+x)\circ y = 0 \text{ or } h(x)+x \in Z(\mathcal{N}) \text{ for all } x, y \in U. \tag{17}$$

If there exists $x_0 \in U$ such that $h(x_0)+x_0 \in Z(\mathcal{N}) \setminus \{0\}$, by (16) together with Lemma 2.1 (iii), we may conclude that

$$y+y \in Z(\mathcal{N}) \text{ for all } y \in U,$$

so that

$$r(y+y) = ry+ry \in Z(\mathcal{N}) \text{ for all } y \in U, r \in \mathcal{N}. \tag{18}$$

Since $\mathcal{N}$ is 2-torsion free, by using (18) and Lemma 2.1 (iii), we obtain $\mathcal{N} \subseteq Z(\mathcal{N})$, which implies that $\mathcal{N}$ is a commutative ring by Lemma 2.1.

In view of (17), we may now assume that $(h(x)+x)\circ y = 0$ for all $x, y \in U$ i.e.

$$y(h(x)+x) = -(h(x)+x)y \text{ for all } x, y \in U.$$

Taking yt instead if y, where $t \in \mathcal{N}$, in the last equation, we obtain

$$\begin{aligned} yt(h(x)+x) &= -(h(x)+x)yt \\ &= (h(x)+x)y(-t) \\ &= (-y(h(x)+x))(-t) \\ &= y(-(h(x)+x))(-t) \quad \text{for all} \quad x, y \in U, t \in \mathcal{N}, \end{aligned}$$

which leads to

$$y(t(h(x)+x) - (-(h(x)+x))(-t)) = 0 \quad \text{for all} \quad x, y \in U, t \in \mathcal{N},$$

thereby obtaining

$$U(-t(-(h(x)+x))+(-(h(x)+x))t)=\{0\} \quad \text{for all} \quad x\in U, t\in \mathcal{N}.$$

By Lemma 2.2 (i), we conclude that $-x+h(-x)\in Z(\mathcal{N})$ for all $x\in U$. Thus By Theorem 3.6, it follows that $\mathcal{N}$ is a commutative ring. □

Theorem 3.12. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation on h which is zero-power valued on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[h(x)+x,y]\in Z(\mathcal{N})$ for all $x,y\in U$.

(ii) $h(xy)+xy\in Z(\mathcal{N})$ for all $x,y\in U$.

(iii) $\mathcal{N}$ is commutative ring.

Proof. It is clear that the implications $(iii)\Rightarrow(i)$ and $(iii)\Rightarrow(ii)$ are trivial.

$(i)\Rightarrow(iii)$ Suppose that

$$[h(x)+x,y]\in Z(\mathcal{N}) \text{ for all } x,y\in U. \tag{19}$$

Replacing y by $(h(x)+x)y$ in (19), we get

$$(h(x)+x)[h(x)+x,y]\in Z(\mathcal{N}) \text{ for all } x,y\in U.$$

By Lemma 2.1(iii), we obtain

$$h(x)+x\in Z(\mathcal{N}) \text{ or } [h(x)+x,y]=0 \text{ for all } x,y\in U.$$

Both cases force that $h(x)+x\in Z(\mathcal{N})$ for all $x\in U$. Using Theorem 3.6, we conclude that $\mathcal{N}$ is a commutative ring.

$(ii)\Longrightarrow(iii)$ Now assume that $h(xy)+xy\in Z(\mathcal{N})$ for all $x,y\in U$. We have

$$\begin{aligned} h(zxy)+zxy &= h(z)h(xy)+h(z)xy+zh(xy)+zxy \\ &= h(z)(h(xy)+xy)+z(h(xy)+xy) \\ &= (h(xy)+xy)(h(z)+z)\in Z(\mathcal{N}) \text{ for all } x,y,z\in U. \end{aligned}$$

Using Lemma 2.1 (iii) implies

$$h(xy)+xy=0 \text{ for all } x,y\in U \text{ or } h(z)+z\in Z(\mathcal{N}) \text{ for all } z\in U. \tag{20}$$

If $h(xy)+xy=0$ for all $x,y\in U$, by recurrence we have $h^k(xy)+(-1)^{k+1}xy=0$ for all $x,y\in U, k\in\mathbb{N}^*$. Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(xy)>1$ such that $h^{k(xy)}(xy)=0$. Replacing k by $k(xy)$ in the above expression we get $xy=0$ for all

$x, y \in U$. Thus by the 3-primeness of $\mathcal{N}$ we get $U = \{0\}$; a contradiction. Hence (19) becomes $h(z) + z \in Z(\mathcal{N})$ for all $z \in U$, and by Theorem 3.6 we proves that $\mathcal{N}$ is commutative ring. □

Theorem 3.13. *Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. There is no nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ such that $h(x \circ y) + x \circ y = [x, y]$ for all $x, y \in U$.*

Proof. Suppose that $h \neq 0$ and

$$h(x \circ y) + x \circ y = [x, y] \quad \text{for all} \quad x, y \in U. \tag{21}$$

Thus

$$\begin{aligned} y[x, y] &= [yx, y] \\ &= h(yx \circ y) + yx \circ y \\ &= h(y(x \circ y)) + y(x \circ y) \\ &= h(y)h(x \circ y) + h(y)x \circ y + yh(x \circ y) + y(x \circ y) \\ &= h(y)(h(x \circ y) + x \circ y) + y(h(x \circ y) + x \circ y) \\ &= h(y)[x, y] + y[x, y] \quad \text{for all} \quad x, y \in U. \end{aligned}$$

This expression gives us $h(y)[x, y] = 0$ for all $x, y \in U$, that is

$$h(y)xy = h(y)yx \quad \text{for all} \quad x, y \in U.$$

Substituting xm in place of x in the last expression, we get

$$\begin{aligned} h(y)xmy &= h(y)yxm \\ &= h(y)xym \quad \text{for all} \quad x, y \in U, m \in \mathcal{N}. \end{aligned}$$

Which can be rewritten as $h(y)U[m, y] = \{0\}$ for all $y \in U, m \in \mathcal{N}$. By Lemma 2.2 (ii), we obtain

$$h(y) = 0 \quad \text{or} \quad y \in Z(\mathcal{N}) \text{ for all} \quad y \in U. \tag{22}$$

Suppose there is an element $y_0 \in U$ such that $y_0 \in Z(\mathcal{N})$. Then (21) becomes $h(2xy_0)+2xy_0 = 0$, for all $x \in U$. By recurrence, it follows that

$$h^k(2xy_0) + (-1)^{k+1}2xy_0 = 0 \text{ for all } x \in U, k \in \mathbb{N}^*. \tag{23}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(2xy_0) > 1$ such that $h^{k(2xy_0)}(2xy_0) = 0$. Replacing k by $k(2xy_0)$ in (23), we get $2xy_0 = 0$ for all $x \in U$, and using 2-torsion freeness of $\mathcal{N}$, it follows that $Uy_0 = \{0\}$. Hence $y_0 = 0$ by Lemma 2.2 (i). In this case, (22) implies that $h(U) = \{0\}$ which gives a contradiction by Lemma 3.3. □

Theorem 3.14. *Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. There is no nonzero homoderivation h satisfying $h([x,y]) + [x,y] = x \circ y$ for all $x, y \in U$.*

Proof. Suppose that $h \neq 0$ and

$$h([x,y]) + [x,y] = x \circ y \quad \text{for all} \quad x, y \in U. \tag{24}$$

Thus

$$\begin{aligned} y(x \circ y) &= yx \circ y \\ &= h([yx,y]) + [yx,y] \\ &= h(y[x,y]) + y[x,y] \\ &= h(y)h([x,y]) + h(y)[x,y] + yh([x,y]) + y[x,y] \\ &= h(y)(h([x,y]) + [x,y]) + y(h([x,y]) + [x,y]) \\ &= h(y)(x \circ y) + y(x \circ y) \quad \text{for all} \quad x, y \in U. \end{aligned}$$

This expression gives us $h(y)(x \circ y) = 0$ for all $x, y \in U$, it follows that

$$h(y)xy = -h(y)yx \quad \text{for all} \quad x, y \in U. \tag{25}$$

Substituting xm in place of x in (25), we get

$$\begin{aligned} h(y)xmy &= -h(y)yxm \\ &= h(y)yx(-m) \\ &= h(y)x(-y)(-m) \quad \text{for all} \quad x, y \in U, m \in \mathcal{N}. \end{aligned}$$

Which can be rewritten as $h(y)U(-m(-y) + (-y)m) = \{0\}$ for all $y \in U$ and $m \in \mathcal{N}$. By Lemma 2.2, we have

$$h(y) = 0 \quad \text{or} \quad -y \in Z(\mathcal{N}) \quad \text{for all} \quad y \in U. \tag{26}$$

Suppose there is an element $y_0 \in U$ such that $-y_0 \in Z(\mathcal{N})$. Replacing y by $-y_0$ in (24) we get $2(-y_0)x = 0$ for all $x \in U$. Using 2-torsion freeness of $\mathcal{N}$, we obtain $-y_0U = \{0\}$ and by Lemma 3.3, we have $y_0 = 0$. In this case, (26) implies that $h(U) = \{0\}$ which gives a contradiction by Lemma 3.3. □

Theorem 3.15. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. There is no nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ such that $h(xy)+xy=[x,y]$ for all $x,y\in U$.*

Proof. Suppose that $h\neq 0$ and

$$h(xy)+xy=[x,y] \quad \text{for all} \quad x,y\in U. \tag{27}$$

Thus

$$\begin{aligned} y[x,y] &= [yx,y] \\ &= h(yxy)+yxy \\ &= h(y(xy))+y(xy) \\ &= h(y)h(xy)+h(y)xy+yh(xy)+y(xy) \\ &= h(y)(h(xy)+xy)+y(h(xy)+xy) \\ &= h(y)[x,y]+y[x,y] \quad \text{for all} \quad x,y\in U. \end{aligned}$$

This expression gives us $h(y)[x,y]=0$ for all $x,y\in U$. As in proof of Theorem 3.12, it follows that

$$h(y)=0 \quad \text{or} \quad y\in Z(\mathcal{N}) \text{ for all } \quad y\in U. \tag{28}$$

Suppose there is an element $y_0\in U$ such that $y_0\in Z(\mathcal{N})$. Then (27) becomes $h(xy_0)+xy_0=0$ for all $x\in U$. By recurrence, it follows that

$$h^k(xy_0)+(-1)^{k+1}xy_0=0 \text{ for all } x\in U, k\in\mathbb{N}^*. \tag{29}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(xy_0)>1$ such that $h^{k(xy_0)}(xy_0)=0$. Replacing k by $k(xy_0)$ in (29), we get $xy_0=0$ for all $x\in U$, so $Uy_0=\{0\}$. Hence $y_0=0$. In this case, (28) implies that $h(U)=\{0\}$ which gives a contradiction by Lemma 3.3. □

Theorem 3.16. *Let $\mathcal{N}$ be a 3-prime zero symmetric near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h such that $h(xy)+xy=x\circ y$ for all $x,y\in U$, then $\mathcal{N}$ is a commutative ring.*

Proof. Suppose that

$$h(xy)+xy=x\circ y \quad \text{for all} \quad x,y\in U. \tag{30}$$

Thus

$$\begin{aligned} y(x\circ y) &= yx\circ y \\ &= h(yxy)+yxy \\ &= h(y(xy))+y(xy) \\ &= h(y)h(xy)+h(y)xy+yh(xy)+y(xy) \\ &= h(y)(h(xy)+xy)+y(h(xy)+xy) \\ &= h(y)(x\circ y)+y(x\circ y) \quad \text{for all} \quad x,y\in U. \end{aligned}$$

This expression gives us $h(y)(x\circ y)=0$ for all $x,y\in U$. As in proof of Theorem 3.17, it follows that

$$h(y)=0 \quad \text{or} \quad -y\in Z(\mathcal{N}) \ \text{ for all} \quad y\in U. \tag{31}$$

Suppose there is an element $y_0\in U$ such that $h(y_0)=0$. Then (30) becomes

$$h(x)y_0+xy_0=x\circ y_0 \quad \text{for all} \quad x\in U. \tag{32}$$

Replacing x by y_0 in (32), we arrive at ${y_0}^2=0$. Substituting xy_0 in place of x in (32), then for $x\in U$, we have

$$\begin{aligned} y_0xy_0 &= x{y_0}^2+y_0xy_0 \\ &= xy_0\circ y_0 \\ &= h(xy_0)y_0+x{y_0}^2 \\ &= 0, \end{aligned}$$

thus $y_0Uy_0=\{0\}$. Hence $y_0=0$. In this case, (31) implies that $-U\subseteq Z(\mathcal{N})$. Since $-U$ is nonzero left semigroup ideal of $\mathcal{N}$, by Lemma 2.3 $\mathcal{N}$ is a commutative ring. □

Theorem 3.17. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation on h which is zero-power valued on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $h(x\circ y)+x\circ y\in Z(\mathcal{N})$ *for all* $x,y\in U$.

(ii) $\mathcal{N}$ *is commutative ring.*

Proof. It is easy to see that $(ii)\Longrightarrow(i)$.

$(i)\Longrightarrow(ii)$ Suppose that

$$h(x\circ y)+x\circ y\in Z(\mathcal{N}) \ \text{ for all } x,y\in U. \tag{33}$$

Since $x \circ xy = x(x \circ y)$ for all $x, y \in U$, replacing y by xy in (33), we obtain

$$h(x)h(x \circ y) + h(x)(x \circ y) + xh(x \circ y) + x(x \circ y) \in Z(\mathcal{N}) \quad \text{for all} \quad x, y \in U.$$

Thus

$$h(x)(h(x \circ y) + x \circ y) + x(h(x \circ y) + x \circ y) \in Z(\mathcal{N}) \quad \text{for all} \quad x, y \in U.$$

By (33) we get

$$(34) \qquad (h(x \circ y) + x \circ y)(h(x) + x) \in Z(\mathcal{N}) \quad \text{for all} \quad x, y \in U.$$

By Lemma 2.1 we have

$$(35) \qquad h(x \circ y) + x \circ y = 0 \quad \text{or} \quad h(x) + x \in Z(\mathcal{N}) \quad \text{for all} \quad x, y \in U.$$

Suppose there is an element $x_0 \in U$ such that $h(x_0 \circ y) + x_0 \circ y = 0$ for all $y \in \mathcal{N}$. Then by recurrence we prove that

$$(36) \qquad h^k(x_0 \circ y) + (-1)^{k+1} x_0 \circ y = 0 \quad \text{for all} \quad y \in U, k \in \mathbb{N}^*.$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(x_0 \circ y) > 1$ such that $h^{k(x_0 \circ y)}(x_0 \circ y) = 0$. Replacing k by $k(x_0 \circ y)$ in (36), it follows that

$$(37) \qquad x_0 \circ y = 0 \quad \text{for all} \quad y \in U.$$

Substituting x_0 in place of y in (37), we get $2(x_0)^2 = 0$. Using 2-torsion freeness of $\mathcal{N}$ we obtain $(x_0)^2 = 0$. Putting $x_0 y$ in place of y in (37), we get $x_0 y x_0 = 0$ for all $y \in U$, so $x_0 U x_0 = \{0\}$. Thus $x_0 = 0$ by Lemma 2.2 (ii). In this case, (35) implies that $h(x) + x \in Z(\mathcal{N})$ for all $x \in U$. By Theorem 3.6 it follows that $\mathcal{N}$ is commutative ring. □

The following examples shows that h is "zero-power valued on $\mathcal{N}$" cannot be omitted in the hypothesis of Theorems 3.6, 3.8, 3.9, 3.11, 3.12 and 3.17.

Example 3.18. Let $\mathcal{N} = U = M_2(\mathbb{Z})$, that is a 2-torsion free prime ring. We consider $h = -id_{\mathcal{N}}$, then it is clear that h is a not "zero-power valued homoderivation on $\mathcal{N}$" which preserve U and satisfy the following conditions:

(i) $h(x) + x \in Z(\mathcal{N})$,
(ii) $-x + h(-x) \in Z(\mathcal{N})$,
(iii) $[h(x), h(y) + y] = 0$,
(iv) $[h(x) + x, y] \in Z(\mathcal{N})$,
(v) $h([x, y]) + [x, y] = [h(x) + x, y]$,
(vi) $(h(x) + x) \circ y \in Z(\mathcal{N})$,
(vii) $h(xy) + xy \in Z(\mathcal{N})$,
(viii) $h(x \circ y) + x \circ y \in Z(\mathcal{N})$,

for all $x, y \in U$, but $\mathcal{N}$ is not commutative.

Example 3.19. Let $\mathcal{N} = U = \mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ be the set of all complex numbers. Addition is the usual addition of complex numbers. Then $(\mathcal{N}, +)$ is a group. Define multiplication $\star$ on $\mathcal{N}$ by $u \star v = |u|v$. Then $(\mathcal{N}, +, \star)$ is a 3-prime near-ring, which is not a ring. We consider $h = -id_{\mathcal{N}}$, then it is clear that h is not "zero-power valued homoderivation on $\mathcal{N}$," which preserve U and satisfy the following conditions:

(i) $h(x) + x \in Z(\mathcal{N})$,
(ii) $-x + h(-x) \in Z(\mathcal{N})$,
(iii) $[h(x), h(y) + y] = 0$,
(iv) $[h(x) + x, y] \in Z(\mathcal{N})$,
(v) $h([x, y]) + [x, y] = [h(x) + x, y]$,
(vi) $(h(x) + x) \circ y \in Z(\mathcal{N})$,
(vii) $h(xy) + xy \in Z(\mathcal{N})$,
$(viii)$ $h(x \circ y) + x \circ y \in Z(\mathcal{N})$,

for all $x, y \in U$, but $\mathcal{N}$ is not commutative ring.

The following example illustrates that the hypothesis "3-primeness of $\mathcal{N}$" is essential in Theorems 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16 and 3.17 of our paper.

Example 3.20. Let S be a zero-symmetric 2-torsion free left near-ring and

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}.$$

$$U = \left\{ \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u, 0 \in S \right\}.$$

Then $\mathcal{N}$ is a 2-torsion left near-ring which is not 3-prime and U is a nonzero semigroup ideal of $\mathcal{N}$. Let us defined $h : \mathcal{N} \to \mathcal{N}$ as follow:

$$h \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that h is a zero-power valued homoderivation on $\mathcal{N}$, which satisfy the following conditions:

(i) $[h(x), h(y) + y] = 0$,

(ii) $[h(x) + x, y] \in Z(\mathcal{N})$,

(iii) $h([x, y]) + [x, y] = [h(x) + x, y]$,

(iv) $(h(x) + x) \circ y \in Z(\mathcal{N})$,

(v) $h\,(xy) + xy \in Z(\mathcal{N})$,

(vi) $h\,(x \circ y) + x \circ y \in Z(\mathcal{N})$,

(vii) $h(x \circ y) + x \circ y = [x, y]$,

$(viii)$ $h([x, y]) = [x, y]$,

(ix) $h([x, y]) + [x, y] = x \circ y$,

(x) $h(xy) + xy = [x, y]$,

(xi) $h(xy) + xy = x \circ y$

for all $x, y \in U$, but $\mathcal{N}$ is not commutative.

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