This paper studies homoderivations satisfying certain conditions on semigroup ideals of near-rings. In addition, we include some examples of the necessity of the hypotheses used in our results.

1. Introduction

An additively written group \((N, +)\) equipped with a binary operation \(\cdot : N \times N \rightarrow N\), \((x, y) \mapsto xy\), such that \((xy)z = x(yz)\) and \(x(y + z) = xy + xz\) for all \(z, y, z \in N\) is called a left near-ring. The results obtained in near-rings can be used in various fields inside and outside of pure mathematics. The best known is to balanced incomplete block designs using planar near-rings. Precisely, we can construct efficient codes and block designs with the help of finite near-rings. Also, there are other applications in cryptography, digital computing, automata theory, sequential mechanics, and combinatorics. For the basic results of near-ring theory and
its applications, we recommend the references of Clay [7], Meldrum [11], Pilz [12], and Lakehal [4].

Throughout this paper, by a near-ring we mean that left near-ring $\mathcal{N}$ with center $Z(\mathcal{N})$. A non empty subset $U$ of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $\mathcal{N} \subseteq U$ ) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. Recall that $\mathcal{N}$ is 3-prime, that is, for all $a, b \in \mathcal{N}$, $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $x0 = 0$). As usual for all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. For $S \subseteq \mathcal{N}$, a mapping $f : \mathcal{N} \rightarrow \mathcal{N}$ is called zero-power valued on $S$ if for each $x \in S$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$. A mapping $f : \mathcal{N} \rightarrow \mathcal{N}$ preserves $S$ if $f(S) \subseteq S$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [13], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. According to [3], an additive mapping $h$ from $\mathcal{N}$ into itself is said to be a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{N}$.

Many results on commutativity in prime and semi-prime rings admitting suitably constrained derivations, generalized derivations, and homoderivations have been published in the literature (see [1], [2], [3], [4], [5], [8], and [11]). Recently, A. Boua has proved comparable results on 3-prime near-rings in [3].

Our aim in this paper is to investigate 3-prime near-rings admitting homoderivations satisfying certain identities, in the case where the constraints are initially assumed to hold on semigroup ideal of near-rings.

2. Preliminaries

In the following, we give some well-known results of near-rings in the literature, which will be used extensively in the proof of our results.

Lemma 2.1. [3, Lemmas 1.2 (i), 1.2 (iii), and 1.3 (iii)] Let $\mathcal{N}$ be a 3-prime near-ring.

(i) If $z \in Z(\mathcal{N}) \setminus \{0\}$, then $z$ is not a zero divisor.
(ii) If $Z(\mathcal{N})$ contains a nonzero element $z$ for which $z + z \in Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.
(iii) If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.

Lemma 2.2. [3, Lemmas 1.3 (i), 1.4 (i), and 1.3 (iii)] Let $\mathcal{N}$ be a 3-prime near-ring.

(i) If $U$ is a nonzero semigroup right (resp. semigroup left ) ideal of $\mathcal{N}$ and $x \in \mathcal{N}$ such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.
(ii) If $U$ is a nonzero semigroup ideal of $\mathcal{N}$ and $x, y \in \mathcal{N}$ such that $xUy = \{0\}$, then $x = 0$ or $y = 0$. 

(iii) If $U$ is a nonzero semigroup right ideal of $\mathcal{N}$ and $x \in \mathcal{N}$ which centralizes $U$, then $x \in Z(\mathcal{N})$.

**Lemma 2.3.** [3, Lemmas 1.5] If $\mathcal{N}$ is a 3-prime near-ring and $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}$ is a commutative ring.

**Lemma 2.4.** [3, Lemma 2.4 (ii)] Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a homoderivation $h$ such that $h^2(\mathcal{N}) = \{0\}$, then $h = 0$.

**Lemma 2.5.** [3, Lemma 2.4] Let $\mathcal{N}$ be a prime 3-near-ring. If $\mathcal{N}$ admits a nonzero homod-erivation $h$, then for all $x, y, a \in \mathcal{N}$ we have

$$h(xy)(h(a) + a) = h(x)h(y)(h(a) + a) + h(x)y(h(a) + a) + xh(y)(h(a) + a).$$

3. Some results for homoderivation and semigroup ideal in 3-prime near-ring

We begin this paragraph with a crucial result, which is necessary for developing the proof of our main results.

**Lemma 3.1.** Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero additive map $f$ on $\mathcal{N}$ which is zero-power valued on $\mathcal{N}$. Then the following assertions are equivalent:

(i) $f(x) + x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.

(ii) $x + f(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$.

(iii) $\mathcal{N}$ is a commutative ring.

**Proof.** It is clear that the implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are trivial.

(i) $\Rightarrow$ (iii) Suppose that

$$f(x) + x \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}.$$  \hspace{1cm} (1)

If $f(x) \neq 0$ for all $x \in \mathcal{N} \setminus \{0\}$. By recurrence we have $f^n(x) \neq 0$ for all $x \in \mathcal{N} \setminus \{0\}$ and $n \in \mathbb{N}^*$. Since $f$ is zero-power valued on $\mathcal{N}$, for each $x \in \mathcal{N}$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$, it follows that for $z = f^{k(x)-1}(x) \neq 0$, $f(z) = f^k(x)(x) = 0$ which is a contradiction. Hence there exists $y \in \mathcal{N} \setminus \{0\}$ such that $f(y) = 0$, so we get $y = f(y) + y \in Z(\mathcal{N}) \setminus \{0\}$ and $y + y = f(y + y) + y + y \in Z(\mathcal{N})$, which forces that $\mathcal{N}$ is abelian.

Now by replacing $x$ by $x - f(x) + f^2(x) + \ldots + (-1)^{k(x)-1}f^{k(x)-1}(x)$ in (1) and using $\mathcal{N}$ is abelian we get $x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$, thus $\mathcal{N} \subseteq Z(\mathcal{N})$. Hence $\mathcal{N}$ is commutative ring by Lemma 2.3. \Box

**Lemma 3.2.** Let $\mathcal{N}$ be a 2-torsion free near-ring. If $\mathcal{N}$ admits a nonzero homoderivation $h$ which is zero-power valued on $\mathcal{N}$, then $\mathcal{N}$ is zero symmetric near-ring.
Proof. We have for all \( z \in \mathcal{N} \)

\[
h(0z) = h(0)h(z) + h(0)z + 0h(z) \]
\[
= 0h(z) + 0z + 0h(z).
\]

On the other hand

\[
h(0z) = h(0(0z))
\]
\[
= h(0)h(0z) + h(0)0z + 0h(0z)
\]
\[
= 0h(0z) + 0z + 0h(0z)
\]
\[
= 0h(z) + 0z + 0h(z) + 0z + 0h(z).
\]

Comparing the last two expressions, we find \( 2(0h(z) + 0z) = 0 \) for all \( z \in \mathcal{N} \). Using 2-torsion freeness of \( \mathcal{N} \) we obtain \( 0h(z) + 0z = 0 \) for all \( z \in \mathcal{N} \). By recurrence, we obtain

\[
0h^n(z) + (-1)^{n+1}0z = 0 \text{ for all } z \in \mathcal{N} \text{ and } n \in \mathbb{N}^*.
\]

Since \( h \) is zero-power valued on \( \mathcal{N} \), there exists an integer \( k(z) > 1 \) such that \( h^{k(z)}(z) = 0 \). Replacing \( n \) by \( k(z) \) in (2), we get \( (-1)^{k(z)+1}0z = 0 \) for all \( z \in \mathcal{N} \). Thus \( \mathcal{N} \) is zero symmetric near-ring. \( \square \)

Lemma 3.3. Let \( \mathcal{N} \) be a 3-prime near-ring and \( h \) be a nonzero homoderivation of \( \mathcal{N} \).

(i) If \( \mathcal{N} \) is zero symmetric and \( U \) is a nonzero semigroup right ideal of \( \mathcal{N} \), then \( h(U) \neq \{0\} \).

(ii) If \( U \) is a nonzero semigroup left ideal of \( \mathcal{N} \), then \( h(U) \neq \{0\} \).

Proof. (i) Let \( U \) be a nonzero semigroup right ideal. Suppose that \( h(U) = \{0\} \). Then for all \( u \in U \) and \( x \in \mathcal{N} \) we have \( 0 = h(ux) = h(u)h(x) + h(u)x + uh(x) = uh(x) \), that is \( uh(x) = 0 \) for all \( u \in U, x \in \mathcal{N} \). This implies that \( Uh(x) = \{0\} \) for all \( x \in \mathcal{N} \). Hence \( h = 0 \) by Lemma 2.2 (i).

(ii) The argument for semigroup left ideal is similar. \( \square \)

Lemma 3.4. Let \( \mathcal{N} \) be a 3-prime near-ring admitting a nonzero homoderivation \( h \) and \( U \) a nonzero semigroup right ideal of \( \mathcal{N} \).

(i) If \( x \in \mathcal{N} \) and \( h(U)(h(x) + x) = \{0\} \), then \( h(x) + x = 0 \).

(ii) If \( x \in \mathcal{N} \) and \( xh(U) = \{0\} \), then \( x = 0 \).

(iii) If \( \mathcal{N} \) is a 2-torsion free, then \( h^2(U) \neq \{0\} \).
Proof. (i) Let \( x \in \mathcal{N} \) and \( h(U)(h(x) + x) = \{0\} \). We have

\[
0 = h(yu)(h(x) + x) = (h(y)h(u) + h(y)u + yh(u))(h(x) + x) = h(y)u(h(x) + x) \quad \text{for all} \quad u, y \in \mathcal{N}.
\]

Then \( h(y)U(h(x) + x) = \{0\} \) for all \( y \in \mathcal{N} \) and by Lemma 2.2 (ii) and Lemma 3.5, we conclude that \( h(x) + x = 0 \).

For (ii), suppose \( xh(U) = \{0\} \). For all \( u \in U \) and \( y \in \mathcal{N} \), we have

\[
0 = xh(uy) - x(h(u)h(y) + h(u)y + uh(y)) = xuh(y).
\]

Hence \( xUh(y) = \{0\} \) for all \( y \in \mathcal{N} \) and \( x = 0 \) by Lemma 2.2 (ii) and Lemma 3.5.

For (iii), assume that \( h^2(U) = \{0\} \), then \( 0 = h^2(uv) = 2h(u)h(v) \) for all \( u, v \in U \), since \( \mathcal{N} \) is a 2-torsion free, we get \( h(u)h(v) = 0 \), thus \( h(U)(h^2(v) + h(v)) = \{0\} \) for all \( v \in U \), in view of Lemma 3.5 (i), we obtain \( h^2(v) + h(v) = h(v) = 0 \). Thus, part (ii) \( h(U) = \{0\} \) which is a contradiction by Lemma 2.2. \( \square \)

Lemma 3.5. Let \( \mathcal{N} \) be a 3-prime near-ring and \( U \) a nonzero semigroup ideal of \( \mathcal{N} \) and \( h \) a nonzero homoderivation on \( \mathcal{N} \) which preserves \( U \). If \( a \in \mathcal{N} \) and \( [h(a) + a, h(U)] = \{0\} \), then \( h(a) + a \in Z(\mathcal{N}) \).

Proof. Let \( a \in \mathcal{N} \) and \( [h(a) + a, h(U)] = \{0\} \).

We set \( C(a) = \{ x \in \mathcal{N} \mid [h(a) + a, x] = 0 \} \). Note that \( h(U) \subset C(a) \cap U \). Thus, if \( y \in C(a) \cap U \) and \( u \in U \), then both \( h(yu) \), \( h(u) \), \( h(y) \) and \( yh(u) \) are in \( C(a) \). Therefore, \( h(y)u \in C(a) \) for all \( u \in U, y \in C(a) \cap U \). Hence, \( h(y)uv \in C(a) \) for all \( u, v \in U, y \in C(a) \cap U \) and so, \( 0 = [h(a) + a, h(y)uv] = h(y)u[(h(a) + a), v] \). Thus, \( h(y)U[(h(a) + a), v] = \{0\} \) for all \( v \in U, y \in C(a) \cap U \). Since \( h(U) \subset C(a) \cap U \), then \( h^2(y)U[(h(a) + a), v] = \{0\} \) for all \( y, v \in U \). Since, by Lemma 2.2 (iii), \( h^2(U) \neq \{0\} \), by Lemma 2.2 (ii) we get \( [h(a) + a, U] = \{0\} \), and \( h(a) + a \in Z(\mathcal{N}) \) by Lemma 2.2 (iii). \( \square \)

Theorem 3.6. Let \( \mathcal{N} \) be a 3-prime near-ring and let \( U \) be a nonzero semigroup left ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero homoderivation \( h \) which is zero-power valued on \( \mathcal{N} \). Then the following assertions are equivalent:

(i) \( h(u) + u \in Z(\mathcal{N}) \) for all \( u \in U \).
(ii) \(-u + h(-u) \in Z(\mathcal{N}) \) for all \( u \in U \).
(iii) \( \mathcal{N} \) is a commutative ring.
Proof. It is clear that the implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are trivial.

$(i) \Rightarrow (iii)$ Let $U \neq \{0\}$ a semigroup left ideal such that $h(u) + u \in Z(N)$ for all $u \in U$. Since $xu \in U$, we get $h(xu) + xu \in Z(N)$. Thus

$$h(xu) + xu = h(x)h(u) + h(x)u + xh(u) + xu = (h(u) + u)(h(x) + x) \in Z(N) \text{ for all } u \in U, x \in N.$$ 

Since $h(u) + u \in Z(N)$, it follows that $h(u) + u = 0$ for all $u \in U$ or $h(x) + x \in Z(N)$ for all $x \in N$.

Suppose that $h(u) + u = 0$ for all $u \in U$. By recurrence, it follows that

$$(3) \quad h^n(u) + (-1)^n + 1 u = 0 \text{ for all } u \in U \text{ and } n \in \mathbb{N}^*.$$ 

Since $h$ is zero-power valued on $N$, there exists an integer $k(u) > 1$ such that $h^{k(u)}(u) = 0$. Replacing $n$ by $k(u)$ in $(3)$, we get $(-1)^{k(u)+1} u = 0$ for all $u \in U$, so $U = \{0\}$ which is a contradiction. Hence

$$h(x) + x \in Z(N) \text{ for all } x \in N.$$ 

By Lemma 3.1, we conclude that $N$ is commutative ring.

$(ii) \Rightarrow (iii)$ Let $U \neq \{0\}$ a semigroup left ideal such that $-u + h(-u) \in Z(N)$ for all $u \in U$. Since $xu \in U$ it follows that $-(h(xu) + xu) \in Z(N)$. Thus

$$-(h(xu) + xu) = -(h(x)h(u) + h(x)u + xh(u) + xu) = -xu - xh(u) - h(x)u - h(x)h(u) = x(-u) + xh(-u) + h(x)(-u) + h(x)h(-u) = x(-u - h(u)) + h(x)(-u - h(u)) = (-u - h(u))(x + h(x)) \in Z(N) \text{ for all } u \in U, x \in N.$$ 

Since $-u - h(u) \in Z(N)$, it follows that $-u - h(u) = 0$ for all $u \in U$ or $x + h(x) \in Z(N)$ for all $x \in N$.

First suppose that $-u - h(u) = 0$ for all $u \in U$. Thus $h(u) + u = 0$ for all $u \in U$. As above, it follows that $U = \{0\}$ which is a contradiction. So

$$(4) \quad x + h(x) \in Z(N) \text{ for all } x \in N.$$ 

From Lemma 3.1, we find that $N$ is commutative ring. \[\Box\]

Remark 3.7. Using Theorem 3.6 and Lemma 3.5, we can easily find the following Theorem.
Theorem 3.8. Let $N$ be a 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a nonzero homoderivation $h$ which is zero-power valued on $N$, that preserves $U$ and satisfies $[h(U), h(U) + U] = \{0\}$, then $N$ is a commutative ring.

Theorem 3.9. Let $N$ be a 2-torsion free 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a nonzero homoderivation $h$ which is zero-power valued on $N$ and preserves $U$, then the following assertions are equivalent:

(i) $h([x, y]) + [x, y] = [h(x) + x, y]$ for all $x, y \in U$.

(ii) $N$ is a commutative ring.

Proof. It is clear that (ii) $\implies$ (i).

(i) $\implies$ (ii) Assume that

$h([x, y]) + [x, y] = [h(x) + x, y]$ \hspace{1cm} \text{for all } x, y \in U. \hspace{1cm} (5)$

Replacing $y$ by $xy$ in (5), and using the fact that $[h(x) + x, x] = 0$ for all $x \in U$, we get

$h(x)h([x, y]) + h(x)[x, y] + x(h([x, y]) + [x, y]) = x[h(x) + x, y]$ \hspace{1cm} \text{for all } x, y \in U.

By using (5), we find

$h(x)[h(x) + x, y] = 0$ \hspace{1cm} \text{for all } x, y \in U. \hspace{1cm} (6)$

Hence

$h(x)y(h(x) + x) = h(x)(h(x) + x)y$ \hspace{1cm} \text{for all } x, y \in U. \hspace{1cm} (7)$

Putting $yt$ instead of $y$ in (7), we arrive at

$h(x)y[t, h(x) + x] = 0$ \hspace{1cm} \text{for all } x, y \in U, t \in N, \hspace{1cm} (8)$

which leads to

$h(x)U[t, h(x) + x] = \{0\}$ \hspace{1cm} \text{for all } x \in U, t \in N.

By Lemma 2.2 (ii), we obtain

$h(x) = 0$ or $h(x) + x \in Z(N)$ \hspace{1cm} \text{for all } x \in U. \hspace{1cm} (8)$

If there exists $x_0 \in U$ such that $h(x_0) = 0$, using (8) we get $h([x_0, y]) = 0$ for all $y \in U$, thus

$x_0h(y) = h(y)x_0$ \hspace{1cm} \text{for all } y \in U. \hspace{1cm} (9)$

Which means that $(h(x_0) + x_0)h(y) = h(y)(h(x_0) + x_0)$ for all $y \in U$. Taking $h(y)t$ instead of $y$, then by Lemma 2.3, we have

$x_0h^2(y)h(t) + x_0h^2(y)t + x_0h(y)h(t) = h^2(y)h(t)x_0 + h^2(y)tx_0 + h(y)h(t)x_0 \hspace{1cm} (10)$
for all \( y, t \in U \). Using (9), (11) becomes

\[
x_0h^2(y)t = h^2(y)tx_0 \quad \text{for all } y, t \in U.
\]

Replacing \( t \) by \( tm \) in (11) and using it again, we get \( h^2(y)t[x_0, m] = 0 \) for all \( y, t \in U, m \in \mathcal{N} \), ie. \( h^2(y)U[x_0, m] = \{0\} \) for all \( y \in U, m \in \mathcal{N} \). By Lemma 4(iii) and Lemma 2(ii), \( x_0 \in Z(\mathcal{N}) \). In this case, (8) becomes \( h(x) + x \in Z(\mathcal{N}) \) for all \( x \in U \) which forces that \( \mathcal{N} \) is a commutative ring by Theorem 5.3. \( \Box \)

**Theorem 3.10.** Let \( \mathcal{N} \) be a 2-torsion free 3-prime near-ring and \( U \) be a nonzero semigroup ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero homoderivation \( h \) which is zero-power valued on \( \mathcal{N} \) and preserves \( U \), then the following assertions are equivalent:

(i) \( h([x, y]) = [x, y] \) for all \( x, y \in U \).

(ii) \( \mathcal{N} \) is a commutative ring.

**Proof.** It is obvious that (ii) implies (i).

(i) \( \Rightarrow \) (ii) Assume that

\[
h([x, y]) = [x, y] \quad \text{for all } x, y \in U.
\]

Putting \( xy \) in place of \( y \) in (12), and using the fact that \([x, xy] = x[x, y], \) we get

\[
x[x, y] = h(x[x, y])
\]

\[
= h(x)h([x, y]) + h(x)[x, y] + xh([x, y])
\]

\[
= 2h(x)[x, y] + x[x, y] \quad \text{for all } x, y \in U.
\]

Which implies that \( 2h(x)[x, y] = 0 \) for all \( x, y \in U \). By 2-torsion freeness of \( \mathcal{N} \), we finds \( h(x)[x, y] = 0 \) for all \( x, y \in U \), which implies that

\[
h(x)xy = h(x)yx \quad \text{for all } x, y \in U.
\]

Substituting \( yt \) for \( y \) in (13) and using it again, we obtain \( h(x)y[x, t] = 0 \) for all \( x, y \in U, t \in \mathcal{N}, \) ie. \( h(x)U[x, t] = \{0\} \) for all \( x \in U, t \in \mathcal{N} \). By Lemma 2(ii), we arrive at

\[
h(x) = 0 \quad \text{or } x \in Z(\mathcal{N}) \quad \text{for all } x \in U.
\]

Suppose there is an element \( x_0 \) of \( U \) such that \( h(x_0) = 0 \), by (13) we can easily see that \([x_0, h(y)] = [x_0, y] \) for all \( y \in U \) and invoking the definition of \( h \). By recurrence we arrive at

\[
[x_0, h^k(y)] = [x_0, y] \quad \text{for all } y \in U, k \in \mathbb{N}^*.
\]

Using the fact that \( h \) is zero-power valued on \( \mathcal{N} \), there exists an integer \( k(y) > 1 \) such that \( h^{k(y)}(y) = 0 \). Replacing \( k \) by \( k(y) \) in (13), we obviously get \( x_0 \in Z(\mathcal{N}) \). In this case, (13) becomes \( x \in Z(\mathcal{N}) \) for all \( x \in U \) which forces that \( \mathcal{N} \) is a commutative ring by Lemma 2.3. \( \Box \)
Theorem 3.11. Let $N$ be a 2-torsion free 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a non zero homoderivation $h$ which is zero-power valued on $N$, then the following assertions are equivalent:

(i) $(h(x) + x) \circ y \in Z(N)$ for all $x, y \in U$.

(ii) $N$ is a commutative ring.

Proof. It is obvious that (ii) implies (i).

(i) $\Rightarrow$ (ii) Assume that

(16) $(h(x) + x) \circ y \in Z(N)$ for all $x, y \in U$.

Replacing $y$ by $(h(x) + x)y$ in (16), we get

$$(h(x) + x)((h(x) + x) \circ y) \in Z(N)$$

for all $x, y \in U$.

By Lemma 2.1, it follows that

(17) $(h(x) + x) \circ y = 0$ or $h(x) + x \in Z(N)$ for all $x, y \in U$.

If there exists $x_0 \in U$ such that $h(x_0) + x_0 \in Z(N) \setminus \{0\}$, by (17) together with Lemma 2.1 (iii), we may conclude that

$y + y \in Z(N)$ for all $y \in U$,

so that

(18) $r(y + y) = ry + ry \in Z(N)$ for all $y \in U, r \in N$.

Since $N$ is 2-torsion free, by using (18) and Lemma 2.1 (iii), we obtain $N \subseteq Z(N)$, which implies that $N$ is a commutative ring by Lemma 2.1.

In view of (17), we may now assume that $(h(x) + x) \circ y = 0$ for all $x, y \in U$ i.e.

$y(h(x) + x) = -(h(x) + x)y$ for all $x, y \in U$.

Taking $yt$ instead if $y$, where $t \in N$, in the last equation, we obtain

$$yt(h(x) + x) = -(h(x) + x)yt$$

$$= (h(x) + x)y(-t)$$

$$= (-y(h(x) + x))(-t)$$

$$= y(-(h(x) + x))(-t)$$

for all $x, y \in U, t \in N$,

which leads to

$$y(t(h(x) + x) - -(h(x) + x))(-t)) = 0$$

for all $x, y \in U, t \in N$. 

thereby obtaining
\[ U(-t(-(h(x) + x)) + (-h(x) + x))t) = \{0\} \quad \text{for all } x \in U, t \in \mathcal{N}. \]

By Lemma 2.2 (i), we conclude that \(-x + h(-x) \in Z(\mathcal{N})\) for all \(x \in U\). Thus By Theorem 3.6, it follows that \(\mathcal{N}\) is a commutative ring. \(\square\)

**Theorem 3.12.** Let \(\mathcal{N}\) be a 3-prime near-ring and \(U\) be a nonzero semigroup ideal of \(\mathcal{N}\). If \(\mathcal{N}\) admits a nonzero homoderivation on \(h\) which is zero-power valued on \(\mathcal{N}\), then the following assertions are equivalent:

(i) \([h(x) + x, y] \in Z(\mathcal{N})\) for all \(x, y \in U\).

(ii) \(h(xy) + xy \in Z(\mathcal{N})\) for all \(x, y \in U\).

(iii) \(\mathcal{N}\) is commutative ring.

**Proof.** It is clear that the implications (iii) \(\Rightarrow\) (i) and (iii) \(\Rightarrow\) (ii) are trivial.

(i) \(\Rightarrow\) (iii) Suppose that
\[
[h(x) + x, y] = Z(\mathcal{N}) \quad \text{for all } x, y \in U.
\]
Replacing \(y\) by \((h(x) + x)y\) in (13), we get
\[
(h(x) + x)[h(x) + x, y] = Z(\mathcal{N}) \quad \text{for all } x, y \in U.
\]
By Lemma 2.1 (iii), we obtain
\[
h(x) + x \in Z(\mathcal{N}) \quad \text{or} \quad [h(x) + x, y] = 0 \quad \text{for all } x, y \in U.
\]
Both cases force that \(h(x) + x \in Z(\mathcal{N})\) for all \(x \in U\). Using Theorem 3.6, we conclude that \(\mathcal{N}\) is a commutative ring.

(ii) \(\Rightarrow\) (iii) Now assume that \(h(xy) + xy \in Z(\mathcal{N})\) for all \(x, y \in U\). We have
\[
h(zxy) + zxy = h(z)h(xy) + h(z)xy + zh(xy) + zxy
= h(z)(h(xy) + xy) + z(h(xy) + xy)
= (h(xy) + xy)(h(z) + z) \in Z(\mathcal{N}) \quad \text{for all } x, y, z \in U.
\]
Using Lemma 2.1 (iii) implies
\[
h(xy) + xy = 0 \quad \text{for all } x, y \in U \quad \text{or} \quad h(z) + z \in Z(\mathcal{N}) \quad \text{for all } z \in U.
\]
If \(h(xy) + xy = 0\) for all \(x, y \in U\), by recurrence we have \(h^k(xy) + (-1)^{k+1}xy = 0\) for all \(x, y \in U, k \in \mathbb{N}^*\). Since \(h\) is zero-power valued on \(\mathcal{N}\), there exists an integer \(k(xy) > 1\) such that \(h^k(xy)(xy) = 0\). Replacing \(k\) by \(k(xy)\) in the above expression we get \(xy = 0\) for all
Thus by the 3-primeness of $N$ we get $U = \{0\}$; a contradiction. Hence (21) becomes $h(z) + z \in Z(N)$ for all $z \in U$, and by Theorem we prove that $N$ is commutative ring.

**Theorem 3.13.** Let $N$ be a 2-torsion 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. There is no nonzero homoderivation $h$ which is zero-power valued on $N$ such that $h(x \circ y) + x \circ y = [x, y]$ for all $x, y \in U$.

**Proof.** Suppose that $h \neq 0$ and

$$h(x \circ y) + x \circ y = [x, y] \quad \text{for all } x, y \in U. \quad (21)$$

Thus

$$y[x, y] = [yx, y]$$
$$= h(yx \circ y) + yx \circ y$$
$$= h(y(x \circ y)) + y(x \circ y)$$
$$= h(y)h(x \circ y) + h(y)x \circ y + yh(x \circ y) + y(x \circ y)$$
$$= h(y)(h(x \circ y) + x \circ y) + y(h(x \circ y) + x \circ y)$$
$$= h(y)[x, y] + y[x, y] \quad \text{for all } x, y \in U.$$

This expression gives us $h(y)[x, y] = 0$ for all $x, y \in U$, that is

$$h(y)xy = h(y)yx \quad \text{for all } x, y \in U.$$

Substituting $xm$ in place of $x$ in the last expression, we get

$$h(y)xy = h(y)xym \quad \text{for all } x, y \in U, m \in N.$$

Which can be rewritten as $h(y)U[m, y] = \{0\}$ for all $y \in U, m \in N$. By Lemma (ii), we obtain

$$h(y) = 0 \quad \text{or } y \in Z(N) \quad \text{for all } y \in U. \quad (22)$$

Suppose there is an element $y_0 \in U$ such that $y_0 \in Z(N)$. Then (21) becomes $h(2xy_0) + 2xy_0 = 0$, for all $x \in U$. By recurrence, it follows that

$$h^k(2xy_0) + (-1)^{k+1}2xy_0 = 0 \quad \text{for all } x \in U, k \in \mathbb{N}^* \quad (23).$$

Since $h$ is zero-power valued on $N$, there exists an integer $k(2xy_0) > 1$ such that $h^{k(2xy_0)}(2xy_0) = 0$. Replacing $k$ by $k(2xy_0)$ in (23), we get $2xy_0 = 0$ for all $x \in U$, and
using 2-torsion freeness of \( \mathcal{N} \), it follows that \( Uy_0 = \{ 0 \} \). Hence \( y_0 = 0 \) by Lemma 2.2 (i). In this case, (22) implies that \( h(U) = \{ 0 \} \) which gives a contradiction by Lemma 3.3. \( \square \)

**Theorem 3.14.** Let \( \mathcal{N} \) be a 2-torsion 3-prime near-ring and \( U \) be a nonzero semigroup ideal of \( \mathcal{N} \). There is no nonzero homoderivation \( h \) satisfying \( h([x,y]) + [x,y] = x \circ y \) for all \( x, y \in U \).

**Proof.** Suppose that \( h \neq 0 \) and

\[
(24) \quad h([x,y]) + [x,y] = x \circ y \quad \text{for all} \quad x, y \in U.
\]

Thus

\[
y(x \circ y) = yx \circ y = h([yx,y]) + [yx,y] = h(y[x,y]) + y[x,y] = h(y)h([x,y]) + h(y)[x,y] + yh([x,y]) + y[x,y] = h(y)(h([x,y]) + [x,y]) + yh([x,y]) + [x,y]) = h(y)(x \circ y) + y(x \circ y) \quad \text{for all} \quad x, y \in U.
\]

This expression gives us \( h(y)(x \circ y) = 0 \) for all \( x, y \in U \), it follows that

\[
(25) \quad h(y)xy = -h(y)yx \quad \text{for all} \quad x, y \in U.
\]

Substituting \( xm \) in place of \( x \) in (24), we get

\[
h(y)xmy = -h(y)yxm = h(y)yx(-m) = h(y)x(-y)(-m) \quad \text{for all} \quad x, y \in U, m \in \mathcal{N}.
\]

Which can be rewritten as \( h(y)U(-m(-y) + (-y)m) = \{ 0 \} \) for all \( y \in U \) and \( m \in \mathcal{N} \). By Lemma 2.2, we have

\[
(26) \quad h(y) = 0 \quad \text{or} \quad -y \in Z(\mathcal{N}) \quad \text{for all} \quad y \in U.
\]

Suppose there is an element \( y_0 \in U \) such that \( -y_0 \in Z(\mathcal{N}) \). Replacing \( y \) by \( -y_0 \) in (24) we get \( 2(-y_0)x = 0 \) for all \( x \in U \). Using 2-torsion freeness of \( \mathcal{N} \), we obtain \( -y_0U = \{ 0 \} \) and by Lemma 3.3, we have \( y_0 = 0 \). In this case, (24) implies that \( h(U) = \{ 0 \} \) which gives a contradiction by Lemma 3.3. \( \square \)
Theorem 3.15. Let \( \mathcal{N} \) be a 3-prime near-ring and \( U \) be a nonzero semigroup ideal of \( \mathcal{N} \). There is no nonzero homoderivation \( h \) which is zero-power valued on \( \mathcal{N} \) such that \( h(xy) + xy = [x, y] \) for all \( x, y \in U \).

Proof. Suppose that \( h \neq 0 \) and

\[
h(xy) + xy = [x, y] \quad \text{for all} \quad x, y \in U.
\]

Thus

\[
y[x, y] = [yx, y] = h(yxy) + yxy = h(y(xy)) + y(xy)
\]

\[
= h(y)h(xy) + h(y)xy +yh(xy) + y(xy)
\]

\[
= h(y)[h(xy) + xy] + y(h(xy) + xy)
\]

\[
= h(y)[x, y] + y[x, y] \quad \text{for all} \quad x, y \in U.
\]

This expression gives us \( h(y)[x, y] = 0 \) for all \( x, y \in U \). As in proof of Theorem 3.12, it follows that

\[
h(y) = 0 \quad \text{or} \quad y \in Z(\mathcal{N}) \quad \text{for all} \quad y \in U.
\]

Suppose there is an element \( y_0 \in U \) such that \( y_0 \in Z(\mathcal{N}) \). Then \((28)\) becomes \( h(xy_0) + xy_0 = 0 \) for all \( x \in U \). By recurrence, it follows that

\[
h^k(xy_0) + (-1)^{k+1}xy_0 = 0 \quad \text{for all} \quad x \in U, k \in \mathbb{N}^*.
\]

Since \( h \) is zero-power valued on \( \mathcal{N} \), there exists an integer \( k(xy_0) > 1 \) such that \( h^{k(xy_0)}(xy_0) = 0 \). Replacing \( k \) by \( k(xy_0) \) in \((28)\), we get \( xy_0 = 0 \) for all \( x \in U \), so \( Uy_0 = \{0\} \). Hence \( y_0 = 0 \). In this case, \((28)\) implies that \( h(U) = \{0\} \) which gives a contradiction by Lemma 3.3. □

Theorem 3.16. Let \( \mathcal{N} \) be a 3-prime zero symmetric near-ring and \( U \) be a nonzero semigroup ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero homoderivation \( h \) such that \( h(xy) + xy = x \circ y \) for all \( x, y \in U \), then \( \mathcal{N} \) is a commutative ring.

Proof. Suppose that

\[
h(xy) + xy = x \circ y \quad \text{for all} \quad x, y \in U.
\]
Thus

\[
y(x \circ y) = yx \circ y \\
= h(yxy) + yxy \\
= h(y(xy)) + y(xy) \\
= h(y)h(xy) + h(y)xy + yh(xy) + y(xy) \\
= h(y)(h(xy) + xy) + y(h(xy) + xy) \\
= h(y)(x \circ y) + y(x \circ y) \quad \text{for all } x, y \in U.
\]

This expression gives us \( h(y)(x \circ y) = 0 \) for all \( x, y \in U \). As in proof of Theorem 3.17, it follows that

\[(31) \quad h(y) = 0 \quad \text{or} \quad -y \in Z(N) \quad \text{for all } y \in U.\]

Suppose there is an element \( y_0 \in U \) such that \( h(y_0) = 0 \). Then (31) becomes

\[(32) \quad h(x)y_0 + xy_0 = x \circ y_0 \quad \text{for all } x \in U.\]

Replacing \( x \) by \( y_0 \) in (32), we arrive at \( y_0^2 = 0 \). Substituting \( xy_0 \) in place of \( x \) in (32), then for \( x \in U \), we have

\[
y_0xy_0 = xy_0^2 + y_0xy_0 \\
= xy_0 \circ y_0 \\
= h(xy_0)y_0 + xy_0^2 \\
= 0,
\]

thus \( y_0Uy_0 = \{0\} \). Hence \( y_0 = 0 \). In this case, (31) implies that \( -U \subseteq Z(N) \). Since \( -U \) is nonzero left semigroup ideal of \( N \), by Lemma 2.3 \( N \) is a commutative ring. \( \square \)

**Theorem 3.17.** Let \( N \) be a 2-torsion free 3-prime near-ring and \( U \) be a nonzero semigroup ideal of \( N \). If \( N \) admits a nonzero homoderivation on \( h \) which is zero-power valued on \( N \), then the following assertions are equivalent:

(i) \( h(x \circ y) + x \circ y \in Z(N) \) for all \( x, y \in U \).

(ii) \( N \) is commutative ring.

**Proof.** It is easy to see that \( (ii) \implies (i) \).

\( (i) \implies (ii) \) Suppose that

\[(33) \quad h(x \circ y) + x \circ y \in Z(N) \quad \text{for all } x, y \in U.\]
Since \( x \circ xy = x(x \circ y) \) for all \( x, y \in U \), replacing \( y \) by \( xy \) in (33), we obtain
\[
h(x)h(x \circ y) + h(x)(x \circ y) + xh(x \circ y) + x(x \circ y) \in Z(N) \quad \text{for all } x, y \in U.
\]
Thus
\[
h(x)(h(x \circ y) + x \circ y) + x(h(x \circ y) + x \circ y) \in Z(N) \quad \text{for all } x, y \in U.
\]
By (33) we get
\[
(34) \quad (h(x \circ y) + x \circ y)(h(x) + x) \in Z(N) \quad \text{for all } x, y \in U.
\]
By Lemma 2.1 we have
\[
(35) \quad h(x \circ y) + x \circ y = 0 \quad \text{or} \quad h(x) + x \in Z(N) \quad \text{for all } x, y \in U.
\]
Suppose there is an element \( x_0 \in U \) such that \( h(x_0 \circ y) + x_0 \circ y = 0 \) for all \( y \in N \). Then by recurrence we prove that
\[
(36) \quad h^k(x_0 \circ y) + (-1)^{k+1}x_0 \circ y = 0 \quad \text{for all } y \in U, k \in \mathbb{N}^*.
\]
Since \( h \) is zero-power valued on \( N \), there exists an integer \( k(x_0 \circ y) > 1 \) such that \( h^{k(x_0 \circ y)}(x_0 \circ y) = 0 \). Replacing \( k \) by \( k(x_0 \circ y) \) in (36), it follows that
\[
(37) \quad x_0 \circ y = 0 \quad \text{for all } y \in U.
\]
Substituting \( x_0 \) in place of \( y \) in (37), we get \( 2(x_0)^2 = 0 \). Using 2-torsion freeness of \( N \) we obtain \( (x_0)^2 = 0 \). Putting \( x_0y \) in place of \( y \) in (37), we get \( x_0yx_0 = 0 \) for all \( y \in U \), so \( x_0Ux_0 = \{0\} \). Thus \( x_0 = 0 \) by Lemma 2.1 (ii). In this case, (35) implies that \( h(x) + x \in Z(N) \) for all \( x \in U \). By Theorem 3.6 it follows that \( N \) is commutative ring.

The following examples shows that \( h \) is "zero-power valued on \( N" cannot be omitted in the hypothesis of Theorems 3.6, 3.8, 3.11, 3.17, 3.19 and 3.20.

**Example 3.18.** Let \( N = U = M_2(\mathbb{Z}) \), that is a 2-torsion free prime ring. We consider \( h = -id_N \), then it is clear that \( h \) is a not "zero-power valued homoderivation on \( N" which preserve \( U \) and satisfy the following conditions:

(i) \( h(x) + x \in Z(N) \),
(ii) \( -x + h(-x) \in Z(N) \),
(iii) \( h(x), h(y) + y = 0 \),
(iv) \( h(x) + x, y \in Z(N) \),
(v) \( h([x, y]) + [x, y] = [h(x) + x, y] \),
(vi) \( h(x) + x \circ y \in Z(N) \),
(vii) \( h(xy) + xy \in Z(N) \),
(viii) \( h(x \circ y) + x \circ y \in Z(N) \),
for all \(x, y \in U\), but \(N\) is not commutative.

**Example 3.19.** Let \(N = U = \mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}\) be the set of all complex numbers. Addition is the usual addition of complex numbers. Then \((N, +)\) is a group. Define multiplication \(*\) on \(N\) by \(u * v = |u|v\). Then \((N, +, *)\) is a 3-prime near-ring, which is not a ring. We consider \(h = -id_N\), then it is clear that \(h\) is not "zero-power valued homoderivation on \(N\)," which preserve \(U\) and satisfy the following conditions:

(i) \(h(x) + x \in Z(N)\),
(ii) \(-x + h(-x) \in Z(N)\),
(iii) \([h(x), h(y) + y] = 0\),
(iv) \([h(x) + x, y] \in Z(N)\),
(v) \(h([x, y]) + [x, y] = [h(x) + x, y]\),
(vi) \((h(x) + x) \circ y \in Z(N)\),
(vii) \(h(xy) + xy \in Z(N)\),
(viii) \(h(x \circ y) + x \circ y \in Z(N)\),

for all \(x, y \in U\), but \(N\) is not commutative ring.

The following example illustrates that the hypothesis ”3-primeness of \(N\)” is essential in Theorems 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16 and 3.17 of our paper.

**Example 3.20.** Let \(S\) be a zero-symmetric 2-torsion free left near-ring and

\[
N = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}.
\]

\[
U = \left\{ \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u, 0 \in S \right\}.
\]

Then \(N\) is a 2-torsion left near-ring which is not 3-prime and \(U\) is a nonzero semigroup ideal of \(N\). Let us defined \(h : N \rightarrow N\) as follow:

\[
h\begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is clear that \(h\) is a zero-power valued homoderivation on \(N\), which satisfy the following conditions:
\( i \) \[ [h(x), h(y) + y] = 0, \]
\( ii \) \[ [h(x) + x, y] \in Z(\mathcal{N}), \]
\( iii \) \[ h([x, y]) + [x, y] = [h(x) + x, y], \]
\( iv \) \[ (h(x) + x) \circ y \in Z(\mathcal{N}), \]
\( v \) \[ h(xy) + xy \in Z(\mathcal{N}), \]
\( vi \) \[ h(x \circ y) + x \circ y \in Z(\mathcal{N}), \]
\( vii \) \[ h(x \circ y) + x \circ y = [x, y], \]
\( viii \) \[ h([x, y]) = [x, y], \]
\( ix \) \[ h([x, y]) + [x, y] = x \circ y, \]
\( x \) \[ h(xy) + xy = [x, y], \]
\( xi \) \[ h(xy) + xy = x \circ y \]

for all \( x, y \in U \), but \( \mathcal{N} \) is not commutative.

4. ACKNOWLEDGMENTS

The authors wish to sincerely thank the referees for several useful comments.

REFERENCES


Samir Mouhssine
University Sidi Mohammed Ben Abdellah,
Polydisciplinary Faculty, Department of Mathematics,
Physics and Computer Science, LSI, Taza; Morocco.
samir.mouhssine@usmba.ac.ma, samirfes27@gmail.com

Abdelkarim Boua
University Sidi Mohammed Ben Abdellah,
Polydisciplinary Faculty, Department of Mathematics,
Physics and Computer Science, LSI, Taza; Morocco.
abdelkarim.boua@usmba.ac.ma, abdelkarimboua@yahoo.fr