

Research Paper

**CHARACTERIZATIONS OF ORDERED SEMIHYPERGROUPS VIA  
(M, N)-INT-SOFT BI-HYPERIDEALS**

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ABSTRACT. The aim of this article is to study ordered semihypergroups in the framework of  $(M, N)$ -int-soft bi-hyperideals. In this paper, we introduce the notion of  $(M, N)$ -int-soft bi-hyperideals of ordered semihypergroups. Some properties of  $(M, N)$ -int-soft bi-hyperideals in ordered semihypergroups are provided. We show that every int-soft bi-hyperideal is an  $(M, N)$ -int-soft bi-hyperideals of  $S$  over  $U$  but the converse is not true which is shown with help of an example. We characterize left  $(M, N)$  simple and completely regular ordered semihypergroups by means of  $(M, N)$ -int-soft bi-hyperideals.

1. INTRODUCTION

The traditional classical models often fail to overcome the complexities arising in the modeling of uncertain data in many fields like economics, engineering, environmental science, sociology, medical science etc. Molodtsov [21], proposed the concept of soft set theory which is

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a completely new mathematical approach for modeling vagueness and uncertainty. At present works on the soft set theory are progressing very rapidly. Maji [18], presented some definitions on soft sets. Further, Ali and Sezgin et al. [11, 3, 24], introduced some new operations on soft sets and obtained some important properties. Simultaneously, this theory is very much useful in some different research areas such as information sciences with intelligent systems, approximate reasoning, expert and decision support systems and decision making etc., for examples, see [12, 19]. Recently, the algebraic structures of soft sets dealing with uncertainties have been studied by many authors. Feng [10], introduced the concepts of soft semirings and idealistic soft semirings, and investigated some characteristics of them. Jun [13], applied soft set theory to BCK/BCI-algebras. Aktas [2], discussed some important properties of soft groups. Based on soft sets many algebraic structures such as soft rings [1], soft ordered semigroups [14], soft hemirings [28, 17] and soft int-groups [4], etc., have been introduced. Khan et al. [16], applied soft set theory to ordered semihypergroups and introduced the notions of  $(M, N)$ -int-soft hyperideals and  $(M, N)$ -int-soft interior hyperideals.

The concept of hyperstructure was first introduced by Marty [20], at the eighth Congress of Scandinavian Mathematicians in 1934, when he defined hypergroups and started to analyze its properties. The core cause which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. Now, the theory of algebraic hyperstructures has become a well-established branch in algebraic theory and it has extensive applications in many branches of mathematics and applied science. In a recent monograph [6], Corsini and Leoreanu have presented numerous applications of algebraic hyperstructures. Later on, people have developed the semihypergroups, which are the simplest algebraic hyperstructures having closure and associative properties. A comprehensive review of the theory of hyperstructures can be found in [22, 25, 26, 9, 15, 27, 8, 7].

In this paper, we study the notion of  $(M, N)$ -int-soft bi-hyperideals of ordered semihypergroups and present some examples of this notion. We investigate several related properties of  $(M, N)$ -int-soft bi-hyperideals. We give the main Theorems which characterize bi-hyperideals in terms of  $(M, N)$ -int-soft bi-hyperideals. Finally we characterize left (resp. right) simple and completely regular ordered semihypergroups by means of  $(M, N)$ -int-soft bi-hyperideals.

## 2. PRELIMINARIES

By an ordered semihypergroup we mean a structure  $(S, \circ, \leq)$  in which the following conditions are satisfied:

- (i)  $(S, \circ)$  is a semihypergroup.

(ii)  $(S, \leq)$  is a poset.

(iii)  $(\forall a, b, x \in S) a \leq b$  implies  $x \circ a \leq x \circ b$  and  $a \circ x \leq b \circ x$ .

A nonempty subset  $A$  of an ordered semihypergroup  $S$  is called a subsemihypergroup of  $S$  if  $A \circ A \subseteq A$ .

A nonempty subset  $A$  of  $S$  is called a left (resp. right) hyperideal of  $S$  if it satisfies the following conditions:

(i)  $S \circ A \subseteq A$  (resp.  $A \circ S \subseteq A$ ).

(ii) If  $a \in A, b \in S$  and  $b \leq a$ , implying  $b \in A$ .

By a two sided hyperideal or simply a hyperideal of  $S$  we mean a nonempty subset of  $S$  which is both a left hyperideal and a right hyperideal of  $S$ .

A subsemihypergroup  $A$  of  $S$  is called a bi-hyperideal of  $S$  if it satisfies the following conditions:

(i)  $A \circ S \circ A \subseteq A$ .

(ii) If  $a \in A, b \in S$  and  $b \leq a$ , implying  $b \in A$ .

An ordered semihypergroup  $(S, \circ, \leq)$  is called *regular* if for every  $a \in S$  there exists  $x \in S$  such that  $a \leq a \circ x \circ a$ . Equivalent Definitions:

(1)  $a \in (a \circ S \circ a) \forall a \in S$ . (2)  $A \subseteq (A \circ S \circ A) \forall A \subseteq S$ .

An ordered semihypergroup  $(S, \circ, \leq)$  is called *left (resp. right) regular* if for each  $a \in S$  there exists  $x \in S$  such that  $a \leq x \circ a \circ a$  (resp.  $a \leq a \circ a \circ x$ ). Equivalent Definitions:

(1)  $a \in (S \circ a^2]$  (resp.  $a \in (a^2 \circ S]$ ). (2)  $A \subseteq (S \circ A^2]$  (resp.  $A \subseteq (A^2 \circ S]$ ) for all  $A \subseteq S$ .

An ordered semihypergroup  $(S, \circ, \leq)$  is called *completely regular* if it is both left regular and right regular.

For  $A \subseteq S$ , we denote  $(A] := \{t \in S : t \leq h \text{ for some } h \in A\}$ .

For  $A, B \subseteq S$ , we have  $A \circ B := \bigcup \{a \circ b : a \in A, b \in B\}$ . For subsets  $A$  and  $B$  of an ordered semihypergroup  $S$  we have  $A \subseteq (A]$  and if  $A \subseteq B$ , then  $(A] \subseteq (B], (A] \circ (B] \subseteq (A \circ B], ((A]) = (A]$ .

**Lemma 2.1.** (see [23]). *An ordered semihypergroup  $(S, \circ, \leq)$  is simple if it has no a proper hyperideal, that is for any hyperideal  $A \neq \emptyset$  of  $S$  we have  $A = S$ .*

**Lemma 2.2.** (see [23]). *An ordered semihypergroup  $(S, \circ, \leq)$  is left (resp. right) simple if  $(S \circ a] = S$  (resp.  $(a \circ S] = S$ ) only if for every  $a \in S$ .*

**Lemma 2.3.** (see [23]). *An ordered semihypergroup  $(S, \circ, \leq)$  is a simple ordered semihypergroup if and only if for every  $a \in S, (S \circ a \circ S] = S$ .*

**Example 2.4.** Let  $S = \{1, 2, 3, 4, 5\}$  be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	1	2	3	4	5
1	$\{1, 5\}$	$\{2, 5\}$	$\{1, 2, 3, 5\}$	$\{4\}$	$\{5\}$
2	$\{2, 5\}$	$\{1, 5\}$	$\{1, 2, 3, 5\}$	$\{4\}$	$\{5\}$
3	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$
4	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 5\}$
5	$\{5\}$	$\{5\}$	$\{1, 2, 3, 5\}$	$\{4\}$	$\{5\}$

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (5, 1), (5, 2), (5, 3), (4, 3)\}.$$

We can easily verify that  $(S, \circ, \leq)$  is a left and right simple ordered semihypergroup.

**Lemma 2.5.** (see [23]). Let  $(S, \circ, \leq)$  be an ordered semihypergroup. Then the following statements are equivalent:

- (1)  $S$  is completely regular.
- (2)  $A \subseteq (A^2 \circ S \circ A^2)$  for all  $A \subseteq S$ .
- (3)  $a \in (a^2 \circ S \circ a^2)$  for all  $a \in S$ .

**Example 2.6.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup where the hyperoperation and order relation are defined:

$\circ$	$a$	$b$	$c$
$a$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$
$b$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$c$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}.$$

It is easy to see that  $(S, \circ, \leq)$  is a left and right simple and also complete regular ordered semihypergroup

### 3. SOFT SETS

In what follows, we take  $E = S$  as the set of parameters, which is an ordered semihypergroup, unless otherwise specified.

From now on,  $U$  is an initial universe set,  $E$  is a set of parameters,  $P(U)$  is the power set of  $U$  and  $A, B, C \dots \subseteq E$ .

**Definition 3.1.** (see [21, 5]). A *soft set*  $f_A$  over  $U$  is defined as

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Hence  $f_A$  is also called an *approximation function*.

A soft set  $f_A$  over  $U$  can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}.$$

It is clear that a soft set is a *parameterized family* of subsets of  $U$ . Note that the set of all soft sets over  $U$  will be denoted by  $S(U)$ .

**Definition 3.2.** (see [5]). Let  $f_A, f_B \in S(U)$ . Then  $f_A$  is called a *soft subset* of  $f_B$ , denoted by  $f_A \tilde{\subseteq} f_B$  if  $f_A(x) \subseteq f_B(x)$  for all  $x \in E$ .

**Definition 3.3.** (see [5]). Two soft sets  $f_A$  and  $f_B$  are said to be equal soft sets if  $f_A \tilde{\subseteq} f_B$  and  $f_B \tilde{\subseteq} f_A$  and is denoted by  $f_A \tilde{=} f_B$ .

**Definition 3.4.** (see [5]). Let  $f_A, f_B \in S(U)$ . Then the *soft union* of  $f_A$  and  $f_B$ , denoted by  $f_A \tilde{\cup} f_B = f_{A \cup B}$ , is defined by  $(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$  for all  $x \in E$ .

**Definition 3.5.** (see [5]). Let  $f_A, f_B \in S(U)$ . Then the *soft intersection* of  $f_A$  and  $f_B$ , denoted by  $f_A \tilde{\cap} f_B = f_{A \cap B}$ , is defined by  $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$  for all  $x \in E$ .

For  $x \in S$ , we define

$$A_x = \{(y, z) \in S \times S \mid x \leq y \circ z\}.$$

**Definition 3.6.** (see [15]). For a nonempty subset  $A$  of  $S$  the characteristic soft set is defined to be the soft set  $\mathcal{S}_A$  of  $A$  over  $U$  in which  $\mathcal{S}_A$  is given by

$$\mathcal{S}_A : S \longmapsto P(U). \quad x \longmapsto \begin{cases} U, & \text{if } x \in A \\ \emptyset, & \text{otherwise} \end{cases}$$

**Definition 3.7.** (see [15]). Let  $f_A$  be a soft set of an ordered semihypergroup  $S$  over  $U$  a subset  $\delta$  such that  $\delta \in P(U)$ . The  $\delta$ -inclusive set of  $f_A$  is denoted by  $i_A(f_A, \delta)$  and defined to be the set

$$i_A(f_A, \delta) = \{x \in S \mid \delta \subseteq f_A(x)\}.$$

**Definition 3.8.** (see [9]). A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an *int-soft subsemihypergroup* of  $S$  over  $U$  if:

$$(\forall x, y \in S) \bigcap_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(x) \cap f_A(y).$$

**Definition 3.9.** (see [9]). Let  $f_A$  be a soft set of an ordered semihypergroup  $S$  over  $U$ . Then  $f_A$  is called an int-soft left (resp. right) hyperideal of  $S$  over  $U$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in S) \bigcap_{\alpha \in xoy} f_A(\alpha) \supseteq f_A(y)$  (resp.  $\bigcap_{\alpha \in xoy} f_A(\alpha) \supseteq f_A(x)$ ).
- (2)  $(\forall x, y \in S) x \leq y \implies f_A(x) \supseteq f_A(y)$ .

A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an *int-soft hyperideal* (or int-soft two-sided hyperideal) of  $S$  over  $U$  if it is both an int-soft left hyperideal and an int-soft right hyperideal of  $S$  over  $U$ .

**Definition 3.10.** (see [9]). A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an int-soft bi-hyperideal of  $S$  over  $U$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in S) \bigcap_{\alpha \in xoy} f_A(\alpha) \supseteq f_A(x) \cap f_A(y)$ .
- (2)  $(\forall x, y, z \in S) \bigcap_{\alpha \in xoyoz} f_A(\alpha) \supseteq f_A(x) \cap f_A(z)$ .
- (3)  $(\forall x, y \in S) x \leq y \implies f_A(x) \supseteq f_A(y)$ .

#### 4. $(M, N)$ -INT-SOFT BI-HYPERIDEALS

In this section, we introduce the notion of  $(M, N)$ -int-soft bi-hyperideals of ordered semihypergroups and investigate some related properties. From now on,  $\emptyset \subseteq M \subset N \subseteq U$ .

**Definition 4.1.** (see [16]). A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an  $(M, N)$ -int-soft subsemihypergroup of  $S$  over  $U$  if:

$$(\forall x, y \in S) \left( \bigcap_{\alpha \in xoy} f_A(\alpha) \right) \cup M \supseteq f_A(x) \cap f_A(y) \cap N.$$

**Definition 4.2.** (see [16]). A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an  $(M, N)$ -int-soft left (resp. right) hyperideal of  $S$  over  $U$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in S) \left( \bigcap_{\alpha \in xoy} f_A(\alpha) \right) \cup M \supseteq f_A(y) \cap N$  (resp.  $\left( \bigcap_{\alpha \in xoy} f_A(\alpha) \right) \cup M \supseteq f_A(x) \cap N$ ).
- (2)  $(\forall x, y \in S) x \leq y \implies f_A(x) \cup M \supseteq f_A(y) \cap N$ .

A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an  $(M, N)$ -int-soft hyperideal of  $S$  over  $U$ , if it is both an  $(M, N)$ -int-soft left hyperideal and an  $(M, N)$ -int-soft right hyperideal of  $S$  over  $U$ .

**Definition 4.3.** A soft set  $f_A$  of an ordered semihypergroup  $S$  over  $U$  is called an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in S) \left( \bigcap_{\alpha \in xoy} f_A(\alpha) \right) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$ .

- (2)  $(\forall x, y, z \in S) (\bigcap_{\alpha \in xoyoz} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N.$
- (3)  $(\forall x, y \in S) x \leq y \implies f_A(x) \cup M \supseteq f_A(y) \cap N.$

**Example 4.4.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	$a$	$b$	$c$	$d$
$a$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a\}$
$c$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$
$d$	$\{a\}$	$\{a, d\}$	$\{a\}$	$\{a\}$

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d)\}.$$

Suppose  $U = \{1, 2, 3\}$ ,  $A = \{a, c, d\}$ ,  $M = \{2\}$  and  $N = \{1, 2\}$ . Let us define  $f_A(a) = \{1, 2, 3\}$ ,  $f_A(b) = \emptyset$ ,  $f_A(c) = \{2, 3\}$  and  $f_A(d) = \{1, 3\}$ . Then  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .

**Remark 4.5.** Every int-soft bi-hyperideal is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . But the converse is not true. We can illustrate it by the following example.

**Example 4.6.** Let  $S = \{1, 2, 3, 4, 5\}$  be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	1	2	3	4	5
1	$\{1\}$	$\{1, 2, 4\}$	$\{1\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
2	$\{1\}$	$\{2\}$	$\{1\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
3	$\{1\}$	$\{1, 2, 4\}$	$\{1, 3\}$	$\{1, 2, 4\}$	$S$
4	$\{1\}$	$\{1, 2, 4\}$	$\{1\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
5	$\{1\}$	$\{1, 2, 4\}$	$\{1, 3\}$	$\{1, 2, 4\}$	$S$

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5), (4, 5)\}.$$

Suppose  $U = \{p_1, p_2, p_3\}$ ,  $A = \{1, 2, 4\}$ ,  $M = \{p_2\}$  and  $N = \{p_2, p_3\}$ . Let us define  $f_A(1) = \{p_1, p_2, p_3\}$ ,  $f_A(2) = \{p_1, p_2\}$ ,  $f_A(3) = \emptyset$ ,  $f_A(4) = \{p_2\}$  and  $f_A(5) = \emptyset$ . Then  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . This is not int-soft bi-hyperideal of  $S$  over  $U$ , as

$$\bigcap_{\alpha \in 1 \circ 1 \circ 2 = \{1, 2, 4\}} f_A(\alpha) = f_A(1) \cap f_A(2) \cap f_A(4) = \{p_2\} \not\supseteq \{p_1, p_2\} = f_A(1) \cap f_A(2).$$

**Theorem 4.7.** If  $\{f_{A_i} \mid i \in I\}$  is a family of  $(M, N)$ -int-soft bi-hyperideal of an ordered semihypergroup  $S$  over  $U$ . Then  $f_A = \bigcap_{i \in I} f_{A_i}$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .

*Proof.* Let  $\{f_{A_i} \mid i \in I\}$  be a family of  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Let  $x, y \in S$ . Then, since each  $f_{A_i}$  ( $i \in I$ ) is an  $(M, N)$ -int-soft bi-hyperideals of  $S$  over  $U$ , so  $\bigcap_{\alpha \in x \circ y} f_{A_i}(\alpha) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(y) \cap N$ . Thus for any  $\alpha \in x \circ y$ ,  $f_{A_i}(\alpha) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(y) \cap N$ . Thus we have  $f_A(\alpha) \cup M = \left( \bigcap_{i \in I} f_{A_i} \right)(\alpha) \cup M = \bigcap_{i \in I} (f_{A_i}(\alpha) \cup M) \supseteq \bigcap_{i \in I} (f_{A_i}(x) \cap f_{A_i}(y) \cap N) = \left( \bigcap_{i \in I} (f_{A_i}(x)) \right) \cap \left( \bigcap_{i \in I} (f_{A_i}(y)) \right) \cap N = \left( \bigcap_{i \in I} f_{A_i} \right)(x) \cap \left( \bigcap_{i \in I} f_{A_i} \right)(y) \cap N = f_A(x) \cap f_A(y) \cap N$ . Thus  $\bigcap_{\alpha \in x \circ y} f_A(\alpha) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$ . Let  $x, y, z \in S$  and  $(\bigcap_{\beta \in x \circ y \circ z} f_{A_i}(\beta)) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(z) \cap N$ . Since each  $f_{A_i}$  ( $i \in I$ ) is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Thus for any  $\beta \in x \circ y \circ z$ ,  $f_{A_i}(\beta) \cup M \supseteq f_{A_i}(x) \cap f_{A_i}(z) \cap N$ . Then  $f_A(\beta) \cup M = \left( \bigcap_{i \in I} f_{A_i} \right)(\beta) \cup M = \left( \bigcap_{i \in I} f_{A_i}(\beta) \right) \cup M \supseteq \bigcap_{i \in I} (f_{A_i}(x) \cap f_{A_i}(z) \cap N) = \left( \bigcap_{i \in I} f_{A_i} \right)(x) \cap \left( \bigcap_{i \in I} f_{A_i} \right)(z) \cap N = f_A(x) \cap f_A(z) \cap N$ . Thus  $(\bigcap_{\beta \in x \circ y \circ z} f_A(\beta)) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$ . Furthermore, if  $x \leq y$ , then  $f_A(x) \cup M \supseteq f_A(y) \cap N$ . Indeed: Since every  $f_{A_i}$  ( $i \in I$ ) is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , it can be obtained that  $f_{A_i}(x) \cup M \supseteq f_{A_i}(y) \cap N$  for all  $i \in I$ . Thus  $f_A(x) \cup M = \left( \bigcap_{i \in I} f_{A_i} \right)(x) \cup M = \left( \bigcap_{i \in I} (f_{A_i}(x)) \right) \cup M \supseteq \left( \bigcap_{i \in I} (f_{A_i}(y)) \right) \cap N = \left( \bigcap_{i \in I} f_{A_i} \right)(y) \cap N = f_A(y) \cap N$ . Thus  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .  $\square$

**Theorem 4.8.** *A non-empty subset  $A$  of an ordered semihypergroup  $(S, \circ, \leq)$  is a bi-hyperideal of  $S$  if and only if the soft set  $f_A$  is defined by*

$$f_A(x) = \begin{cases} N & \text{if } x \in A \\ M & \text{if } x \notin A \end{cases}$$

is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .

*Proof.* Suppose  $A$  is a bi-hyperideal of  $S$ . If there exist  $x, y \in S$  such that  $x \leq y$ . If  $y \in A$ , then  $x \in A$ . Hence  $f_A(x) = N$ . Therefore  $f_A(x) \cup M \supseteq N = f_A(y) \cap N$ . If  $y \notin A$ , then  $f_A(y) \cap N = M$ . Thus  $f_A(x) \cup M \supseteq M = f_A(y) \cap N$ . Let  $x, y \in S$  such that  $x, y \in A$ . Then  $f_A(x) = N$  and  $f_A(y) = N$ . Hence  $(\bigcap_{\alpha \in x \circ y} f_A(\alpha)) \cup M \supseteq N = f_A(x) \cap f_A(y) \cap N$ . If  $x \notin A$  or  $y \notin A$ , then  $f_A(x) \cap f_A(y) \cap N = M$ . Thus  $(\bigcap_{\alpha \in x \circ y} f_A(\alpha)) \cup M \supseteq M = f_A(x) \cap f_A(y) \cap N$ . Now let  $x, y, z \in S$ , such that  $x, z \in A$ . Then  $f_A(x) = N$  and  $f_A(z) = N$ . Hence for any  $\alpha \in x \circ y \circ z$ ,  $(\bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha)) \cup M \supseteq N = f_A(x) \cap f_A(z) \cap N$ . If  $x \notin A$  or  $z \notin A$  then  $f_A(x) \cap f_A(z) \cap N = M$ .



Thus  $(\bigcap_{\alpha \in xoyoz} f_A(\alpha)) \cup M \supseteq M = f_A(x) \cap f_A(z) \cap N$ . Hence  $(\bigcap_{\alpha \in xoyoz} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N$ . Consequently,  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .  $\square$

**Theorem 4.9.** *Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $A$  be a nonempty subset of  $S$ . Then  $A$  is a bi-hyperideal of  $S$  if and only if the characteristic function  $\mathcal{S}_A$  of  $A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .*

*Proof.* Suppose that  $A$  is a bi-hyperideal of  $S$ . Let  $x, y \in S$ . Then we have,  $(\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha)) \cup M \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(y) \cap N$ . Indeed, if  $x \circ y \not\subseteq A$ , then there exists  $\alpha \in x \circ y$  such that  $\alpha \notin A$ , and we have  $\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha) = \emptyset$ . Besides that  $x \circ y \not\subseteq A$  implies that  $x \notin A$  or  $y \notin A$ . Then  $\mathcal{S}_A(x) = \emptyset$  or  $\mathcal{S}_A(y) = \emptyset$  and hence  $(\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha)) \cup M \supseteq \emptyset = \mathcal{S}_A(x) \cap \mathcal{S}_A(y) \cap N$ . Let  $x \circ y \subseteq A$ . Then  $\mathcal{S}_A(\alpha) = U$  for any  $\alpha \in x \circ y$ . It implies that  $\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha) = U$ . Since we have  $\mathcal{S}_A(x) \subseteq U$  for any  $x \in A$ . Thus  $(\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha)) \cup M = U \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(y) \cap N$ . Let  $x, y$  and  $z$  be any elements of  $S$ . Then  $(\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha)) \cup M \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(z) \cap N$ . Indeed, If  $x, z \in A$ , then  $\mathcal{S}_A(x) = U$  and  $\mathcal{S}_A(z) = U$ . Since  $A$  is a bi-hyperideal of  $S$ , we have  $\alpha \in x \circ y \circ z \subseteq A \circ S \circ A \subseteq A$  we have  $\mathcal{S}_A(\alpha) = U$  and  $\emptyset \subseteq M \subset N \subseteq U$ . Thus  $(\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha)) \cup M = U \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(z) \cap N$ . If  $x \notin A$  or  $z \notin A$  then  $\mathcal{S}_A(x) = \emptyset$  or  $\mathcal{S}_A(z) = \emptyset$ . Since  $\mathcal{S}_A(p) \supseteq \emptyset$  for all  $p \in S$ . Thus  $(\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha)) \cup M \supseteq \emptyset = \mathcal{S}_A(x) \cap \mathcal{S}_A(z) \cap N$ . Let  $x, y \in S$  with  $x \leq y$ . Then  $\mathcal{S}_A(x) \cup M \supseteq \mathcal{S}_A(y) \cap N$ . Indeed, if  $y \notin A$  then  $\mathcal{S}_A(y) = \emptyset$  and  $\emptyset \subseteq M \subset N \subseteq U$  so  $\mathcal{S}_A(x) \cup M \supseteq \emptyset = \mathcal{S}_A(y) \cap N$ . If  $y \in A$  then  $\mathcal{S}_A(y) = U$ . Since  $x \leq y$  and  $A$  is a bi-hyperideal of  $S$ , we have  $x \in A$  and thus  $\mathcal{S}_A(x) \cup M = U \supseteq \mathcal{S}_A(y) \cap N$ .

Conversely, let  $\emptyset \neq A \subseteq S$  such that  $\mathcal{S}_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . We claim that  $A \circ A \subseteq A$ . To prove the claim, let  $x, y \in A$ . By hypothesis,  $(\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha)) \cup M \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(y) \cap N = U$  which implies that  $(\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha)) \cup M \supseteq U \cap U \cap N = N$ . Thus by  $\emptyset \subseteq M \subset N \subseteq U$ ,  $\bigcap_{\alpha \in xoy} \mathcal{S}_A(\alpha) \supseteq N \supset \emptyset$ . On the other hand  $\mathcal{S}_A(x) \subseteq U$  for all  $x \in S$ . Thus for any  $\alpha \in x \circ y$ ,  $\mathcal{S}_A(\alpha) = U$  implies that  $\alpha \in A$ . It thus follows that  $A \circ A \subseteq A$ . Let  $\alpha \in A \circ S \circ A$ , then there exist  $x, z \in A$  and  $y \in S$  such that  $\alpha \in x \circ y \circ z$ . Since  $(\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha)) \cup M \supseteq \mathcal{S}_A(x) \cap \mathcal{S}_A(z) \cap N$ , and  $x, z \in A$  we have  $\mathcal{S}_A(x) = U$  and  $\mathcal{S}_A(z) = U$ . Hence for each  $\alpha \in A \circ S \circ A$ , we have  $(\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha)) \cup M \supseteq U \cap U \cap N = N$ . Thus by  $\emptyset \subseteq M \subset N \subseteq U$ ,  $\bigcap_{\alpha \in xoyoz} \mathcal{S}_A(\alpha) \supseteq N \supset \emptyset$ . On the other hand  $\mathcal{S}_A(x) \subseteq U$  for all  $x \in S$ . Thus

for any  $\alpha \in x \circ y \circ z$ ,  $\mathcal{S}_A(\alpha) = U$  implies that  $\alpha \in A$ . Thus  $A \circ S \circ A \subseteq A$ . Furthermore, let  $x \in A$ ,  $S \ni y \leq x$ . Then  $y \in A$ . Indeed, it is enough to prove that  $\mathcal{S}_A(y) = U$ . By  $x \in A$  we have  $\mathcal{S}_A(x) = U$ . Since  $\mathcal{S}_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$  and  $y \leq x$ , we have  $\mathcal{S}_A(y) \cup M \supseteq \mathcal{S}_A(x) \cap N = U \cap N = N$ . Notice that  $\emptyset \subseteq M \subset N \subseteq U$ , we conclude that  $\mathcal{S}_A(y) \supseteq \emptyset$ . Thus  $\mathcal{S}_A(y) = U$ . Therefore  $A$  is a bi-hyperideal of  $S$ .  $\square$

**Theorem 4.10.** *Let  $f_A$  be a soft set of an ordered semihypergroup  $S$  over  $U$  and  $\delta \in P(U)$ . Then  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$  if and only if each nonempty  $\delta$ -inclusive set  $i_A(f_A, \delta)$  of  $f_A$  is a bi-hyperideal of  $S$ .*

*Proof.* Assume that  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , and  $i_A(f_A, \delta) \neq \emptyset$ . Let  $x, y \in i_A(f_A, \delta)$ . Then  $f_A(x) \supseteq \delta$  and  $f_A(y) \supseteq \delta$ . By hypothesis, we have  $(\bigcap_{\alpha \in x \circ y} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(y) \cap N \supseteq \delta \cap \delta \cap N = \delta$ . Since  $M \subset \delta$  we can write as  $\bigcap_{\alpha \in x \circ y} f_A(\alpha) \supseteq \delta$ . Thus for any  $\alpha \in x \circ y$ , we have  $f_A(\alpha) \supseteq \delta$ , implies that  $\alpha \in i_A(f_A, \delta)$ . It follows that  $x \circ y \subseteq i_A(f_A, \delta)$ . Hence  $i_A(f_A, \delta)$  is a subsemihypergroup of  $S$ . Let  $x, y, z \in S$  and  $x, z \in i_A(f_A, \delta)$  where  $M \subset \delta \subseteq N$ . Then  $f_A(x) \supseteq \delta$  and  $f_A(z) \supseteq \delta$ . Since  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Thus  $(\bigcap_{w \in x \circ y \circ z} f_A(w)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N \supseteq \delta \cap \delta \cap N = \delta$ . Since  $\emptyset \subseteq M \subset \delta \subseteq N \subseteq U$ , we can write as  $\bigcap_{w \in x \circ y \circ z} f_A(w) \supseteq \delta$ . Hence  $f_A(w) \supseteq \delta$  for any  $w \in x \circ y \circ z$  implies that  $w \in i_A(f_A, \delta)$ . Thus  $i_A(f_A, \delta) \circ S \circ i_A(f_A, \delta) \subseteq i_A(f_A, \delta)$ . Furthermore, let  $x \in i_A(f_A, \delta)$ ,  $S \ni y \leq x$ . Then  $y \in i_A(f_A, \delta)$ . Indeed, since  $x \in i_A(f_A, \delta)$ ,  $f_A(x) \supseteq \delta$  and  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , we have  $f_A(y) \cup M \supseteq f_A(x) \cap N \supseteq \delta \cap N = \delta$ . By  $M \subset \delta$ , we have  $f_A(y) \supseteq \delta$ , i.e.,  $y \in i_A(f_A, \delta)$ . Therefore  $i_A(f_A, \delta)$  is a bi-hyperideal of  $S$ .

Conversely, suppose that  $i_A(f_A, \delta) \neq \emptyset$  is a bi-hyperideal of  $S$  for all  $M \subset \delta \subseteq N$ . If there exist  $x_0, y_0 \in S$  such that  $(\bigcap_{\alpha \in x_0 \circ y_0} f_A(\alpha)) \cup M \subset f_A(x_0) \cap f_A(y_0) \cap N$ . Then there exists  $M \subset \delta \subseteq N$  such that  $(\bigcap_{\alpha \in x_0 \circ y_0} f_A(\alpha)) \cup M \subset \delta \subseteq f_A(x_0) \cap f_A(y_0) \cap N$ , and we have  $f_A(x_0) \supseteq \delta$ ,  $f_A(y_0) \supseteq \delta$  and  $\bigcap_{\alpha \in x_0 \circ y_0} f_A(\alpha) \subset \delta$  which implies that  $x_0, y_0 \in i_A(f_A, \delta)$  and  $x_0 \circ y_0 \not\subseteq i_A(f_A, \delta)$ . It contradicts the fact that  $i_A(f_A, \delta)$  is a bi-hyperideal of  $S$ . Consequently,  $(\bigcap_{\alpha \in x \circ y} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(y) \cap N$  for all  $x, y \in S$ . Now let  $x, y, z \in S$ . We will prove that  $(\bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha)) \cup M \supseteq f_A(x) \cap f_A(z) \cap N$  for all  $x, y, z \in S$ . If there exist  $x_0, y_0, z_0$  such that  $(\bigcap_{\alpha \in x_0 \circ y_0 \circ z_0} f_A(\alpha)) \cup M \subset f_A(x_0) \cap f_A(z_0) \cap N$ , and  $M \subset \delta \subseteq N$  such that  $(\bigcap_{\alpha \in x_0 \circ y_0 \circ z_0} f_A(\alpha)) \cup M \subset \delta \subseteq f_A(x_0) \cap f_A(z_0) \cap N$ , so  $f_A(x_0) \supseteq \delta$ ,  $f_A(z_0) \supseteq \delta$  and  $\bigcap_{\alpha \in x_0 \circ y_0 \circ z_0} f_A(\alpha) \subset \delta$  then

$x_0, z_0 \in i_A(f_A, \delta)$  and  $x_0 \circ y_0 \circ z_0 \notin i_A(f_A, \delta)$ . This is a contradiction that  $i_A(f_A, \delta)$  is a bi-hyperideal of  $S$ . Moreover if  $x \leq y$  then  $f_A(x) \cup M \supseteq f_A(y) \cap N$ . Indeed, if there exist  $x_0, y_0 \in S$  such that  $x_0 \leq y_0$  and  $f_A(x_0) \cup M \subset f_A(y_0) \cap N$ ,  $M \subset \delta \subseteq N$  such that  $f_A(x_0) \cup M \subset \delta \subseteq f_A(y_0) \cap N$  and we have  $f_A(y_0) \supseteq \delta$  and  $f_A(x_0) \subset \delta$ . Then  $y_0 \in i_A(f_A, \delta)$  and  $x_0 \notin i_A(f_A, \delta)$ . This is a contradiction that  $i_A(f_A, \delta)$  is a bi-hyperideal of  $S$ . Thus if  $x \leq y$  then  $f_A(x) \cup M \supseteq f_A(y) \cap N$ .  $\square$

**Theorem 4.11.** *Every  $(M, N)$ -int-soft left (resp. right) hyperideal of  $S$  over  $U$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .*

*Proof.* Let  $f_A$  is an  $(M, N)$ -int-soft left hyperideal of  $S$  over  $U$ . Let  $x, y \in S$ . Then  $(\bigcap_{\alpha \in xoy} f_A(\alpha)) \cup M \supseteq f_A(y) \cap N \supseteq f_A(x) \cap f_A(y) \cap N$ . Let  $x, y, z \in S$ . Then  $(\bigcap_{\alpha \in xoyz} f_A(\alpha)) \cup M = (\bigcap_{\substack{\alpha \in \gamma oz \\ \gamma \in xoy}} f_A(\alpha)) \cup M \supseteq f_A(z) \cap N \supseteq f_A(x) \cap f_A(z) \cap N$ . Thus  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .  $\square$

**Definition 4.12.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup. Let  $f_A$  be a soft set of  $S$  over  $U$ . We define the the soft set  $f_A^*$  of  $S$  over  $U$  as follows:

$$f_A^*(x) = \{\forall x \in S \mid (f_A(x) \cap N) \cup M\}.$$

for all  $x \in S$ .

**Definition 4.13.** If  $\mathcal{S}_A$  is the characteristic soft function of  $A$ . Then  $\mathcal{S}_A^*$  is defined over  $U$  in which  $\mathcal{S}_A^*$  is given by

$$\mathcal{S}_A^*(x) = \begin{cases} N & \text{if } x \in A \\ M & \text{if } x \notin A \end{cases}$$

**Theorem 4.14.** *The characteristic function  $\mathcal{S}_A^*$  of  $A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , if and only if  $A$  is a bi-hyperideal of  $S$ .*

*Proof.* Suppose that  $A$  is a bi-hyperideal of  $S$ . Then by Theorem 4.9,  $\mathcal{S}_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .

Conversely, assume that  $\mathcal{S}_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Let  $x, y \in S$  be such that  $x, y \in A$ . It implies that  $\mathcal{S}_A^*(x) = N$  and  $\mathcal{S}_A^*(y) = N$ . Since  $\mathcal{S}_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Therefore

$$\begin{aligned} \bigcap_{\alpha \in xoy} \mathcal{S}_A^*(\alpha) \cup M &\supseteq \mathcal{S}_A^*(x) \cap \mathcal{S}_A^*(y) \cap N \\ &= N \cap N \cap N = N. \end{aligned}$$

Since  $M \subseteq N$ . So  $\bigcap_{\alpha \in x \circ y} \mathcal{S}_A^*(\alpha) = N$ . Thus  $\mathcal{S}_A^*(\alpha) = N$ . It shows that  $x \circ y \subseteq A$ . Let  $x, y \in S$  such that  $x \leq y \in A$ . It implies that  $\mathcal{S}_A^*(y) = N$ . Since  $\mathcal{S}_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Therefore  $\mathcal{S}_A^*(x) \cup M \supseteq \mathcal{S}_A^*(y) \cap N = N \cap N = N$ . Since  $M \subset N$ . Hence  $\mathcal{S}_A^*(x) = N$ . Implies that  $x \in A$ . Now if there exist  $x, y, z \in S$  such that  $x, z \in A$ . Then  $\mathcal{S}_A^*(x) = N$  and  $\mathcal{S}_A^*(z) = N$ . Since  $\mathcal{S}_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . We have

$$\begin{aligned} \bigcap_{\alpha \in x \circ y \circ z} \mathcal{S}_A^*(\alpha) \cup M &\supseteq \mathcal{S}_A^*(x) \cap \mathcal{S}_A^*(z) \cap N \\ &= N \cap N \cap N \\ &= N. \end{aligned}$$

Since  $M \subset N$ . Hence  $\mathcal{S}_A^*(\alpha) = N$ . Thus  $\alpha \in x \circ y \circ z \subseteq A$ . Consequently,  $A$  is a bi-hyperideal of  $S$ .  $\square$

**Proposition 4.15.** *If  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Then  $f_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .*

*Proof.* Assume that  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Since  $f_A$  is an  $(M, N)$ -int-soft subsemihypergroup of  $S$  over  $U$ . Let  $x, y \in S$ . Then

$$\begin{aligned} \bigcap_{\alpha \in x \circ y} f_A^*(\alpha) \cup M &= \left[ \bigcap_{\alpha \in x \circ y} \{(f_A(\alpha) \cap N) \cup M\} \right] \cup M \\ &= \left[ \bigcap_{\alpha \in x \circ y} (f_A(\alpha) \cup M) \cap (N \cup M) \right] \cup M \\ &= \left[ \bigcap_{\alpha \in x \circ y} (f_A(\alpha) \cup M) \cap N \right] \cup M \\ &\supseteq \{(f_A(x) \cap f_A(y) \cap N) \cap N\} \cup M \\ &= \{(f_A(x) \cap N) \cap (f_A(y) \cap N) \cap N\} \cup M \\ &= \{(f_A(x) \cap N) \cup M\} \cap \{(f_A(y) \cap N) \cup M\} \cap (N \cup M) \\ &= f_A^*(x) \cap f_A^*(y) \cap N. \end{aligned}$$

Hence  $f_A^*$  is an  $(M, N)$ -int-soft subsemihypergroup of  $S$  over  $U$ . Now let  $x, y, z \in S$ , then

$$\begin{aligned}
 \bigcap_{\alpha \in x \circ y \circ z} f_A^*(\alpha) \cup M &= \left\{ \left( \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \cap N \right) \cup M \right\} \cup M \\
 &= \left( \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \cap N \right) \cup M \\
 &= \left( \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \cup M \right) \cap (N \cup M) \\
 &= \left( \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \cup M \right) \cap N \\
 &= \left\{ \left( \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \cup M \right) \cup M \right\} \cap N \\
 &\supseteq \{ (f_A(x) \cap f_A(z) \cap N) \cup M \} \cap N \\
 &= \{ (f_A(x) \cap f_A(z) \cap N \cap N) \cup M \cup M \} \cap N \\
 &= \{ (f_A(x) \cap N) \cup M \} \cap \{ (f_A(z) \cap N) \cup M \} \} \cap N \\
 &= [f_A^*(x) \cap f_A^*(z)] \cap N \\
 &= f_A^*(x) \cap f_A^*(z) \cap N.
 \end{aligned}$$

Let  $x, y \in S$  such that  $x \leq y$ . Then  $f_A^*(x) \cup M \supseteq f_A^*(y) \cap N$ . Indeed. Thus

$$\begin{aligned}
 f_A^*(x) \cup M &= \{ (f_A(x) \cap N) \cup M \} \cup M \\
 &= \{ (f_A(x) \cap N) \cup M \} \\
 &= \{ (f_A(x) \cup M) \cap (N \cup M) \} \\
 &= \{ (f_A(x) \cup M) \cap N \} \\
 &= \{ (f_A(x) \cup M) \cup M \} \cap N \\
 &\supseteq \{ (f_A(y) \cap N) \cup M \} \cap N \\
 &= f_A^*(y) \cap N.
 \end{aligned}$$

Hence  $f_A^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .  $\square$

**Corollary 4.16.** *If  $\{f_{A_i} \mid i \in I\}$  is a family of  $(M, N)$ -int-soft bi-hyperideal of an ordered semihypergroup  $S$  over  $U$ . Then  $f_A^* = \bigcap_{i \in I} f_{A_i}^*$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ .*

**Theorem 4.17.** *An ordered semihypergroup  $S$  is regular, left and right simple if and only if for every  $(M, N)$ -int-soft bi-hyperideal  $f_A$  of  $S$  over  $U$ ,  $f_A^*$  of  $S$  over  $U$  is constant function .*

*Proof.* Assume that  $S$  is regular, left and right simple and  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . We consider

$$\Omega = \{B \subseteq S \mid B \leq B^2\} \text{ and } E_S = \bigcup_{B \in \Omega} B.$$

Let  $a \in S$ . Since  $S$  is regular, there exists  $x \in S$  such that  $a \leq a \circ x \circ a$  and  $a \circ x \leq a \circ x \circ a \circ x = (a \circ x)^2$ . Hence  $a \circ x \in \Omega$ . This means  $E_S \neq \emptyset$ . First we show that  $f_A^*$  is constant in  $E_S$ . Let  $e, a \in E_S$ . By Lemma 2.2, we obtain that  $(S \circ a] = S = (a \circ S]$ . Since  $e \in \Omega$ , we have  $e \leq B \leq B^2$  for some  $B \in \Omega$ . Then  $e \leq \beta \in p \circ q$  for some  $p, q \in B$ . By assumption we consider  $p \in (a \circ S]$  and  $q \in (S \circ a]$ . Then  $p \leq x$  and  $q \leq y$  for some  $x \in a \circ S$  and  $y \in S \circ a$ . Thus  $e \leq \beta \leq p \circ q \leq x \circ y \leq (a \circ s_1) \circ (s_2 \circ a) = a \circ (s_1 \circ s_2) \circ a$  for some  $s_1, s_2 \in S$ . That is  $e \leq a \circ s \circ a$  for some  $s \in s_1 \circ s_2$ . So  $e \leq \alpha$  for some  $\alpha \in a \circ s \circ a$ . Since  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , we have

$$\begin{aligned} \bigcap_{\alpha \in a \circ s \circ a} f_A^*(\alpha) &= \bigcap_{\alpha \in a \circ s \circ a} (f_A(\alpha) \cap N) \cup M \\ &= \bigcap_{\alpha \in a \circ s \circ a} (f_A(\alpha) \cup M) \cap (N \cup M) \\ &= \bigcap_{\alpha \in a \circ s \circ a} (f_A(\alpha) \cup M) \cap N \\ &= \left( \bigcap_{\alpha \in a \circ s \circ a} (f_A(\alpha) \cup M) \cup M \right) \cap N \\ &\supseteq \{(f_A(a) \cap f_A(a) \cap N) \cup M\} \cap N \\ &= \{(f_A(a) \cap f_A(a) \cap N) \cap N\} \cup (N \cap M) \\ &= (f_A(a) \cap N) \cup M \\ &= f_A^*(a). \end{aligned}$$

Hence  $f_A^*(e) \supseteq f_A^*(a)$ . Similarly we can prove that  $f_A^*(a) \supseteq f_A^*(e)$ . Therefore  $f_A^*(a) = f_A^*(e)$ , i.e.,  $f_A^*$  is constant in  $E_S$ . Next we show that  $f_A^*$  is constant in  $S$ . Let  $a \in S$  and  $e \in E_S$ . Since  $S$  is regular, there exists  $x \in S$  such that  $a \leq a \circ x \circ a$  and so,  $a \circ x \leq a \circ x \circ a \circ x = (a \circ x) \circ (a \circ x) = (a \circ x)^2$  and  $x \circ a \leq x \circ a \circ x \circ a = (x \circ a) \circ (x \circ a) = (x \circ a)^2$ . Thus  $a \circ x$  and  $x \circ a \in E_S$ . Then  $f_A^*(b) = f_A^*(e) = f_A^*(c)$  for all  $b \in a \circ x$ ,  $c \in x \circ a$  by the previous arguments. Since  $a \leq a \circ x \circ a \leq (a \circ x \circ a) \circ x \circ a = a \circ x \circ a \circ x \circ a$ . So there exist

$u \in a \circ x \circ a \circ x \circ a$  such that  $a \leq u$ . We get

$$\begin{aligned}
 f_A^*(a) &= (f_A(a) \cap N) \cup M \\
 &= (f_A(a) \cup M) \cap (N \cup M) \\
 &= (f_A(a) \cup M) \cap N \\
 &= \{(f_A(a) \cup M) \cup M\} \cap N \\
 &\supseteq \{(f_A(u) \cap N) \cup M\} \cap N \\
 &= \{(f_A(u) \cup M) \cap (N \cup M)\} \cap N \\
 &= \{(f_A(u) \cup M) \cap N\} \cap N \\
 &= (f_A(u) \cup M) \cap N \\
 &\supseteq \bigcap_{u \in a \circ x \circ a \circ x \circ a} \{(f_A(u) \cup M) \cap N\} \\
 &= \bigcap_{u \in a \circ x \circ a \circ x \circ a} \{(f_A(u) \cup M) \cup M\} \cap N \\
 &= \bigcap_{\substack{u \in b \circ a \circ c \\ b \in a \circ x, c \in x \circ a}} \{(f_A(u) \cup M) \cup M\} \cap N \\
 &\supseteq \{(f_A(b) \cap f_A(c) \cap N) \cup M\} \cap N \\
 &= [\{f_A(b) \cap f_A(c) \cap N\} \cap N] \cup (M \cap N) \\
 &= \{f_A(b) \cap f_A(c) \cap N\} \cup M \\
 &= [\{f_A(b) \cap N\} \cup M] \cap [\{f_A(c) \cap N\} \cup M] \\
 &= f_A^*(b) \cap f_A^*(c) = f_A^*(b).
 \end{aligned}$$

Thus  $f_A^*(a) \supseteq f_A^*(b)$ . Similarly we can show that  $f_A^*(b) \supseteq f_A^*(a)$ . Therefore  $f_A^*(a) = f_A^*(b) = f_A^*(e)$ . Thus  $f_A^*$  is constant in  $S$ .

Conversely, let  $f_A^*$  be an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$  and  $f_A^*$  is a constant. Let  $a \in S$ . We have  $(S \circ a]$  is a left hyperideal and  $(a \circ S]$  is a right hyperideal of  $S$ . It is easy to prove that both are bi-hyperideals. By Theorem 4.14, we have the characteristic functions  $\mathcal{S}_{(S \circ a]}^*$  and  $\mathcal{S}_{(a \circ S]}^*$  are  $(M, N)$ -int-soft bi-hyperideals of  $S$  over  $U$ . By assumption  $\mathcal{S}_{(S \circ a]}^*$  and  $\mathcal{S}_{(a \circ S]}^*$  are constant. Thus,  $(S \circ a] = S = (a \circ S]$ . By Lemma 2.2,  $S$  is a left and right simple. Since  $a \in S$  and  $(a \circ S] = S = (S \circ a]$ , we have  $a \in (a \circ S] = (a \circ (S \circ a]) \subseteq (a \circ S \circ a]$ . Thus  $S$  is regular.  $\square$

**Theorem 4.18.** *If  $f_A$  is a soft set of  $S$  over  $U$ . Then  $S$  is completely regular if and only if for every  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ , we have*

$$f_A^*(a) = \bigcap_{\alpha \in a^2} f_A^*(\alpha) \text{ for all } a \in S.$$

*Proof.* Let us suppose that  $S$  is completely regular and  $a \in S$ , then by Lemma 2.5,  $a \in (a^2 \circ S \circ a^2]$ . It implies that there exist  $x \in S$  such that  $a \leq a^2 \circ x \circ a^2$ . Then there exist  $\alpha \in a^2$  and  $v \in \alpha \circ x \circ \alpha$  such that  $a \leq v$ . Since  $f_A$  is an  $(M, N)$ -int-soft bi-hyperideal of  $S$  over  $U$ . Therefore

$$\begin{aligned} f_A^*(a) &= \{(f_A(a) \cap N) \cup M\} \\ &= \{(f_A(a) \cup M) \cap (N \cup M)\} \\ &= \{(f_A(a) \cup M) \cap N\} \\ &= \{(f_A(a) \cup M) \cup M\} \cap N \\ &\supseteq \{(f_A(v) \cap N) \cup M\} \cap N \\ &= \{(f_A(v) \cup M) \cap (N \cup M)\} \cap N \\ &= \{(f_A(v) \cup M) \cap N\} \cap N \\ &= \{(f_A(v) \cup M) \cap N\} \\ &\supseteq \bigcap_{v \in \alpha \circ x \circ \alpha} \{(f_A(v) \cup M) \cap N\} \\ &= \bigcap_{v \in \alpha \circ x \circ \alpha} \{(f_A(v) \cup M) \cup M\} \cap N \\ &\supseteq \{(f_A(\alpha) \cap f_A(\alpha) \cap N) \cup M\} \cap N \\ &= \{(f_A(\alpha) \cap N) \cup M\} \cap N \\ &= \{(f_A(\alpha) \cap N) \cap N\} \cup (M \cap N) \\ &= (f_A(\alpha) \cap N) \cup M \\ &= f_A^*(\alpha). \end{aligned}$$

Hence  $f_A^*(a) \supseteq \bigcap_{\alpha \in a^2} f_A^*(\alpha)$ . Similarly we can show that  $\bigcap_{\alpha \in a^2} f_A^*(\alpha) \supseteq f_A^*(a)$ . Therefore  $\bigcap_{\alpha \in a^2} f_A^*(\alpha) = f_A^*(a)$ .

Conversely, let  $a \in S$ . We consider the bi-hyperideal  $B(a^2) = (a^2 \cup a^4 \cup a^2 \circ S \circ a^2]$  of  $S$  generated by  $a^2$ . By Theorem 4.14,  $\mathcal{S}_{B(a^2)}^*$  is an  $(M, N)$ -int-soft bi-hyperideals of  $S$  over  $U$ . Since  $a^2 \subseteq A$ . By assumption we have  $\mathcal{S}_{B(a)}^* = \bigcap_{\alpha \in a^2} \mathcal{S}_{B(a^2)}^* = N$ . Then  $a \in B(a^2)$  and so,  $a \leq y$ ,



for some  $y \in (a^2 \cup a^4 \cup a^2 \circ S \circ a^2)$ . If  $y \in a^2$ , then  $a \leq y \leq a^2 \leq a^3 \leq a^4 \leq a^5 = a^2 \circ a \circ a^2 \subseteq a^2 \circ S \circ a^2$ . Hence  $a \in (a^2 \circ S \circ a^2)$ . If  $y \in a^4$ , then  $a \leq y \leq a^4 \leq a^7 \leq a^2 \circ a^3 \circ a^2$ . Hence  $a \in (a^2 \circ S \circ a^2)$ . If  $y \in a^2 \circ S \circ a^2$ , then  $a \leq y \leq a^2 \circ S \circ a^2$ . Hence  $a \in (a^2 \circ S \circ a^2)$ . By Lemma 2.5, is completely regular.  $\square$

## 5. CONCLUSION

In this paper, we have introduced the notion of  $(M, N)$ -int-soft bi-hyperideals of ordered semihypergroups and provided several examples. We have considered the characterization of  $(M, N)$ -int-soft bi-hyperideals of ordered semihypergroups. We have shown the following.

(1) We proved that every int-soft bi-hyperideal is an  $(M, N)$ -int-soft bi-hyperideals of  $S$  over  $U$  but the converse is not true.

(2) We have provided an example which shows that every  $(M, N)$ -int-soft bi-hyperideal is not int-soft bi-hyperideals of  $S$  over  $U$ .

(3) When  $M = \emptyset$  and  $N = U$ , we meet int-soft bi-hyperideals. From this analysis, we say that  $(M, N)$ -int-soft bi-hyperideals are more general concept than ordinary int-soft ones.

(4) We characterized left  $(M, N)$  simple and completely regular ordered semihypergroups by means of  $(M, N)$ -int-soft bi-hyperideals.

Future research will focus on applying the idea/result in this paper to the ideal theory of hyperrings.

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