



Research Paper

**ON PSEUDO-CONTRACTIBILITY OF CERTAIN ALGEBRAS RELATED
TO A DISCRETE SEMIGROUP**

AMIR SAHAMI*, MEHDI ROSTAMI AND SHAHAB KALANTARI

ABSTRACT. In this paper, we introduce a notion of ultra central approximate identity for Banach algebras which is a generalization of the bounded approximate identity and the central approximate identity. Using this concept we study pseudo-contractibility of some matrix algebras among ℓ^1 -Munn algebras. As an application, for the Brandt semigroup $S = M^0(G, I)$ over a non-empty set I , we show that $\ell^1(S)$ has an ultra central approximate identity if and only if I is finite. Also we show that the notion of pseudo-contractibility and contractibility are the same on $\ell^1(S)^{**}$, where S is the Brandt semigroup.

1. INTRODUCTION AND PRELIMINARIES

Johnson introduced and studied the concept of amenability for Banach algebras. A Banach algebra A is called amenable (contractible) if there exists a bounded net (an element) m_α (m)

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*Corresponding author

in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ ($a \cdot m = m \cdot a$) and $\pi_A(m_\alpha)a \rightarrow a$ ($\pi_A(m_\alpha)a = a$) for all $a \in A$, respectively, where $A \otimes_p A$ is denoted for projective tensor product A with A and also $\pi_A : A \otimes_p A \rightarrow A$ is given for the product morphism, defined by $\pi_A(a \otimes b) = ab$. It is well-known that every amenable Banach algebra A has a bounded approximate identity, that is a bounded net (e_α) in A such that $ae_\alpha \rightarrow a$ for all $a \in A$. So for study the amenability, it is important to study the existence of a bounded approximate identity. For the history of amenability see [15].

By easing the boundedness condition of amenability, Ghahramani and Zhang gave two concepts of amenability, say pseudo-amenability and pseudo-contractibility. A Banach algebra A is called pseudo-amenable(pseudo-contractible) if there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ ($a \cdot m_\alpha = m_\alpha \cdot a$) and $\pi_A(m_\alpha)a \rightarrow a$ for all $a \in A$, respectively. It is known that every pseudo-amenable(pseudo-contractible) Banach algebra A has an approximate identity(central approximate identity), that is a net (e_α) in A which $ae_\alpha \rightarrow a$ ($ae_\alpha = e_\alpha a \rightarrow a$) for all $a \in A$, respectively.

Motivated by these considerations, if A^{**} is pseudo-contractible, then A^{**} has a central approximate identity. Then there exists a central approximate identity (e_α) in A^{**} which

$$ae_\alpha = e_\alpha a \rightarrow a, \quad (a \in A^{**}).$$

By relaxing the condition $a \in A^{**}$ and since A can be embedded in A^{**} , we have the following definition:

Definition 1.1. Let A be a Banach algebra. We say that A has an ultra central approximate identity if there exists a net (e_α) in A^{**} such that $ae_\alpha = e_\alpha a$ and $e_\alpha a \rightarrow a$, for every $a \in A$.

Remark 1.2. Note that if we assume that the net (e_α) is bounded, then the assumption of boundedness of (e_α) in the definition 1.1 is in fact equivalent to the existence of a bounded approximate identity. For this, let $(e_\alpha) \subseteq A^{**}$ be bounded in definition 1.1 and $E = \text{wk}^* - \lim_\alpha e_\alpha$. It is known that for $a \in A$ the maps $A^{**} \rightarrow A^{**} : x \mapsto ax$ and $A^{**} \rightarrow A^{**} : x \mapsto xa$ are weak*-continuous. Using this we obtain that $aE = Ea = a$, for all $a \in A$. By Goldstine's Theorem there exists a net (ξ_α) in A such that $\|\xi_\alpha\| \leq \|E\|$ and $\widehat{\xi}_\alpha \xrightarrow{\text{wk}^*} E$. Thus $a\widehat{\xi}_\alpha - \widehat{\xi}_\alpha a \xrightarrow{\text{wk}^*} 0$ and $a\widehat{\xi}_\alpha \xrightarrow{\text{wk}^*} a$ for all $a \in A$. So $a\xi_\alpha - \xi_\alpha a \xrightarrow{\text{wk}} 0$ and $a\xi_\alpha \xrightarrow{\text{wk}} a$, for all $a \in A$. Applying Mazur's Theorem, we may assume that $a\xi_\alpha - \xi_\alpha a \rightarrow 0$ and $a\xi_\alpha \rightarrow a$, for all $a \in A$. Thus the net $(\xi_\alpha) \subseteq A$ is a bounded approximate identity for A .

We show that a Banach algebra with central approximate identity or bounded approximate identity has an ultra central approximate identity. In the theory of Banach algebras, the existence of a bounded (central) approximate identity may characterize the structure of Banach algebra, respectively. For instance, the Fourier algebra $A(G)$ has a bounded approximate

identity if and only if G is amenable (Leptin’s theorem). Also the Segal algebra $S(G)$ has a central approximate identity if and only if G is a SIN group, see [12]. Ramsden in [14, Proposition 2.9] showed that if a semigroup algebra $\ell^1(S)$ has bounded approximate identity, then the set of idempotent elements of S , say $E(S)$, is finite, provided that S is uniformly locally finite semigroup. In fact, he relate the topological notion of bounded approximate identity to the algebraic notion of idempotent set.

Motivated by these considerations, the following question raised:

” Whenever the semigroup algebra $\ell^1(S)$ has an ultra central approximate identity?”

Since the structure of the uniformly locally finite inverse semigroup algebra is related to some group algebras, we answer this question for the semigroup algebras associated to a uniformly locally finite inverse semigroups. In fact we show that $M_\Lambda(\mathbb{C})$ (the Banach algebra of $\Lambda \times \Lambda$ -matrices over \mathbb{C} , with finite ℓ^1 -norm and matrix multiplication which belongs to the class of ℓ^1 -Munn algebras) has an ultra central approximate identity if and only if Λ is finite (for further information on ℓ^1 -Munn algebras or namely Esslamzadeh-Munn algebras see [3], [4], [5], [6], [7] and [13]). Using this result, we characterize pseudo-contractibility of $M_\Lambda(\mathbb{C})^{**}$. We study the existence of an ultra central approximate identity for the semigroup algebra $\ell^1(S)$, provided that S is an uniformly locally finite semigroup. As an application, we show that $\ell^1(S)$ has an ultra central approximate identity if and only if I is finite, where $S = M^0(G, I)$ is the Brandt semigroup over a non-empty set I . As the main result of this paper we show that $\ell^1(S)^{**}$ is pseudo-contractible if and only if $\ell^1(S)^{**}$ is contractible (or super amenable), where $S = M^0(G, I)$ is the Brandt semigroup over a non-empty set I .

First we give some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Then A^{**} denotes the second dual of A . It is well-known that we can see A as a subset of A^{**} . If X is a Banach A -bimodule, then X^* is also a Banach A -bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

Let A and B be Banach algebras. The projective tensor product $A \otimes_p B$ with the following multiplication is a Banach algebra

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (a_1, a_2 \in A, b_1, b_2 \in B).$$

The product morphism $\pi_A : A \otimes_p A \rightarrow A$ is specified by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$.

Let A be a Banach algebra and let Λ be a non-empty set. We denote $\varepsilon_{i,j}$ for a matrix belongs to $M_\Lambda(A)$ which (i, j) -entry is 1 and 0 elsewhere. The map $\theta : M_\Lambda(A) \rightarrow A \otimes_p M_\Lambda(\mathbb{C})$ defined by $\theta((a_{i,j})) = \sum_{i,j} a_{i,j} \otimes \varepsilon_{i,j}$ is an isometric algebra isomorphism.

We present some notions of semigroup theory, for the further background see [11]. Let S be a semigroup and let $E(S)$ be the set of its idempotents. There exists a partial order on $E(S)$

which is defined by

$$s \leq t \iff s = st = ts \quad (s, t \in E(S)).$$

A semigroup S is called inverse semigroup, if for every $s \in S$ there exists $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s^*$. If S is an inverse semigroup, then there exists a partial order on S which coincides with the partial order on $E(S)$. Indeed

$$s \leq t \iff s = ss^*t \quad (s, t \in S).$$

For every $x \in S$, we denote $(x) = \{y \in S \mid y \leq x\}$. S is called locally finite (uniformly locally finite) if for each $x \in S$, $|(x)| < \infty$ ($\sup\{|(x)| : x \in S\} < \infty$), respectively.

Suppose that S is an inverse semigroup. Then the maximal subgroup of S at $p \in E(S)$ is denoted by $G_p = \{s \in S \mid ss^* = s^*s = p\}$.

Let S be an inverse semigroup. There exists an equivalence relation \mathfrak{D} on S such that $s\mathfrak{D}t$ if and only if there exists $x \in S$ such that $ss^* = xx^*$ and $t^*t = x^*x$. We denote $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$ for the collection of \mathfrak{D} -classes and $E(\mathfrak{D}_\lambda) = E(S) \cap \mathfrak{D}_\lambda$.

2. BASIC PROPERTIES

In this section we focus on studying some properties of a class of ℓ^1 -Munn algebras, namely $\mathbb{M}_\Lambda(\mathbb{C})$.

Lemma 2.1. *Let Λ be any non-empty set. Then $\mathbb{M}_\Lambda(\mathbb{C})$ has an ultra central approximate identity if and only if Λ is finite.*

Proof. Suppose that $A = \mathbb{M}_\Lambda(\mathbb{C})$ has an ultra central approximate identity. It follows that there exists a net (e_α) in A^{**} such that $a \cdot e_\alpha = e_\alpha \cdot a$ and $e_\alpha a \rightarrow a$ for each $a \in A$. Fix $i_0 \in \Lambda$. Using Mazur's lemma we can find a net $(x_\alpha^\beta)_\beta$ in A such that

$$w - \lim_\beta ax_\alpha^\beta - x_\alpha^\beta a = 0, \quad w^* - \lim_\beta x_\alpha^\beta = e_\alpha, \quad \|x_\alpha^\beta\| \leq \|e_\alpha\|, \quad (a \in A).$$

Suppose that $x_\alpha^\beta = [x_{\alpha,\beta}^{i,j}]_{i,j \in \Lambda}$, where $x_{\alpha,\beta}^{i,j}$ belongs to \mathbb{C} . Set $a = \varepsilon_{i_0,j}$, where $j \in \Lambda$. Put this a at the above fact, we have

$$w - \lim_\beta x_{\alpha,\beta}^{j,j} - x_{\alpha,\beta}^{i_0,i_0} = 0.$$

Since the net (x_α^β) is bounded in A , $(x_{\alpha,\beta}^{i_0,i_0})_\beta$ is a bounded net in \mathbb{C} . Using Heine-Borel theorem we may assume that $(x_{\alpha,\beta}^{i_0,i_0})_\beta$ converges to x . We assume in contradiction that $x = 0$. Hence $|\cdot| - \lim_\beta x_{\alpha,\beta}^{i_0,i_0} = 0$. It follows that $|\cdot| - \lim_\beta x_{\alpha,\beta}^{j,j} = 0$. Define $T_{i_0} : A \rightarrow \mathbb{C}$ by $T_{i_0}([x_{\alpha,\beta}^{i,j}]) = x_{\alpha,\beta}^{i_0,i_0}$. Clearly $T_{i_0} \in A^*$. Since $w^* - \lim_\beta x_\alpha^\beta = e_\alpha$, we have $w^* - \lim_\beta x_\alpha^\beta \varepsilon_{i_0,j} = e_\alpha \varepsilon_{i_0,j}$. So

$$|\cdot| - \lim_\beta x_\alpha^\beta \varepsilon_{i_0,i_0}(T_{i_0}) = e_\alpha \varepsilon_{i_0,i_0}(T_{i_0}).$$

We can see that

$$x_\alpha^\beta \varepsilon_{i_0, i_0}(T_{i_0}) = T_{i_0}(x_\alpha^\beta \varepsilon_{i_0, i_0}) = x_{\alpha, \beta}^{i_0, i_0} \xrightarrow{|\cdot|} 0.$$

On the other hand

$$\lim_\beta x_\alpha^\beta \varepsilon_{i_0, i_0}(T_{i_0}) = \varepsilon_{i_0, i_0}(T_{i_0}) = T_{i_0}(\varepsilon_{i_0, i_0}) = 1.$$

So a contradiction reveals. Therefore $x \neq 0$. Hence $|\cdot| - \lim_\beta x_{\alpha, \beta}^{j, j} = x$. Then there exists β_0 such that $\frac{|x|}{2} < |x_{\alpha, \beta_0}^{j, j}| < \frac{3}{2}|x|$.

We claim that Λ is finite. We assume in contradiction, that Λ is infinite. So Λ has a countable and infinite subset say J . Clearly

$$\sum_{j \in \Lambda} |x_{\alpha, \beta}^{j, j}| \leq \|x_\alpha^\beta\| < \infty,$$

because $(x_\alpha^\beta)_\beta$ is bounded. So $\sum_{j \in J} |x_{\alpha, \beta}^{j, j}|$ is a convergent series. Therefore $\lim_j |x_{\alpha, \beta}^{j, j}| = 0$, for all α and β . Put $\beta = \beta_0$ and let j be big enough. Then we can assume that $|x_{\alpha, \beta}^{j, j}| < \frac{|x|}{2}$. It is a contradiction. So Λ must be finite.

Conversely, suppose that Λ be finite. it is easy to see that $\mathbb{M}_\Lambda(\mathbb{C})$ has an identity. So $\mathbb{M}_\Lambda(\mathbb{C})$ has an ultra central approximate identity. \square

Proposition 2.2. *Let A be a Banach algebra with a bounded approximate identity. Then A has an ultra central approximate identity.*

Proof. Suppose that (e_α) is a bounded approximate identity for A . Put m for w^* -limit point of (e_α) in A^{**} . We can see that $am = ma$ and $ma = a$, for every $a \in A$. Thus A has an ultra central approximate identity. \square

We recall that a Banach algebra A is amenable if and only if there exists an element $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ and $\pi_A^{**}(m)a = a$ for each $a, b \in A$, see [15].

Lemma 2.3. *Let A be an amenable Banach algebra. Then A has an ultra central approximate identity.*

Proof. Since A is amenable, there exists an element $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ and $\pi_A^{**}(m)a = a$ for each $a, b \in A$. It is easy to see that $\pi_A^{**}(m) \in A^{**}$, $a\pi_A^{**}(m) = \pi_A^{**}(a \cdot m) = \pi_A^{**}(m \cdot a) = \pi_A^{**}(m)a$ and $\pi_A^{**}(m)a = a$ for every $a \in A$. So A has an ultra central approximate identity. \square

We recall that a Banach algebra A is called pseudo-contractible if there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A(m_\alpha)a \rightarrow a$ for every $a \in A$, see [10].

Lemma 2.4. *Let A be a pseudo-contractible Banach algebra. Then A has an ultra central approximate identity.*

Proof. Since A is pseudo-contractible, there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A(m_\alpha)a \rightarrow a$ for every $a \in A$. Set $e_\alpha = \pi_A(m_\alpha)$. It is easy to see that

$$ae_\alpha = a\pi_A(m_\alpha) = \pi_A(a \cdot m_\alpha) = \pi_A(m_\alpha \cdot a) = \pi_A(m_\alpha)a = e_\alpha a$$

and $e_\alpha a = \pi_A(m_\alpha)a \rightarrow a$ for every $a \in A$. Since A can be embedded in A^{**} , (e_α) becomes an ultra central approximate identity for A . \square

Theorem 2.5. *Let A be a Banach algebra and also let Λ be a non-empty set. Then $\mathbb{M}_\Lambda(\mathbb{C})^{**}$ is pseudo-contractible if and only if Λ is finite.*

Proof. Suppose that $\mathbb{M}_\Lambda(\mathbb{C})^{**}$ is pseudo-contractible. Then there exists a net (m_α) in $\mathbb{M}_\Lambda(\mathbb{C})^{**} \otimes_p \mathbb{M}_\Lambda(\mathbb{C})^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_{\mathbb{M}_\Lambda(\mathbb{C})^{**}}(m_\alpha)a \rightarrow a$, for each $a \in \mathbb{M}_\Lambda(\mathbb{C})^{**}$. There exists a bounded linear map $\psi : \mathbb{M}_\Lambda(\mathbb{C})^{**} \otimes_p \mathbb{M}_\Lambda(\mathbb{C})^{**} \rightarrow (\mathbb{M}_\Lambda(\mathbb{C}) \otimes_p \mathbb{M}_\Lambda(\mathbb{C}))^{**}$ such that for $a, b \in \mathbb{M}_\Lambda(\mathbb{C})$ and $m \in \mathbb{M}_\Lambda(\mathbb{C})^{**} \otimes_p \mathbb{M}_\Lambda(\mathbb{C})^{**}$, the following holds;

- (i) $\psi(a \otimes b) = a \otimes b$,
- (ii) $\psi(m) \cdot a = \psi(m \cdot a), \quad a \cdot \psi(m) = \psi(a \cdot m)$,
- (iii) $\pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m)) = \pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(m)$,

see [9, Lemma 1.7]. Since $\pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}$ is a $\mathbb{M}_\Lambda(\mathbb{C})$ -bimodule morphism, we have

$$a\pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m_\alpha)) = \pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(a\psi(m_\alpha)) = \pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m_\alpha)a) = \pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m_\alpha))a$$

and

$$\pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m_\alpha))a = \pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(m_\alpha)a \rightarrow a,$$

for each $a \in \mathbb{M}_\Lambda(\mathbb{C})^{**}$. Clearly $\pi_{\mathbb{M}_\Lambda(\mathbb{C})}^{**}(\psi(m_\alpha))$ is a net in $\mathbb{M}_\Lambda(\mathbb{C})^{**}$. It follows that $\mathbb{M}_\Lambda(\mathbb{C})$ has an ultra central approximate identity. Applying Lemma 2.1 gives that Λ is finite.

For converse, if Λ is finite then $\mathbb{M}_\Lambda(\mathbb{C})^{**} = \mathbb{M}_\Lambda(\mathbb{C}^{**}) = \mathbb{M}_\Lambda(\mathbb{C})$. So $\mathbb{M}_\Lambda(\mathbb{C})^{**}$ is contractible. Then $\mathbb{M}_\Lambda(\mathbb{C})^{**}$ is pseudo-contractible. \square

Using the similar argument as in the proof of previous theorem we have

Lemma 2.6. *Let A^{**} be a pseudo-contractible Banach algebra. Then A has an ultra central approximate identity.*

We give some examples among the semigroup algebras which have an ultra central approximate identity.

Example 2.7. Suppose that $S = \mathbb{N} \cup \{0\}$. With the following product S becomes a semigroup:

$$m * n = \begin{cases} m & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Since $\sup\{|(s)| : s \in S\} = 2$, one can easily see that S is a uniformly locally finite semigroup. Note that S is a commutative semigroup which $E(S) = S$ (then S is a semilattice). Applying [8, Corollary 2.7], $\ell^1(S)$ is pseudo-contractible. Thus by Lemma 2.4, $\ell^1(S)$ has an ultra central approximate identity.

Example 2.8. Let $S = \mathbb{N}$. With \min as its multiplication, S becomes a commutative semigroup. Let $w : S \rightarrow [1, \infty)$ be any function. It is easy to show that $w(st) \leq w(s)w(t)$ for each $s, t \in S$. So w is a weight on S . Set $A = \ell^1(S, w)$, the weighted semigroup algebra with respect to S . Suppose that $w(n) = e^n$ for each $n \in S$. Clearly $\lim w(n) = \infty$. So by [1, Proposition 3.3.1] A doesn't have a bounded approximate identity but it has a central approximate identity. Then we have a Banach algebra with an ultra central approximate identity but it doesn't have bounded approximate identity.

Let G be a locally compact non-SIN group. Then by the main result of [12], we have $L^1(G)$ doesn't have central approximate identity. On the other hand it is well-known that every group algebra on a locally compact group G has a bounded approximate identity. Then $L^1(G)$ has an ultra central approximate identity but it doesn't have a central approximate identity.

Lemma 2.9. *Let A and B be Banach algebras which A is unital. If $A \otimes_p B$ has an ultra central approximate identity, then B has an ultra central approximate identity.*

Proof. Suppose that $A \otimes_p B$ has an ultra central approximate identity. Then there exists a net (e_α) in $(A \otimes_p B)^{**}$ such that $xe_\alpha = e_\alpha x$ and $e_\alpha x \rightarrow x$ for every $x \in A \otimes_p B$. Using the following actions one may consider $A \otimes_p B$ as a Banach B -bimodule:

$$b_1 \cdot (a \otimes b_2) = a \otimes b_1 b_2, \quad (a \otimes b_2) \cdot b_1 = a \otimes b_2 b_1 \quad (a \in A, b_1, b_2 \in B).$$

Let e be the identity of A . By Hahn-Banach theorem we can find $\varphi_e \in A^*$ such that $\varphi_e(e) = 1$. Define $\varphi_e \otimes id_B$ from $A \otimes_p B$ into B by $\varphi_e \otimes id_B(a \otimes b) = \varphi_e(a)b$ for every $a \in A$ and $b \in B$, where id_B is denoted for the identity map on B . It is easy to see that $\varphi_e \otimes id_B$ is a bounded linear map. We claim that $((\varphi_e \otimes id_B)^{**}(e_\alpha)_\alpha)$ is an ultra central approximate identity for B . To see this consider

$$b(\varphi_e \otimes id_B)^{**}(x) = (\varphi_e \otimes id_B)^{**}(b \cdot x), \quad (\varphi_e \otimes id_B)^{**}(x)b = (\varphi_e \otimes id_B)^{**}(x \cdot b)$$

for all $(x \in (A \otimes_p B)^{**}, b \in B)$. Then we have

$$\begin{aligned}
b(\varphi_e \otimes id_B)^{**}(e_\alpha) &= (\varphi_e \otimes id_B)^{**}(b \cdot e_\alpha) = (\varphi_e \otimes id_B)^{**}((e \otimes b)e_\alpha) \\
&= (\varphi_e \otimes id_B)^{**}(e_\alpha(e \otimes b)) \\
&= (\varphi_e \otimes id_B)^{**}(e_\alpha \cdot b) \\
&= (\varphi_e \otimes id_B)^{**}(e_\alpha)b
\end{aligned}$$

and

$$(\varphi_e \otimes id_B)^{**}(e_\alpha)b = (\varphi_e \otimes id_B)^{**}(e_\alpha(e \otimes b)) \rightarrow (\varphi_e \otimes id_B)^{**}(e \otimes b) = \varphi_e \otimes id_B(e \otimes b) = b,$$

for each $b \in B$. Thus B has an ultra central approximate identity. \square

3. APPLICATIONS TO CERTAIN SEMIGROUP ALGEBRAS

In this section we study the existence of an ultra central approximate identity for certain semigroup algebras.

Theorem 3.1. *Let S be an inverse semigroup such that $E(S)$ is uniformly locally finite. Then the following are equivalent:*

- (i) $\ell^1(S)$ has an ultra central approximate identity;
- (ii) Each \mathfrak{D} -class has finitely many idempotent elements.

Proof. Suppose that $\ell^1(S)$ has an ultra central approximate identity. Then there exists a net (e_α) in $\ell^1(S)^{**}$ such that $ae_\alpha = e_\alpha a$ and $e_\alpha a \rightarrow a$ for each $a \in \ell^1(S)$. Using [14, Theorem 2.18] since S is a uniformly locally finite inverse semigroup, we have

$$\ell^1(S) \cong \ell^1 - \bigoplus \{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))\},$$

where $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$ is a \mathfrak{D} -class and G_{p_λ} is a maximal subgroup at p_λ . We claim that $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$ has an ultra central approximate identity. To see this let P_λ be the projection map from $\ell^1(S)$ onto $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$. It is easy to see that

$$aP_\lambda^{**}(e_\alpha) = P_\lambda^{**}(ae_\alpha) = P_\lambda^{**}(e_\alpha a) = P_\lambda^{**}(e_\alpha)a$$

and

$$P_\lambda^{**}(e_\alpha)a = P_\lambda^{**}(e_\alpha a) \rightarrow P_\lambda^{**}(a) = a,$$

for every $a \in \mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$. Then $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$ has an ultra central approximate identity. On the other hand we know that $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda})) \cong \ell^1(G_{p_\lambda}) \otimes_p \mathbb{M}_{E(\mathfrak{D}_\lambda)}(\mathbb{C})$. Since $\ell^1(G_{p_\lambda})$ is a unital Banach algebra, by Lemma 2.9, we have $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\mathbb{C})$ has an ultra central approximate identity. Now applying Lemma 2.1 implies that $E(\mathfrak{D}_\lambda)$ is finite.

Conversely, suppose that $E(\mathfrak{D}_\lambda)$ is finite. Since each $\ell^1(G_{p_\lambda})$ is unital, each $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda})) \cong \ell^1(G_{p_\lambda}) \otimes_p \mathbb{M}_{E(\mathfrak{D}_\lambda)}(\mathbb{C})$ is a unital Banach algebra. So one can easily see that

$$\ell^1(S) \cong \ell^1 - \bigoplus \{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))\},$$

has a central approximate identity. Therefore $\ell^1(S)$ has an ultra central approximate identity.

□

For a locally compact group G and non-empty set I , set

$$M^0(G, I) = \{(g)_{i,j} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{i,j}$ denotes the $I \times I$ matrix with g in (i, j) -position and zero elsewhere. With the following multiplication $M^0(G, I)$ becomes a semigroup

$$(g)_{i,j} * (h)_{k,l} = \begin{cases} (gh)_{il} & j = k \\ 0 & j \neq k, \end{cases}$$

It is well known that $M^0(G, I)$ is an inverse semigroup with $(g)_{i,j}^* = (g^{-1})_{j,i}$. This semigroup is called Brandt semigroup over G with the index set I .

Theorem 3.2. *Let $S = M^0(G, I)$ be a Brandt semigroup over a group G with index set I . Then the following are equivalent:*

- (i) $\ell^1(S)$ has an ultra central approximate identity;
- (ii) I is finite;

Proof. (i) \Rightarrow (ii) Suppose that $\ell^1(S)$ has an ultra central approximate identity. Using [2, Remark, p 315], we know that $\ell^1(S)$ is isometrically isomorphic with $[\mathbb{M}_I(\mathbb{C}) \otimes_p \ell^1(G)] \oplus_1 \mathbb{C}$. Put P_1 for the projection from $\ell^1(S)$ onto $\mathbb{M}_I(\mathbb{C}) \otimes_p \ell^1(G)$. So it is easy to see that $P_1^{**}(e_\alpha)$ is an ultra central approximate identity for $\mathbb{M}_I(\mathbb{C}) \otimes_p \ell^1(G)$, where (e_α) is an ultra central approximate identity for $\ell^1(S)$. Since $\ell^1(G)$ is unital, Lemma 2.9 follows that $\mathbb{M}_I(\mathbb{C})$ has an ultra approximate identity. Applying Lemma 2.1, implies that I is finite.

(ii) \Rightarrow (i) Since I is finite, $[\mathbb{M}_I(\mathbb{C}) \otimes_p \ell^1(G)] \oplus_1 \mathbb{C}$ has an identity. So $\ell^1(S)$ is unital. Clearly $\ell^1(S)$ has an ultra central approximate identity. □

Theorem 3.3. *Let $S = M^0(G, I)$ be a Brandt semigroup over a group G with index set I . Then the following are equivalent:*

- (i) $\ell^1(S)^{**}$ pseudo-contractible.
- (ii) $\ell^1(S)^{**}$ contractible.

Proof. Let $\ell^1(S)^{**}$ be pseudo-contractible. Then Lemma 2.6 follows that $\ell^1(S)$ has an ultra central approximate identity. Applying previous theorem gives that I must be finite. Since $\ell^1(S)$ is isometrically isomorphic with $[\mathbb{M}_I(\mathbb{C}) \otimes_p \ell^1(G)] \oplus_1 \mathbb{C}$, $\ell^1(S)$ becomes unital. It follows that $\ell^1(S)^{**}$ is unital. By [10, Theorem 2.4] pseudo-contractibility of $\ell^1(S)^{**}$ gives that $\ell^1(S)^{**}$ is contractible.

Converse is clear. \square

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Amir Sahami

Department of Mathematics, Faculty of Basic Sciences,
Ilam University
P.O. Box 69315-516, Ilam, Iran.
a.sahami@ilam.ac.ir

Mehdi Rostami

Faculty of Mathematics and Computer Science
Amirkabir University of Technology
424 Hafez Avenue, 15914 Tehran, Iran.
mross@aut.ac.ir

Shahab Kalantari

Department of Basic Science
Babol Noshirvani University of Technology,
Shariati Avenue, Babol 47148-71167, Iran.
shahab.kalantari@nit.ac.ir