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Research Paper

SOME REMARKS ON GOURSAT LEMMA

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ABSTRACT. In this article,we give a characterization of containment of subgroups in a direct product $A \times B \times C$. Other potential generalizations are investigated and applications characterizing different types of groups and modules are given. Most of applications are simple while somewhat deeper applications occur in the case of cyclic modules.

1. Introduction

Over the past years various authors have investigated the famous elementary result in group theory called Goursat's lemma for characterizing the subgroups of the direct product $A \times B$ of two groups A, B. This has the advantage that it generalize easily to higher direct product. Given a family of groups $(A_i)_{1\leq i\leq n}$, the direct product $A_1 \times \cdot \times A_n$ of A_i is the set of ordered pairs $\{(a_1, \cdots, a_n)|a_i \in A_i\}$ with coordinate-wise product $(a_1, \cdots, a_n)(b_1, \cdots, b_n) = (a_1b_1, \cdots, a_nb_n)$. Here $(1, \cdots, 1)$ is the identity element and $(a_1, \cdots, a_n)^{-1} = (a_1^{-1}, \cdots, a_n^{-1})$.

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If H_i is a subgroup of A_i, then $H_1 \times \cdot \times H_n$ is easily checked to be a subgroup of $A_1 \times \cdot \times A_n$. Moreover $H_1 \times \cdot \times H_n$ is a normal subgroup of $A_1 \times \cdot \times A_n$ if and only if each $H_i \trianglelefteq A_i$. Let us call a subgroup of $A_1 \times \cdot \times A_n$ of the form $H_1 \times \cdot \times H_n$ a subproduct of $A_1 \times \cdot \times A_n$. The only prerequisites from group theory are a good understanding of subgroups, normal subgroups, factor group, and permutable subgroup of G . Two subgroups M and S of a group G are said to permute, or M permutes with S, if $MS = SM$. Furthermore, M is a permutable subgroup of G if M permutes with every subgroup of G.

There are a number of interesting possibilities for generalizing this useful lemma. The first is to subgroups of a semi-direct product, and this is studied in [21]. The second is to other categories besides groups. Indeed, it is proved for modules in [9], and this implies that it will hold in any abelian category by applying the embedding theorems .The most general category in which one can hope to have a Goursat lemma is likely an exact Mal'cev and Goursat category Structure of the paper: In section 2 we recall some properties of groups. In section 3, We state this section with module version of a generalized Goursat's lemma and explore relative deeper applications as cyclic modules.

2. Containment of Subgroups of a Direct Product $A \times B \times C$

Goursat's lemma for groups ([15], p. 2) can be stated as follows:

Lemma 2.1. *Let L and R be arbitrary groups. Then there is a bijection between the set $\mathcal{S}$ of all subgroups of $L \times R$ and the set $\mathcal{T}$ of all quintuple (A, B, C, D, θ), where $B \trianglelefteq A \leq L$, $D \trianglelefteq C \leq R$ and $\theta : A/B \to C/D$ is a bijective homomorphism (here $\leq$ denotes subgroup and $\trianglelefteq$ denotes normal subgroup). More precisely, the subgroup corresponding to (A, B, C, D, θ) is*

$$G = \{(g, h) \in A \times C : \theta(gB) = hD\}.$$

Example 2.2. The subgroups of $S_2 \times S_2$. First, the subgroups of S_2 are $\langle(1)\rangle$,$\langle(12)\rangle$. Consider the subnormal quotient groups A/B where $B \trianglelefteq A \subseteq S_2$. If $|A/B| = 1$, one has $\langle(1)\rangle/\langle(1)\rangle; \langle(12)\rangle/\langle(12)\rangle$ It has only the identity maps between the 2 different quotients;so there are 4 different isomorphisms $\theta : A/B \to C/D$ yielding the 4 different subproducts $\langle(1)\rangle \times \langle(1)\rangle, v_1 = \langle(1)\rangle \times S_2, v_2 = S_2 \times \langle(1)\rangle$ and $S_2 \times S_2$. If $|A/B| = 2$ on has $\langle(12)\rangle/\langle(1)\rangle$; therefore the isomorphism $\langle(12)\rangle/\langle(1)\rangle \to \langle(12)\rangle/\langle(1)\rangle$; gives the subgroup $v_3 = \big\{\big((1), (1)\big), \big((12), (12)\big)\big\}$.

Remark 2.3. *For an arbitrary quadruple $Q_2(G) = (A, C, D, \theta_1)$ and $\theta_1 : A \to C/D$ a surjective homomorphism define*

$$\Psi_2(Q) := p^{-1}(\mathcal{G}_{\theta_1}),$$

where $\mathcal{G}_{\theta_1} \subseteq A \times (C/D)$ is a graph of θ_1 and $p : A \times C \to A \times (C/D)$ is natural surjection. The functions Q_2 and Ψ_2 are inverse to each other.

Lemma 2.4. *Let $H, G \leq A \times B$ where G is given by the quadruple $Q_2(G) = \{\overline{G}_1, \overline{G}_2, G_2, \theta\}$ and H is given by the quadruple $Q_2(H) = \{\overline{H}_1, \overline{H}_2, H_2, \gamma\}$. Suppose further that the following conditions hold:*

(i) $\overline{H}_i \leq \overline{G}_i,\ H_1 \leq G_1,\ i = 1, 2$

(ii) $\theta(\overline{H}_1) = \dfrac{\overline{H}_2 G_2}{G_2}$

(iii) *Define* $\widetilde{\theta} : \overline{H}_1 \to \dfrac{\overline{H}_2 G_2}{G_2}$ *and* $\widetilde{\gamma} : \overline{H}_1 \to \dfrac{\overline{H}_2}{\overline{H}_2 \cap G_2}$ *to be the restrictions of θ and γ respectively. Then $\widetilde{\theta}$ and $\widetilde{\gamma}$ are epimorphisms.*

Note that for $(h, l) \in H, \widetilde{\gamma}(h) = l(\overline{H}_2 \cap G_2)$

Theorem 2.5. *Let $H, G \leq A \times B$ where G is given by the quadruple $Q_2(G) = \{\overline{G}_1, \overline{G}_2, G_2, \theta\}$ and H is given by the quadruple $Q_2(H) = \{\overline{H}_1, \overline{H}_2, H_2, \gamma\}$. $H \leq G$ if and only if*

(i) $\overline{H}_i \leq \overline{G}_i,\ H_1 \leq G_1,\ i = 1, 2$

(ii) $\theta(\overline{H}_1) = \dfrac{\overline{H}_2 G_2}{G_2}$

(iii) *the following diagram commutes .*

$$\begin{array}{ccc} \overline{H}_1 & \xrightarrow{\ \widetilde{\theta}\ } & \dfrac{\overline{H}_2 G_2}{G_2} \\ & {\scriptstyle\widetilde{\gamma}}\searrow & \downarrow{\scriptstyle\varepsilon} \\ & & \dfrac{\overline{H}_2}{\overline{H}_2 \cap G_2} \end{array}$$

Proof. ($\Rightarrow$) Suppose $H \leq G$. It is obvious that $\overline{H}_i \leq \overline{G}_i,\ H_1 \leq G_1,\ i = 1, 2,\ \theta(\overline{H}_1) = \dfrac{\overline{H}_2 G_2}{G_2}$.Hence, it suffices to show that the diagram commutes. More specifically, that $\varepsilon\widetilde{\theta} = \widetilde{\gamma}$ let $(h, l) \in H$ since $H \leq G$ we have $\theta(h) = lG_2$. Then $\varepsilon\widetilde{\theta}(h) = \varepsilon(lG_2) = l(\overline{H}_2 \cap G_2) = \widetilde{\gamma}(h)$.
($\Leftarrow$)Conversely, suppose the containments hold and the diagram commutes. Our aim is to show $H \leq G$. Let $(h, l) \in H$. Then $\widetilde{\theta}(h) = \varepsilon^{-1}\widetilde{\gamma}(h) = \varepsilon^{-1}(l\overline{H}_2 \cap G_2) = lG_2$ .Therefore, by Lemma 2.4 we know $\theta(h) = lG_2$ and $H \leq G$. □

We will state the result (Lemma 2.1 below) for $n \geq 3$, after first introducing some convenient notation

Definition 2.6. Let $\underline{m} = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\} = \underline{n}$, and $j \in \underline{n} \setminus \underline{m}$. Then

$$\begin{aligned} G(j|\underline{m}) \ := \ & \{x_j \in A_j | (x_1, \cdots, x_j, \cdots, x_n) \in G \\ & \text{for some } x_i \in A_i, 1 \leq i \leq n, i \neq j \text{ with } x_i = e \text{ if } i \in \underline{m}\} \end{aligned}$$

For brevity, we extend this notation and let $\overline{G}_k = G(k|\emptyset) = \pi_k(G)$ for all k,where $\pi_i : A_1 \times \cdots \times A_n \to A_i$ is the standard projection onto the ith factors. We next give a generalized of Goursat's lemma as follows:

Theorem 2.7. [15]*(Goursat's Lemma for $n \geq 2$) There is a bijection correspondence between the subgroups $G \leq A_1 \times A_2 \times \cdots \times A_n$ and $(3n-2)$-tuples $Q_n(G) := \{\overline{G}_1, \overline{G}_2, , G(2|1), \theta_1, \cdots, \overline{G}_n, G(n|1, \cdots, n-1), \theta_{n-1}\}$ where $\overline{G}_i \leq A_i, G(i|1, \cdots, i-1) \trianglelefteq \overline{G}_i$ and $\theta_{i-1} : \Omega_{i-1} \twoheadrightarrow \overline{G}_i / G(i|1, \cdots, i-1)$ is a surjective homomorphism. Here $\Omega_i \leq A_1 \times \cdots \times A_i$ is defined recursively,$1 \leq i \leq n$, by setting $\Omega_1 := \overline{G}_1$ and*

$$\Omega_i = \Psi_2\big(\{\Omega_{i-1}, \overline{G}_i, G(i|1, \cdots, i-1), \theta_{i-1}\}\big) \leq (A_1 \times \cdots \times A_{i-1}) \times A_i,$$

with Ψ_2 as defined in remark 2.3.

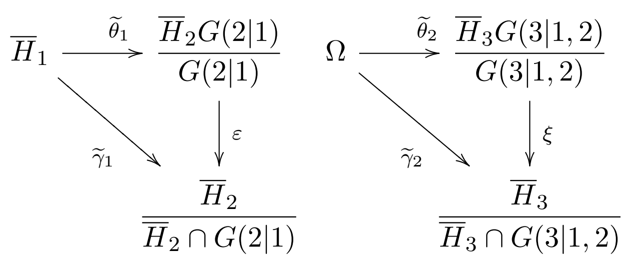
Definition 2.8. For a subgroup $G \leq A_1 \times \cdots \times A_n$, we say that the corresponding $(3n-2)$-tuple $Q_n(G)$ of theorem 2.7 is the Goursat decomposition of G .

Theorem 2.9. *Let $H, G \leq A \times B \times C$ where G is given by the Goursat decomposition $Q_3(G) := \{\overline{G}_1, \overline{G}_2, G(2|1), \theta_1, \overline{G}_3, G(3|1,2), \theta_2, \Lambda\}$ and H is given by the Goursat decomposition $Q_3(H) := \{\overline{H}_1, \overline{H}_2, H(2|1), \gamma_1, \overline{H}_3, H(3|1,2), \gamma_2, \Omega\}$. $H \leq G \Leftrightarrow$*

(i) $\overline{H}_i \leq \overline{G}_i, i = 1, 2, \Omega \leq \Lambda,\ H(2|1) \leq G(2|1), H(3|1,2) \leq G(3|1,2)$

(ii) $\theta_1(\overline{H}_1) = \dfrac{\overline{H}_2 G(2|1)}{G(2|1)}$ *and* $\theta_2(\Omega) = \dfrac{\overline{H}_3 G(3|1,2)}{G(3|1,2)}$

(iii) the following diagram is a commutative



Proof. ($\Rightarrow$) Suppose $H \leq G$. It is obvious that $\overline{H}_i \leq \overline{G}_i, i = 1, 2,\ H(2|1) \leq G(2|1), H(3|1,2) \leq G(3|1,2)$. It suffices to show that $\Omega \leq \Lambda$. But

$$\begin{aligned} \Omega &= \Psi_2(\{\overline{H}_1, \overline{H}_2, H(2|1), \gamma_1\}) \\ &= \{(x,y) \in \overline{H}_1 \times \overline{H}_2 / \gamma_1(x) = yH(2|1)\} \leq A \times B \\ \Lambda &= \{(a,b) \in \overline{G}_1 \times \overline{G}_2 / \theta_1(a) = bG(2|1)\} \leq A \times B \end{aligned}$$

By using Theorem 2.5 we have $\Omega \leq \Lambda$ and

$$\theta_1(\overline{H}_1) = \frac{\overline{H}_2 G(2|1)}{G(2|1)}, \theta_2(\Omega) = \frac{\overline{H}_3 G(3|1,2)}{G(3|1,2)}$$

Against By using Theorem 2.5 show that the first diagram commute :$\varepsilon\widetilde{\theta}_1 = \widetilde{\gamma}_1$. Similarly we show that $\varepsilon\widetilde{\theta}_2 = \widetilde{\gamma}_2$ since

$$\begin{aligned} H &= \Psi_2(\{\Omega, \overline{H}_3, H(3|1,2), \gamma_2\}) \\ &= \{(x,z) \in \Omega \times \overline{H}_3 / \gamma_2(x) = zH(3|1,2)\} \\ G &= \{(a,c) \in \Lambda \times \overline{G}_3 / \theta_2(a) = cG(3|1,2)\}. \end{aligned}$$

$\Leftarrow$ Conversely, suppose the containments hold and the diagram commutes. Our aim is to show that $H \leq G$. The same argument as in the Theorem 2.5 show that $H \leq G$. □

3. Applications

There is a well known characterization of normal subgroups of a direct product. It states that N is a normal subgroup of $G \times H$ if and only if $\pi_G(N)/(N \cap G) \leq Z(G/(N \cap G))$ and $\pi_H(N)/(N \cap H) \leq Z(H/(N \cap H))$, where π_G and π_H are the natural projections of $G \times H$ onto G and H respectively. This prior result allows us to give a version of Goursat's Lemma for normal subgroups of order n. Let $\underline{i} = \{1, \cdots, i\}$, then we have the following lemma.

Lemma 3.1. *Let $G \leq A_1 \times \cdots \times A_n$, with Goursat decomposition*

$$Q_n(G) := \{\overline{G}_1, \overline{G}_2, , G(2|1), \theta_1, \cdots, \overline{G}_n, G(n|1, \cdots, n-1), \theta_{n-1}\}.$$

If $G \trianglelefteq A_1 \times A_2 \times \cdots \times A_n$, then $G(i|\underline{i-1}) \trianglelefteq A_i$ and $\overline{G}_i/G(i|\underline{i-1}) \subseteq Z(A_i/G(i|\underline{i-1}))$ the center of $A_i/G(i|\underline{i-1})$.

Proof. Suppose that $G \trianglelefteq A_1 \times \cdots \times A_n$,show that $G(i|1, \cdots, i-1) \trianglelefteq A_i$. Let $a_i \in G(i|1, \cdots, i-1)$ and $b_j \in A_j$, since $a_i \in G(i|1, \cdots, i-1)$ we have $a = (e, \cdots, e, a_i, \cdots, a_n) \in G$ as $G \trianglelefteq A_1 \times \cdots \times A_n$ then

$$bab^{-1} = (e, \cdots, e, b_i a_i b_i^{-1}, \cdots, b_n a_n b_n^{-1}) \in G$$

with $b = (b_1, \cdots, b_n) \in A_1 \times, \cdots, \times A_n$, thus $G(i|1, \cdots, i-1) \trianglelefteq A_i$.
Show that $\overline{G}_i/G(i|1, \cdots, i-1) \subseteq Z(A_i/G(i|1, ., i-1))$. For $\overline{g}_i = g_i G(i|1, \cdots, i-1)$ with $g_i \in \overline{G}_i$ and $\overline{a}_i = a_i G(i|1, ., i-1))$ with $a_i \in A_i$,since $g_i \in \overline{G}_i$ then there exists $g_j \in A_j$ such that $b = (g_1, ., g_i, ., g_n) \in G$, $(g_1, ., a_i^{-1} g_i a_i, g_{i+1}, ., g_n), b^{-1} \in G$ give $(e, , e, a_i^{-1} g_i a_i, e, ., e) \in G$ and hence $a_i^{-1} g_i a_i \in G(i|1, \cdots, i-1)$. So $\overline{G}_i/G(i|\underline{i-1}) \subseteq Z(A_i/G(i|\underline{i-1}))$. □

Our goal in this section is to provide necessary conditions for a subgroup of a direct product of n groups to be permutable.Let $P = A_1 \times A_2 \times \cdots \times A_n$.

Lemma 3.2. *Let $G \leq P$, with Goursat decomposition*

$$Q_n(G) := \{\overline{G}_1, \overline{G}_2, , G(2|1), \theta_1, \cdots, \overline{G}_n, G(n|1, \cdots, n-1), \theta_{n-1}\}.$$

If G is a permutable subgroup of P, then, for all $a_i \in A_i$,
$\overline{G}_i \leq N_{A_i}\big(G(i|\underline{i-1})\langle a_i\rangle\big)$.

Proof. Without loss of generality assume $i = 2$ .Let $a_2 \in A_2$ and $(v_1, v_2, \cdots, v_n) \in G$. Since G is permutable in P,

$$(1, a_2, 1, \cdots, 1)(v_1, v_2, \cdots, v_n) = (w_1, w_2, \cdots, w_n)(1, a_2^j, 1, \cdots, 1)$$

for some $(w_1, w_2, \cdots, w_n) \in P$ and $j \in \mathbb{Z}$. Thus, $w_i = v_i$ for $i \neq 2$. But then $w_2 = v_2 x$ for some $x \in G(2|1)$. So $v_2^{-1} a_2 v_2 = x a_2^j$. Of course,$\overline{G}_i \leq N_{A_i}\big(G(i|\underline{i-1})\big)$, and therefore $\overline{G}_2 \leq N_{A_2}\big(G(2|1)\langle a_2\rangle\big)$ for $a_2 \in A_2$. □

We continue this section by stating a module asymmetric version of Goursat's lemma, and give generalization of this lemma. throughout this paper, let R be a commutative ring with identity.

Lemma 3.3. *Let R be a ring and N_1 and N_2 R-modules.*

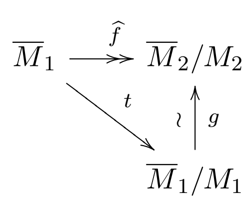
(1) *Let M be an R-submodule of $N_1 \times N_2$. Let $M_2 = \{b \in N_2 | (0, b) \in M\}, \overline{M}_2 = \{b \in N_2 | \exists a \in N_1 : (a, b) \in M\}$ and $\overline{M}_1 = \{a \in N_1 | \exists b \in N_2 : (a, b) \in M\}$. Then $M_2 \subseteq \overline{M}_2$ are R-submodules of N_2 and the map $\widehat{f} : \overline{M}_1 \twoheadrightarrow \overline{M}_2/M_2$ given by $\widehat{f}(a) = b + M_2$ where $(a, b) \in M$ is an R-module epimorphism.*

(2) *Let $M_2 \subseteq \overline{M}_2$ be R-submodules of N_2 with an R-module epimorphism $f : \overline{M}_1 \twoheadrightarrow \overline{M}_2/M_2$. Then $M = \{(a, b) \in \overline{M}_1 \times \overline{M}_2 | f(a) = b + M_2\}$ is an R-module of $N_1 \times N_2$.*

(3) *The constructions given in* (1) *and* (2) *are inverses to each other.*

Proof. It is easy checked that $M_2 \subseteq \overline{M}_2$ are R-submodules of N_2. Define $\widehat{f} : \overline{M}_1 \twoheadrightarrow \overline{M}_2/M_2$ by $\widehat{f}(a) = b + M_2$ where $(a, b) \in M$. If $(a, b), (a, c) \in M$, then $(0, c - b) = (-a, -b) + (a, c) \in M$ implies $c - b \in M_2$ and hence $b + M_2 = c + M_2$ so $\widehat{f}$ is well defined . It is easily checked that $\widehat{f}$ is surjective and R-linear.
(2) and (3) are clears. □

$\widehat{f}$ determines g via the first isomorphism theorem, specifically



Theorem 3.4. *Let V and U be an R-submodules of $N_1 \times N_2$. where V is given by the quintuple $Q_2(V) = \{\overline{V}_1, V_1, \overline{V}_2, V_2, \theta\}$ and U is given by the quintuple $Q_2(U) = \{\overline{U}_1, U_1, \overline{U}_2, U_2, \alpha\}$. Then $V \leq U$ if and only if:*

(a) $\overline{V}_n \leq \overline{U}_n$ *and* $V_n \leq U_n$ *for* $n = 1, 2$

(b) $\alpha(\frac{\overline{V}_1+U_1}{U_1}) = \frac{\overline{V}_2+U_2}{U_2}$ *and* $\theta(\frac{\overline{V}_1 \cap U_1}{V_1}) = \frac{\overline{V}_2 \cap U_2}{V_2}$

(c) *the following diagram is a commutative*

$$\begin{array}{ccc} \dfrac{\overline{V}_1 + U_1}{U_1} & \xrightarrow{\ \widetilde{\alpha}\ } & \dfrac{\overline{V}_2 + U_2}{U_2} \\ \lambda_1 \downarrow & & \downarrow \lambda_2 \\ \dfrac{\overline{V}_1}{\overline{V}_1 \cap U_1} & \xrightarrow[\ \widetilde{\theta}\ ]{} & \dfrac{\overline{V}_2}{\overline{V}_2 \cap U_2} \end{array}$$

Proof. ($\Rightarrow$) Suppose $V \leq U$. It is obvious that $\overline{V}_n \leq \overline{U}_n$ and $V_n \leq U_n$ for $n = 1, 2$, $\alpha(\frac{\overline{V}_1+U_1}{U_1}) = \frac{\overline{V}_2+U_2}{U_2}$ and $\theta(\frac{\overline{V}_1 \cap U_1}{V_1}) = \frac{\overline{V}_2 \cap U_2}{V_2}$. Hence, it suffices to show that the diagram commutes. More specifically, that $\lambda_2\widetilde{\alpha} = \widetilde{\theta}\lambda_1$. Let $c \in \overline{V}_1 = \pi_{N_1}(V)$. Then there exists a $d \in \overline{V}_2 = \pi_{N_2}(V)$ such that $(c, d) \in V$. Since $V \leq U$, we know $\alpha(c + U_1) = d + U_2$. Then $\widetilde{\alpha}$ is a restriction of α. Hence, we can consider $\widetilde{\alpha}(c + U_1)$. Then $\lambda_2(\widetilde{\alpha}(c + U_1) = \lambda_2(d + U_2) = d + \overline{V}_2 \cap U_2$. On the other hand, we get $\widetilde{\theta}(\lambda_1(c + U_1)) = \widetilde{\theta}(c + \overline{V}_1 \cap U_1) = d + \overline{V}_2 \cap U_2$. Therefore, $\lambda_2\widetilde{\alpha} = \widetilde{\theta}\lambda_1$, and the diagram commutes.

($\Leftarrow$) Conversely, suppose the containments hold and the diagram commutes. Our aim is to show $V \leq U$. Let $(c, d) \in V$. Then $\theta(c + V_1) = d + V_2$, where $c \in \overline{V}_1$ and $d \in \overline{V}_2$. Then $\widetilde{\alpha}(c + U_1) = \lambda_2^{-1}\widetilde{\theta}(\lambda_1(c + U_1)) = \lambda_2^{-1}(\widetilde{\theta}(c + \overline{V}_1 \cap U_1))$. We see that $\lambda_2^{-1}(\widetilde{\theta}(c + \overline{V}_1 \cap U_1)) = \lambda_2^{-1}(d + \overline{V}_2 \cap U_2) = d + U_2$. So, $\widetilde{\alpha}(c + U_1) = d + U_2$. Therefore, we know $\alpha(c + U_1) = d + U_2$ and $V \leq U$. □

A cyclic module or monogenous module [17] is a module over a ring that is generated by one element. The concept is analogous to cyclic group, that is, a group that is generated by one element.

Definition 3.5. A left R-module M is called cyclic if M can be generated by a single element i.e.$M = (x) = Rx = \{rx | r \in R\}$ for some x in M.

A left R-module M is called finite cyclic if $(M, +)$ is finite cyclic abelian group. We denote by $\mathbb{O}$ the trivial module .

Example 3.6. (1) Every cyclic group is a cyclic $\mathbb{Z}$-module.

(2) If the ring R is considered as a left module over itself, then its cyclic submodules are exactly its left principal ideals as a ring.

Remark 3.7. *Given a cyclic R-module M that is generated by x, there exists an isomorphism between M and $R/Ann_R x$, where $Ann_R x$ denotes the annihilator of x in R. If R is integral domain and M torsion free then $M \cong R$.*

The next application, that of determining the cyclic submodule of $M_1 \times M_2$, will involve more substantial use of Goursat's lemma. In what follow, let R is integral domain and M torsion free.

Theorem 3.8. *Let M be a left R-submodule of $M_1 \times M_2$ with Goursat quintuple $Q_2(M) = \{\overline{M}_1, M_1, \overline{M}_2, M_2, \theta\}$. The submodule M is cyclic if and only if one of the following three cases occur:*

i) $\overline{M}_1 \approx R$, $\overline{M}_2$ *is finite cyclic, and* $M_2 = \mathbb{O}$,
ii) $\overline{M}_2 \approx R$, $\overline{M}_1$ *is finite cyclic, and* $M_1 = \mathbb{O}$,
iii) $\overline{M}_1 \approx \overline{M}_2 \approx R$ *with* $M_1 = M_2 = \mathbb{O}$.

Proof. If M is $\mathbb{Z}$-module we recover the same theorem in groups (see theorem 4.4 [15], p.9) Since M is a cyclic left R-module and $M \subseteq \overline{M}_1 \times \overline{M}_2$, at least one of $\overline{M}_1, \overline{M}_2$ must be cyclic.Without loss of generality, suppose $\overline{M}_1 \approx R$. Now suppose (α, β) generates the cyclic module M, then α generates $\overline{M}_1$, and β generates $\overline{M}_2$. We claim that $M_2 = \mathbb{O}$. For, if $y \in M_2$ then $y = r\beta$ for some element $r \in R$, whence $(0, y) = r(0, \beta) \in M$. This implies $(0, y) = t(\alpha, \beta) = (t\alpha, t\beta)$ for some element $t \in R$. Therefore $t\alpha = 0$, whence $t = 0$ since R is principal ideal domain and $y = t\beta = 0$. Hence $M_2 = \mathbb{O}$. We now consider separately the cases M_2 cyclic module and M_2 finite cyclic module (the case M_1 finite and $M_2 \approx R$ is symmetric so can be omitted).
Suppose first $M \approx R$ with $M_2 \approx R$. Then the argument in the previous paragraph now also implies $M_1 = \mathbb{O}$. Conversely, suppose $\overline{M}_1 \approx \overline{M}_2 \approx R$ and $M_1 = M_2 = \mathbb{O}$. Then the isomorphisms

$$M/(M_1 \times M_2) \approx \overline{M}_1/M_1 \overset{\approx}{\to} \overline{M}_2/M_2 \text{ reduce to } M \approx \overline{M}_1 \approx \overline{M}_2 \approx R.$$

Secondly, for the remaining case, suppose $M \approx R, M_1 \approx R$ as before and now $\overline{M}_2 \approx \mathbb{Z}_n$ is cyclic of order $n, n \geq 2$. Then $n(\alpha, \beta) = (n\alpha, 0)$ implies $n\alpha \in M_1$ and clearly $i\alpha \notin M_1$ if $i < n$. Thus $M_1 \approx nR$, and as before $M_2 = \mathbb{O}$.
Conversely, suppose $\overline{M}_1 \approx \mathbb{Z}, M_1 \approx n\mathbb{Z}, \overline{M}_2 \approx \mathbb{Z}_n$ and $M_2 = 0$. In this case we have the isomorphism $\theta : \overline{M}_1/M_1 \overset{\approx}{\to} \overline{M}_2/M_2 \approx \overline{M}_2$. Let $\alpha \in M_1$ with $[\alpha]$ generating $\overline{M}_1/M_1$. Then $\theta([\alpha]) = \beta$ generates $\overline{M}_2/M_2 \approx \overline{M}_2 \approx \mathbb{Z}_n$. We claim that M is generated by the single element (α, β), and thus is cyclic module. □

Definition 3.9. Let M be the R-submodule of $N_1 \times N_2 \times \cdots \times N_n$ Let $\underline{m} = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\} = \underline{n}$, and $j \in \underline{n} \setminus \underline{m}$. Then

$$\begin{aligned} M(j|\underline{m}) \;&:=\; \{x_j \in N_j | (x_1, \cdots, x_j, \cdots, x_n) \in M \\ &\quad \text{for some } x_i \in N_i, 1 \leq i \leq n, i \neq j \text{ with } x_i = 0 \text{ if } i \in \underline{m}\} \end{aligned}$$

let $\overline{M}_k = M(k|\emptyset)$.

We next give a generalized of Goursat's lemma as follows:

Lemma 3.10. *(Goursat's Lemma for $n \geq 2$) There is a bijection correspondence between the R-submodule M of $N_1 \times N_2 \times \cdots \times N_n$ and $(3n-2)$-tuples*

$$Q_n(M) := \{\overline{M}_1, \overline{M}_2, \cdots, M(2|1), \theta_1, \cdots, \overline{M}_n, M(n|1, \cdots, n-1), \theta_{n-1}\}$$

where , $M(i|1, \cdots, i-1) \leq \overline{M}_i$ are R-submodules of N_i and $\theta_{i-1} : \Omega_{i-1} \twoheadrightarrow \overline{M}_i / M(i|1, \cdots, i-1)$ is an R-module epimorphism. Here $\Omega_i \leq N_1 \times \cdots \times N_i$ is defined recursively,$1 \leq i \leq n$, by setting $\Omega_1 := \overline{M}_1$ and

$$\Omega_i = \Psi_2\big(\{\Omega_{i-1}, \overline{M}_i, M(i|1, \cdots, i-1), \theta_{i-1}\}\big) \leq N_1 \times \cdots N_i,$$

and Ψ_2 is defined as in remark 2.3.

Proof. Use theorem 2.7 for instance. □

In what follow, let R is integral domain and M torsion free.

Theorem 3.11. *Let $M \leq N_1 \times N_2 \times N_3$, with Goursat decomposition*

$$Q_3(M) := \{\overline{M}_1, \overline{M}_2, M(2|1), \theta_1, \overline{M}_3, M(3|1,2), \theta_2\}.$$

The submodule M is cyclic if and only if one of the following three cases (up to obvious permutation of indices) occur:

i) $\overline{M}_1 \approx R$, $\overline{M}_2$ *and* $\overline{M}_3$ *are finite cyclic,* $M(2|1) = M(3|1) = \mathbb{O}$,
ii) $\overline{M}_1 \approx \overline{M}_2 \approx R, \overline{M}_3$ *finite cyclic, and* $M(2|1) = M(3|1) = M(1|2) = M(3|2) = \mathbb{O}$.
iii) $\overline{M}_i \approx R$ *for* $i = 1, 2, 3$ *and* $M(i|j) = \mathbb{O}$ *for* $1 \leq i \neq j \leq 3$.

Proof. The three cases when M is cyclic module all follow from Theorem 3.8 in obvious ways, namely in (i) we use $N_1 \times N_2 \times N_3 \approx N_1 \times (N_2 \times N_3)$, in (ii) and (iii) we use $N_1 \times N_2 \times N_3 \approx (N_1 \times N_2) \times N_3$. We omit the details. □

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