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Research Paper

THE (p,q,r)-GENERATIONS OF THE SYMPLECTIC GROUP $Sp(6,2)$

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Abstract. A finite group G is called (l,m,n)-*generated*, if it is a quotient group of the triangle group $T(l,m,n) = \langle x,y,z|x^l = y^m = z^n = xyz = 1\rangle$. In [29], Moori posed the question of finding all the (p,q,r) triples, where p, q and r are prime numbers, such that a non-abelian finite simple group G is a (p,q,r)-generated. In this paper we establish all the (p,q,r)-generations of the symplectic group $Sp(6,2)$. GAP [20] and the Atlas of finite group representations [33] are used in our computations.

1. Introduction

Generations of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [34] for details). Also Di Martino et al. [27] established a useful connection between generation

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of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [32], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions. In this paper we are interested in the generation of the symplectic group $Sp(6,2)$ by two elements of prime orders, not necessary distinct primes such that the product is an element of a prime order.

A finite group G is said to be (l,m,n)-*generated*, if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this case G is also a quotient group of the triangular group $T(l,m,n)$ and, by definition of the triangular group, G is also $(\sigma(l),\sigma(m),\sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. In a series of papers [21, 22, 23, 24, 25, 28, 29], Moori and Ganief established all possible (p,q,r)-generations, where p,q and r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22}. Ashrafi in [3, 4] did the same for the sporadic simple groups He and HN. Also Darafsheh and Ashrafi established in [15, 16, 17, 18], the (p,q,r)-generations of the sporadic simple groups Co_1, Ru, $O'N$ and Ly. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

In this paper we intend to establish all the (p,q,r)-generations of the symplectic group $Sp(6,2)$. For more information on (p,q,r)-generations, the reader is referred to [1] and [2]. We follow the methods used in the papers [5, 6, 7, 8, 9, 10, 11] and [12]. Note that, in general, if G is a $(2,2,n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [13], if G is a non-abelian (l,m,n)-generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p,q,r)-generations of $G = Sp(6,2)$ the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The result on the (p,q,r)-generations of $Sp(6,2)$ can be summarized in the following theorem.

Theorem 1.1. *With the notation being as in the Atlas* [14], *the symplectic group* $Sp(6,2)$ *is generated by the triples* (pL, qM, rN), p, q *and* r *primes dividing* $|Sp(6,2)|$, *for the cases* $(pL, qM, rN) \in \{(2D, 3B, 7A), (2X, 5A, 7A), (2X, 7A, 7A), (3C, 5A, 5A), (3Y, 5A, 7A), (3Z, 7A, 7A), (5A, 5A, 7A), (5A, 7A, 7A), (7A, 7A, 7A)\}$, *for all* $X \in \{C, D\}$, $Y \in \{B, C\}$ *and* $Z \in \{A, B, C\}$.

2. Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i,\ 1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta^*_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle . \tag{2}$$

Definition 2.1. If $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be **$(C_1, C_2, \cdots, C_k)$-generated**.

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \cdots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \cdots, h_{k-1})$ are in $C_1 \times C_2 \times \cdots \times C_{k-1}$ such that

$$h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \cdots, h_{k-1} \rangle \leq H, \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [22] and [25, Theorem 2.1]. □

The above number $h(g,H)$ is useful in giving a lower bound for $\Delta^*_G(C_1,C_2,\cdots,C_k)$, namely $\Delta^*_G(C_1,C_2,\cdots,C_k) \geq \Theta_G(C_1,C_2,\cdots,C_k)$, where

$$\Theta_G(C_1,C_2,\cdots,C_k) = \Delta_G(C_1,C_2,\cdots,C_k) - \sum h(g_k,H)\Sigma_H(C_1,C_2,\cdots,C_k), \tag{4}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1,C_2,\cdots,C_k$. Since we have all the maximal subgroups of the sporadic simple groups (except for $G=\mathbb{M}$ the Monster group), it is possible to build a small subroutine in GAP [20] to compute the values of $\Theta_G = \Theta_G(C_1,C_2,\cdots,C_k)$ for any collection of conjugacy classes of the symplectic group $Sp(6,2)$.

The following results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.3. *Let G be a finite centerless group. If $\Delta^*_G(C_1,C_2,\cdots,C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1,C_2,\cdots,C_k) = 0$ and therefore G is not $(C_1,C_2,\cdots,C_k)$-generated.*

Proof. See [10, Lemma 2.7]. □

Theorem 2.4 (Ree [30])**.** *Let G be a transitive permutation group generated by permutations $g_1,g_2,\cdots,g_s$ acting on a set of n elements such that $g_1g_2\cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n+2$.*

Theorem 2.5 (Scott [31])**.** *Let $g_1,g_2,\cdots,g_s$ be elements generating a group G with $g_1g_2\cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim\mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i\rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i\rangle}$ being the trivial character of the cyclic group $\langle g_i\rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([19]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi{\downarrow}^G_{\langle g_i\rangle}, \mathbf{1}_{\langle g_i\rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i\rangle|}\sum_{j=0}^{o(g_i)-1}\chi(g_i^j). \end{aligned} \tag{5}$$

Theorem 2.6. [10, Lemma 2.5] *Let G be a $(2X,sY,tZ)$-generated simple group, then G is $(sY,sY,(tZ)^2)$-generated.*

3. The symplectic group $Sp(6,2)$

In this section we apply the results discussed in Section 2, to the group $Sp(6,2)$. We determine all the (p,q,r)-generations of $Sp(6,2)$, where p, q and r are primes dividing the order of $Sp(6,2)$.

The group $Sp(6,2)$ is a simple group of order $1451520 = 2^9 \times 3^4 \times 5 \times 7$. By the Atlas of finite groups [14], the group $Sp(6,2)$ has exactly 30 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$$\begin{array}{lll} H_1 = U_4(2){:}2, & H_2 = S_8, & H_3 = 2^5{:}S_6, \\ H_4 = U_3(3){:}2, & H_5 = 2^6{:}L_3(2), & H_6 = (2^{1+4} \times 2^2){:}(S_3 \times S_3), \\ H_7 = S_3 \times S_6, & H_8 = L_2(8){:}3. & \end{array}$$

In this section we let $G = Sp(6,2)$. For the sake of computations with GAP, we use a permutation presentation for G. By the electronic Atlas [33], G can be generated in terms of permutations on 28 points. Generators g_1 and g_2 can be taken as follows:

$$\begin{aligned} g_1 &= (2,3)(6,7)(9,10)(12,14)(17,19)(20,22), \\ g_2 &= (1,2,3,4,5,6,8)(7,9,11,13,16,18,14)(10,12,15,17,20,19,21)(22,23,24,25,26,27,28), \end{aligned}$$

with $o(g_1) = 2,\ o(g_2) = 7$ and $o(g_1g_2) = 9$.

The group $Sp(6,2)$ has 7- and 15-dimensional complex irreducible modules $\mathbb{V}_1$ and $\mathbb{V}_2$ respectively. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}_i/C_{\mathbb{V}_i}(nX)),\ i \in \{1,2\}$ denote the codimension of the fixed space (in $\mathbb{V}_i$) of a representative of nX. Using Equation (5) together with the power maps associated with the character table of $Sp(6,2)$ given in the Atlas, we were able to compute all the values of d_{nX} for all non-trivial classes nX of G, with respect to $\mathbb{V}_1$ and $\mathbb{V}_2$ and we list these values in Tables 1 and 2 respectively.

In Table 3, we list the values of the cyclic structure for each conjugacy of G which containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scott theorems, respectively.

In Table 4 we list the representatives of classes of the maximal subgroups together with orbit lengths of these maximal subgroups on their conjugates and respective permutation characters.

Table 5 gives us the partial fusion maps of classes of maximal subgroups into the classes of $Sp(6,2)$. These will be used in our computations.

Tables 6 to 13 that give the structure constants of $Sp(6,2)$ are listed at the end of this paper.

TABLE 1. $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 7$

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$
d_{nX}	6	4	2	4	2	6	4	4	4	6	6	4	4	6	4

nX	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
d_{nX}	6	4	6	6	6	6	6	6	6	6	6	6	6	6

TABLE 2. $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of $Sp(6,2)$ and $\dim(\mathbb{V}) = 15$.

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$	$6C$	$6D$
d_{nX}	10	4	6	8	10	12	8	10	12	10	8	10	12	14	12	12	12

nX	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
d_{nX}	12	10	12	12	12	12	14	14	14	14	14	14

TABLE 3. Cycle structures of prime order conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$1A$	1^{28}	28	0
$2A$	$1^{16}2^{6}$	22	6
$2B$	$1^{4}2^{12}$	16	12
$2C$	$1^{8}2^{10}$	18	10
$2D$	$1^{4}2^{12}$	16	12
$3A$	$1^{10}3^{6}$	16	12
$3B$	$1\ 3^{9}$	10	18
$3C$	$1\ 3^{9}$	10	18
$5A$	$1^{3}5^{5}$	8	20
$7A$	7^{4}	4	24

TABLE 4. Maximal subgroups of $Sp(6,2)$

Maximal Subgroup	Orbit Lengths	Character
$U_4(2){:}2$	[1,27]	$1a + 27a$
S_8	[28]	$1a + 35b$
$2^5{:}S_6$	[12,16]	$1a + 27a + 35b$
$U_3(3){:}2$	[28]	$1a + 35a + 84a$
$2^6{:}L_3(2)$	[28]	$1a + 15a + 35b + 84a$
$(2 \cdot 2^6){:}(S_3 \times S_3)$	[4,24]	$1a + 27a + 35b + 84a + 168a$
$S_3 \times S_6$	[10,18]	$1a + 27a + 35b + 105b + 168a$
$L_2(8){:}3$	[28]	$1a + 70a + 84a + 105b + 280a + 420a$

Table 5. The partial fusion maps into $Sp(6,2)$

$U_4(2)$:2-class	2a	2b	2c	2d	3a	3b	3c	5a
$\to$ $Sp(6,2)$	2C	2A	2B	2D	3A	3C	3B	5A
h								3

Table 5 continued

$U_4(2)$:2-class	2a	2b	2c	2d	3a	3b	3c	5a								
$\to$ $Sp(6,2)$	2C	2A	2B	2D	3A	3C	3B	5A								
h								3								
S_8-class	2a	2b	2c	2d	3a	3b	5a	7a								
$\to$ $Sp(6,2)$	2C	2A	2D	2B	3A	3C	5A	7A								
h							1	1								
2^5:S_6-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	3a	3b	5a			
$\to$ $Sp(6,2)$	2A	2C	2D	2C	2A	2D	2C	2D	2D	2B	3A	3B	5A			
h													3			
$U_3(3)$:2-class	2a	2b	3a	3b	7a											
$\to$ $Sp(6,2)$	2B	2D	3C	3B	7A											
h					1											
2^6:$L_3(2)$-class	2a	2b	2c	2d	2e	2f	2g	3a	7a	7b						
$\to$ $Sp(6,2)$	2A	2C	2D	2B	2C	2D	2B	3C	7A	7A						
h									1	1						
$2 \cdot 2^6$:$(S_3 \times S_3)$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k	2l	2m	3a	3b	3c
$\to$ $Sp(6,2)$	2C	2A	2B	2C	2D	2B	2D	2A	2D	2C	2D	2C	2B	3B	3A	3C
h														9	15	3
$S_3 \times S_6$-class	2a	2b	2c	2d	2e	2f	2g	3a	3b	3c	3d	3e	5a			
$\to$ $Sp(6,2)$	2A	2C	2A	2C	2D	2B	2D	3A	3C	3A	3C	3B	5A			
h													1			
$L_2(8)$:3-class	2a	3a	3b	3c	7a											
$\to$ $Sp(6,2)$	2D	3C	3B	3B	7A											
h					1											

4. The (p,q,r)-generations of $Sp(6,2)$

Let pX, $p \in \{2,3,5,7\}$ be a conjugacy class of G and c_i be the number of disjoint cycles in a representative of pX. For the group G and from the Atlas of finite group representations [33] we have G acting on 28 points, so that $n = 28$ and since our generation is triangular, we have $s = 3$. Hence by Ree's Theorem [30] if G is (l,m,n)-generated, then $\sum c_i \leq 30$.

4.1. $(2,q,r)$**-generations.** Now the $(2,q,r)$-generations of G comprises the cases $(2,3,r)$-, $(2,5,r)$- and $(2,7,r)$-generations.

4.1.1. $(2,3,r)$*-generations.* The condition $\frac{1}{2}+\frac{1}{3}+\frac{1}{r}<1$ shows that r must be 7. Thus we have to consider the cases $(2X,3Y,7A)$ for $X \in \{A,B,C,D\}$ and $Y \in \{A,B,C\}$.

Proposition 4.1. *The group G is*
(i) neither $(2X,3Y,7A)$- nor $(2D,3Z,7A)$-generated group for all $X,Y \in \{A,B,C\}$ and $Z \in \{A,C\}$,
(ii) $(2D,3B,7A)$-generated.

Proof. (i) Since by [26, Theorem 2], G is a Hurwitz group, we have to consider the triples $(2X, 3Y, 7A)$ for $X \in \{A, B, C, D\}$ and $Y \in \{A, B, C\}$. If G is a $(2A, 3A, 7A)$-generated group, then we must have $c_{2A}+c_{3A}+c_{7A} \leq 30$. From Table 3 we see that $c_{2A}+c_{3A}+c_{7A} = 22+16+4 = 42 > 30$ and by Ree's Theorem [30], it follows that G is not $(2A, 3A, 7A)$-generated. The same applies to the triples $(2A, 3B, 7A)$, $(2A, 3C, 7A)$, $(2B, 3A, 7A)$, $(2C, 3A, 7A)$, $(2C, 3B, 7A)$, $(2C, 3C, 7A)$ and $(2D, 3A, 7A)$. By Table 7 we have $\Delta_G(2B, 3B, 7A) = 0$ and by Lemma 2.3 it follows that G is not $(2B, 3B, 7A)$-generated.

By Table 5 we see that only four maximal subgroups of G have each an element of order 7 viz. H_2, H_4, H_5 and H_8. The intersection of these four maximal subgroups contains only the identity element. The intersection of any three maximal subgroups does not contain an element of order 7. We also noticed that $H_2 \cap H_5 \cong 2^4{:}S_4$ and $H_5 \cap H_8 \cong A_4$ do not contain an element of order 7. Thus, subgroups H_2, H_4, H_5, H_8, $H_2 \cap H_4 \cong L_3(2){:}2$, $H_2 \cap H_8 \cong 7{:}6$, $H_4 \cap H_5 \cong L_3(2)$ and $H_4 \cap H_8 \cong 7{:}6$ contain elements of order 7. We obtained that $H_2 \cap H_8$ and $H_4 \cap H_8$ will not have any contributions here since their relevant structure constants are all zero.

By Table 2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{2B} + d_{3C} + d_{7A} = 4 + 8 + 12 = 24 < 2 \times 15$. By applying Scott's Theorem [31], we conclude that G is not $(2B, 3C, 7A)$-generated.

We now investigate the $(2D, 3C, 7A)$ generation of G. By the same Table 2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{2D}+d_{3C}+d_{7A} = 8+8+12 = 28 < 2 \times 15$. Again by applying Scott's Theorem, we conclude that G is not $(2D, 3C, 7A)$-generated.

(ii) By Table 9 we have $\Delta_G(2D, 3B, 7A) = 7$. Although H_4 and H_8 are the only subgroups meeting the $2D$, $3B$, $7A$ classes of G, none of them will have any contributions because their relevant structure constants are all zero. We then obtained $\Delta^*_G(2D, 3B, 7A) = 7 > 0$, proving that G is $(2D, 3B, 7A)$-generated. □

4.1.2. $(2, 5, r)$-*generations.* The condition $\frac{1}{2} + \frac{1}{5} + \frac{1}{r} < 1$ shows that r must be 5 or 7. Thus we have to consider the cases $(2X, 5A, 5A)$ and $(2X, 5A, 7A)$, for $X \in \{A, B, C, D\}$.

Proposition 4.2. *The group G is not a $(2X, 5A, 5A)$-generated group for all $X \in \{A, B, C, D\}$.*

Proof. If G is a $(2X, 5A, 5A)$-generated group, then we must have $c_{2X} + c_{5A} + c_{5A} \leq 30$. Since by Table 3 we have $c_{2X} \in \{16, 18, 22\}$, it follows that $c_{2X} + c_{5A} + c_{5A} = c_{2X} + 8 + 8 > 30$ for any $X \in \{A, B, C, D\}$ and by Ree's Theorem [30] we conclude that G is not $(2X, 5A, 5A)$-generated group, for all $X \in \{A, B, C, D\}$. □

Proposition 4.3. *The group G is*
(i) not $(2X, 5A, 7A)$-generated for $X \in \{A, B, C\}$,
(ii) $(2D, 5A, 7A)$-generated.

Proof. (i) By Table 6 we see that $\Delta_G(2A, 5A, 7A) = 0$, it follows that G is not $(2A, 5A, 7A)$-generated.

We prove that G is not $(2B, 5A, 7A)$-generated. By Proposition 4.1, we see that the groups $2^3{:}L_3(2)$, $L_3(2){:}2$, $L_3(2)$ and 7:6 have elements of order 7 and no one will contribute here because none have elements of order 5. By Table 4 we noticed that H_2 is the only maximal subgroup containing elements of orders 2, 5 and 7. We obtained that $\Sigma_{H_2}(2b, 5a, 7a) = 7$ and $h(7A, H_2) = 1$. Since by Table 7 we have $\Delta_G(2B, 5A, 7A) = 7$, it follows that

$$\Delta^*_G(2B, 5A, 7A) = \Delta_G(2B, 5A, 7A) - \sum_{H_2}(2b, 5a, 7a) = 7 - 7 = 0,$$

proving that G is not $(2B, 5A, 7A)$-generated.

(ii) As stated ealier, only H_2 will have a contribution because it contains elements of orders 2, 5 and 7. For the case $(2C, 5A, 7A)$, we obtained that $\Sigma_{H_2}(2c, 5a, 7a) = 7$ and by Table 8, we have $\Delta_G(2C, 5A, 7A) = 7$ so that

$$\Delta^*_G(2C, 5A, 7A) = \Delta_G(2C, 5A, 7A) - \Sigma_{H_2}(2c, 5a, 7a) = 14 - 7 = 7 > 0,$$

proving that G is $(2C, 5A, 7A)$-generated.

By Table 9 we have $\Delta_G(2D, 5A, 7A) = 98$. Although H_2 is the only maximal subgroup meeting the $2D$, $5A$, $7A$ classes of G, it will not have any contribution since its relevant structure constant is zero. We then obtained that $\Delta^*_G(2D, 5A, 7A) = \Delta_G(2D, 5A, 7A) = 98 > 0$, proving that G is $(2D, 5A, 7A)$-generated. □

4.1.3. *$(2, 7, r)$-generations.* We have to check the generation of G through the triples $(2A, 7A, 7A)$, $(2B, 7A, 7A)$, $(2C, 7A, 7A)$ and $(2D, 7A, 7A)$.

Proposition 4.4. *The group G is*
(i) not $(2X, 7A, 7A)$-generated for $X \in \{A, B\}$
(ii) $(2Y, 7A, 7A)$-generated for $Y \in \{C, D\}$.

Proof. (i) As in Proposition 4.1, subgroups H_2, H_4, H_5, H_8, $H_2 \cap H_4$, $H_2 \cap H_8$, $H_4 \cap H_5$ and $H_4 \cap H_8$ are the only ones having elements of order 7.

By Table 6 we have $\Delta_G(2A, 7A, 7A) = 14$. Out of all the subgroups having elements of order 7, only H_2 and H_5 meet the $2A$, $7A$ classes of G. The maximal subgroup H_2 will not have any contribution here since its relevant structure constant is zero. We obtained that

$\Sigma_{H_5}(2a, 7x, 7x) = \Delta_{H_5}(2a, 7a, 7a) + \Delta_{H_5}(2a, 7a, 7b) + \Delta_{H_5}(2a, 7b, 7b) = 7 + 0 + 7 = 14.$, Since the value of h for each of these contributing subgroups is 1, we then obtain that $\Delta^*_G(2A, 7A, 7A) = \Delta_G(2A, 7A, 7A) - \Sigma_{H_5}(2a, 7x, 7x) = 14 - 14 = 0$, proving that G is not $(2A, 7A, 7A)$-generated.

By Table 2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{2B} + d_{7A} + d_{7A} = 4 + 12 + 12 = 28 < 2 \times 15$. By applying Scott's Theorem, we conclude that G is not $(2B, 7A, 7A)$-generated.

(ii) By Table 8 we have $\Delta_G(2C, 7A, 7A) = 210$. Of all the subgroups of G having elements of order 7, only H_2, H_5, $H_2 \cap H_4$ and $H_4 \cap H_5$ meet the $2C$, $7A$ classes of G. The subgroup $H_4 \cap H_5$ will not have any contributions since its relevant structure constant is zero. We obtained that $\Sigma_{H_2}(2a, 7a, 7a) = 70$, $\sum_{H_5}(2x, 7y, 7z) = \Delta_{H_5}(2b, 7a, 7a) + \Delta_{H_5}(2b, 7a, 7b) + \Delta_{H_5}(2b, 7b, 7b) + \Delta_{H_5}(2e, 7a, 7a) + \Delta_{H_5}(2e, 7a, 7b) + \Delta_{H_5}(2e, 7b, 7b) = 21 + 0 + 21 + 0 + 28 + 0 = 70$ and $\sum_{H_2 \cap H_4}(2b, 7a, 7a) = 7$. The value of h for all contributing subgroups is 1. We then get that

$$\begin{aligned}
\Delta^*_G(2C, 7A, 7A) &= \Delta_G(2C, 7A, 7A) - \sum_{H_2}(2a, 7a, 7a) \\
&- \sum_{H_5}(2x, 7y, 7z) + \sum_{H_2 \cap H_4}(2b, 7a, 7a) \\
&= 210 - 70 - 70 + 7 = 77 > 0,
\end{aligned}$$

proving that G is $(2C, 7A, 7A)$-generated.

By Table 9 we have $\Delta_G(2D, 7A, 7A) = 560$. Although H_2, H_4, H_5 and H_8 are the only subgroups meeting the $2D$, $7A$ classes of G, only H_5 and H_8 have contributions since the relevant structure constants of H_2 and H_4 are all zero. We obtained that $\sum_{H_5}(2x, 7y, 7z) = \Delta_{H_5}(2c, 7a, 7a) + \Delta_{H_5}(2c, 7a, 7b) + \Delta_{H_5}(2c, 7b, 7b) + \Delta_{H_5}(2f, 7a, 7a) + \Delta_{H_5}(2f, 7a, 7b) + \Delta_{H_5}(2f, 7b, 7b) = 21 + 0 + 21 + 0 + 28 + 0 = 70$ and $\sum_{H_8}(2a, 7a, 7a) = 28$. The value of h for all contributing subgroups is 1. We then get

$$\begin{aligned}
\Delta^*_G(2D, 7A, 7A) &= \Delta_G(2D, 7A, 7A) - \sum_{H_5}(2x, 7y, 7z) - \sum_{H_8}(2a, 7a, 7a) \\
&= 560 - 112 - 28 = 420 > 0,
\end{aligned}$$

proving that G is $(2D, 7A, 7A)$-generated. $\square$

4.2. $(3, q, r)$**-generations.** The condition $\frac{1}{3} + \frac{1}{3} + \frac{1}{r} < 1$ shows that r must be 5 or 7. For the $(3, q, r)$-generations, we end up having the following cases: $(3X, 3Y, 5A)$-, $(3X, 3Y, 7A)$-, $(3X, 5A, 5A)$-, $(3X, 5A, 7A)$- and $(3X, 7A, 7A)$-
generations.

4.2.1. $(3, 3, r)$-*generations.*

Proposition 4.5. *The group G is not $(3X, 3Y, 5A)$-generated group for all $X, Y \in \{A, B, C\}$.*

Proof. The group G acts on a 7-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem [31] applied to the module $\mathbb{V}$ and using the Atlas of finite groups [14], we get:

$$\begin{aligned} d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) &= \frac{2(7-4)}{3} = 2, \\ d_{3B} = \dim(\mathbb{V}/C_{\mathbb{V}}(3B)) &= \frac{2(7+2)}{3} = 6. \\ d_{3C} = \dim(\mathbb{V}/C_{\mathbb{V}}(3C)) &= \frac{2(7-1)}{3} = 4, \\ d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) &= \frac{4(7-2)}{5} = 4. \end{aligned}$$

Since $d_{3X} \in \{2, 4, 6\}$ above, it follows that $d_{3A} + d_{3X} + d_{5A} < 14$ and by Scott's Theorem G is not $(3A, 3X, 5A)$-generated for all $X \in \{A, B, C\}$. Again by Scott's Theorem, G is not $(3C, 3C, 5A)$-generated because $d_{3C} + d_{3C} + d_{5A} = 12 < 14$. By Table 11 we see that $\Delta_G(3B, 3B, 5A) = \Delta_G(3B, 3C, 5A) = 10 < 30 = |C_G(5A)|$, proving that G is not $(3B, 3X, 5A)$-generated for $X \in \{B, C\}$. □

Proposition 4.6. *The group G is not $(3X, 3Y, 7A)$-generated for $X \in \{A, B, C\}$.*

Proof. By Table 10 we have $\Delta_G(3A, 3A, 7A) = \Delta_G(3A, 3B, 7A) = 0$, it follows that G is not $(3A, 3X, 7A)$-generated for all $X \in \{A, B\}$. As in Proposition 4.1, subgroups H_2, H_4, H_5, H_8, $H_2 \cap H_4$, $H_2 \cap H_8$, $H_4 \cap H_5$ and $H_4 \cap H_8$ are the only ones having elements of order 7.

Again, by Table 10 we have $\Delta_G(3A, 3C, 7A) = 7$. The maximal subgroup H_2 is the only one meeting the classes $3A$, $3C$ and $7A$ of G. We obtained that $\sum_{H_2}(3a, 3b, 7a) = 7$ and we have $h(7A, H_2) = 1$. We obtain $\Delta^*_G(3A, 3C, 7A) = 7 - 7 = 0$, proving that the group G is not $(3A, 3C, 7A)$-generated.

By Table 11 we have $\Delta_G(3B, 3B, 7A) = 7$. Although the maximal subgroups H_4 and H_8 are the only ones meeting the $3B$, $7A$ classes of G, the maximal subgroup H_4 will not contribution because its relevant structure constant is zero. Although the intersection of H_4 and H_4 contains an element of order 7, it will not have any contributions because none of its element of order 3 fuses to $3B$. We obtained that $\sum_{H_8}(3c, 3c, 7a) = 7$ and we have $h(7A, H_8) = 1$. Thus we obtain

$$\Delta^*_G(3B, 3B, 7A) = \Delta_G(3B, 3B, 7A) - \sum_{H_8}(3c, 3c, 7a) = 7 - 7 = 0,$$

proving that G is not $(3B, 3B, 7A)$-generated.

By Table 11 we have $\Delta_G(3B, 3C, 7A) = 7$. Although H_4 and H_8 meet the $3B$, $3C$, $7A$ classes of G, only H_4 has a contribution because the relevant structure constant of H_8 is zero. We obtained that $\sum_{H_4}(3a, 3b, 7a) = 7$. Thus we obtain

$$\Delta_G^*(3B, 3C, 7A) = \Delta_G(3B, 3C, 7A) - \sum_{H_4}(3b, 3a, 7a) = 7 - 7 = 0,$$

proving that G is not $(3B, 3C, 7A)$-generated.

By Table 2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{3C} + d_{3C} + d_{7A} = 8 + 8 + 12 = 28 < 2 \times 15$. By applying Scott's Theorem, we conclude that G is not $(3C, 3C, 7A)$-generated. $\square$

4.2.2. $(3, 5, r)$-*generations.*

Proposition 4.7. *The group G is*

(i) not $(3X, 5A, 5A)$-generated for all $X \in \{A, B\}$,

(ii) $(3C, 5A, 5A)$-generated.

Proof. (i) If G is a $(3A, 5A, 5A)$-generated group, then we must have $c_{3A} + c_{5A} + c_{5A} \leq 30$. Since by Table 3 we have $c_{3A} + c_{5A} + c_{5A} = 16 + 8 + 8 = 32 > 30$ and by Ree's Theorem we conclude that G is not $(3A, 5A, 5A)$-generated group.

From Table 5 we see that only four maximal subgroups of G, namely H_1, H_2, H_3 and H_7, that each one has elements of order 5. The intersection of these four maximal subgroups contains the identity element and the intersection of any three maximal subgroups does not contain an element of order 5. The subgroups $H_1 \cap H_7 \cong ((S_3 \times S_3{:}2) \times S_3)$, $H_2 \cap H_3 \cong (S_4 \times S_4){:}2$, $H_2 \cap H_7 \cong 2 \times ((S_3 \times S_3){:}2)$ and $H_3 \cap H_7 \cong 2 \times 2 \times 4$ will not have any contributions because they do not contain elements of order 5. Thus, subgroups H_1, H_2, H_3, H_7, $H_1 \cap H_2 \cong 2 \times S_6$ and $H_1 \cap H_3 \cong 2 \times S_6$ are the only ones containing elements of order 5. We have $h(5A, H_1) = h(5A, H_3) = h(5A, H_1 \cap H_2) = h(5A, H_1 \cap H_3) = 3$ and $h(5A, H_2)H_2 = h(5A, H_7) = 1$.

By Table 11 we have $\Delta_G(3B, 5A, 5A) = 30$. Out of the above subgroups having elements of order 5, only H_1 and H_7 meet the $3B$, $5A$ classes of G. The maximal subgroup H_7 has no contributions because its structure constant is zero. We obtained that $\sum_{H_1}(3c, 5a, 5a) = 10$. We obtain

$$\Delta_G^*(3B, 5A, 5A) = \Delta_G(3B, 5A, 5A) - 3 \cdot \sum_{H_1}(3c, 5a, 5a) = 30 - 3(10) = 0,$$

proving that G is not $(3B, 5A, 5A)$-generated.

(ii) By Table 12 we have $\Delta_G(3C, 5A, 5A) = 690$. Out of the above subgroups having elements of order 5, only H_1, H_2, H_7, $H_1 \cap H_2$ and $H_1 \cap H_7$ meet the $3C$, $5A$ classes of G. We obtained that $\sum_{H_1}(3b, 5a, 5a) = 105$, $\sum_{H_2}(3b, 5a, 5a) = 135$, $\sum_{H_7}(3x, 5a, 5a) = \Delta_{H_7}(3b, 5a, 5a) + \Delta_{H_7}(3d, 5a, 5a) = 0 + 15 = 15$ and $\sum_{H_1 \cap H_7}(3b, 5a, 5a) = 15$. We get

$$\begin{aligned} \Delta^*_G(3C, 5A, 5A) &= \Delta_G(3C, 5A, 5A) - 3 \cdot \sum_{H_1}(3b, 5a, 5a) - \sum_{H_2}(3b, 5a, 5a) \\ &- \sum_{H_7}(3x, 5a, 5a) + 3 \cdot \sum_{H_1 \cap H_7}(3a, 5b, 5a) \\ &= 690 - 3(105) - 135 - 15 + 3(15) = 270 > 0, \end{aligned}$$

proving (ii). □

Proposition 4.8. *The group G is*
(i) not $(3A, 5A, 7A)$-generated
(ii) $(3X, 5A, 7A)$-generated, where $X \in \{B, C\}$.

Proof. By Table 10 we have $\Delta_G(3A, 5A, 7A) = 7$, From Table 5, we see that H_2 is the only maximal subgroup of G having elements of orders 3, 5 and 7. We obtained that $\sum_{H_2}(3a, 5a, 7a) = 7$ and we have $h(7A, H_2) = 1$. We then obtain

$$\Delta^*_G(3A, 5A, 7A) = \Delta_G(3A, 5A, 7A) - \sum_{H_2}(3a, 5a, 7a) = 7 - 7 = 0,$$

proving that G is not $(3A, 5A, 7A)$-generated.
(ii) By Table 11 we have $\Delta_G(3B, 5A, 7A) = 77$. None of the elements of H_2 meet the $3B$, $5A$, $7A$ classes of G, so we obtained that $\Delta^*_G(3B, 5A, 7A) = \Delta_G(3B, 5A, 7A) = 77 > 0$, proving that G is $(3B, 5A, 7A)$-generated.
By Table 12 we have $\Delta_G(3C, 5A, 7A) = 441$. Although H_2 is the only maximal subgroup meeting the classes $3C$, $5A$ and $7A$ of G, it has no contribution because its relevant structure constant is zero. We obtain that $\Delta^*_G(3C, 5A, 7A) = \Delta_G(3C, 5A, 7A) = 441 > 0$. Thus, the group G is $(3C, 5A, 7A)$-generated. □

4.2.3. $(3, 7, r)$-*generations.* In this subsection we discuss the cases $(3, 7, r)$-generations. It follows that we will end up with 3 cases, namely $(3A, 7A, 7A)$-, $(3B, 7A, 7A)$- and $(3C, 7A, 7A)$-generation.

Proposition 4.9. *The group G is $(3X, 7A, 7A)$-generated for all $X \in \{A, B, C\}$.*

Proof. As in Proposition 4.1, subgroups $H_2,\ H_4,\ H_5,\ H_8,\ H_2 \cap H_4,\ H_2 \cap H_8,$ $H_4 \cap H_5$ and $H_4 \cap H_8$ are the only ones having elements of order 7.
By Table 10 we have $\Delta_G(3A, 7A, 7A) = 133$. Only H_2 has a contribution because it meets the $3A$, $7A$ classes of G. We obtained that $\sum_{H_2}(3a, 7a, 7a) = 42$ and $h(7A, H_2) = 1$. We then obtain that $\Delta^*_G(3A, 7A, 7A) = \Delta_G(3A, 7A, 7A) - \sum_{H_2}(3a, 7a, 7a) = 133 - 42 = 91 > 0$. This shows that G is $(3A, 7A, 7A)$-generated.

By Table 11 we have $\Delta_G(3B, 7A, 7A) = 245$. Out of all the subgroups of G having elements of order 7, only H_4 and H_8 meet the $3B$, $7A$ classes of G. The maximal subgroup H_8 has no contributions since its relevant structure constant is zero. We obtained that $\sum_{H_4}(3b, 7a, 7a) = 7$ and $h(7A, H_4) = 1$. We then obtain $\Delta^*_G(3B, 7A, 7A) = \Delta_G(3B, 7A, 7A) - \sum_{H_4}(3b, 7a, 7a) = 245 - 7 = 238 > 0$, proving that G is $(3B, 7A, 7A)$-generated.

By Table 12 we have $\Delta_G(3C, 7A, 7A) = 2289$. All these subgroups $H_2,\ H_4,\ H_5,\ H_8,$ $H_2{\cap}H_4,\ H_2{\cap}H_8,\ H_4{\cap}H_5$ and $H_4{\cap}H_8$ meet the $3C$, $7A$ classes of G. Although $H_2{\cap}H_8$ and $H_4{\cap}H_8$ meet the $3C$, $7A$ classes of G, they will not have any contributions because their relevant structure contants are all zero. We obtained that $\sum_{H_2}(3b, 7a, 7a) = 294$, $\sum_{H_4}(3a, 7a, 7a) = 189$, $\sum_{H_5}(3a, 7x, 7y) = \Delta_{H_5}(3a, 7a, 7a) + \Delta_{H_5}(3a, 7a, 7b) + \Delta_{H_5}(3a, 7b, 7b) = 112 + 112 + 112 = 336$, $\sum_{H_8}(3a, 7a, 7a) = 21$, $\sum_{H_2\cap H_4}(3a, 7a, 7a) = 14$ and $\sum_{H_4\cap H_5}(3a, 7x, 7y) = \Delta_{H_4\cap H_5}(3a, 7a, 7a) + \Delta_{H_4\cap H_5}(3a, 7a, 7b) + \Delta_{H_4\cap H_5}(3a, 7b, 7b) = 7 + 7 + 7 = 21$. The value of h for all contributing subgroups is 1. We then get

$$\begin{aligned}
\Delta^*_G(3C, 7A, 7A) &= \Delta_G(3C, 7A, 7A) - \sum_{H_2}(3b, 7a, 7a) - \sum_{H_4}(3a, 7a, 7a) \\
&- \sum_{H_5}(3a, 7x, 7y) - \sum_{H_8}(3a, 7a, 7a) + \sum_{H_2\cap H_4}(3a, 7a, 7a) \\
&+ \sum_{H_2\cap H_4}(3b, 3a, 7a) + \sum_{H_4\cap H_5}(3a, 7x, 7y) \\
&= 2289 - 294 - 189 - 336 - 21 + 14 + 21 = 1484 > 0.
\end{aligned}$$

Hence G is $(3C, 7A, 7A)$-generated. □

4.3. **Other results.** In this subsection we handle all the remaining cases, namely the $(5, q, r)$- and $(7, q, r)$-generations. This will end up with four cases, namely $(5A, 5A, 5A)$-, $(5A, 5A, 7A)$-, $(5A, 7A, 7A)$- and $(7A, 7A, 7A)$-generation.

4.3.1. $(5, 5, r)$**-generations.** We have to check the generation of G through the triples $(5A, 5A, 5A)$ and $(5A, 5A, 7A)$.

Proposition 4.10. *The group G is not $(5A, 5A, 5A)$-generated.*

Proof. By Table 1, the group G acts on a 7-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{5A} + d_{5A} + d_{5A} + d_{5A} = 3 \times 4 < 2 \times 7$. By applying Scott's Theorem, we conclude that G is not $(5A, 5A, 5A)$-generated. □

Proposition 4.11. *The group G is $(5A, 5A, 7A)$-generated.*

Proof. By Proposition 4.3, G is $(2C, 5A, 7A)$-generated. It follows by Theorem 2.6 that G is $(5A, 5A, (7A)^2)$-generated. Since G has one class of order 7, we must have $(7A)^2 = 7A$. The group G will become $(5A, 5A, 7A)$-generated. □

4.3.2. $(5, 7, r)$**- and** $(7, 7, r)$**-generations.**

Proposition 4.12. *The group G is $(5A, 7A, 7A)$-generated.*

Proof. By Table 5, we see that H_2 is the only subgroup of G having elements of orders 5 and 7. We then obtained that $\sum_{H_2}(5a, 7a, 7a) = 91$. Since by Table 13 we have $\Delta_G(5A, 7A, 7A) = 7483 - 91 = 7392 > 0$. Hence the group G is $(5A, 7A, 7A)$-generated. □

We conclude our investigation on the (p, q, r)-generation of the symplectic group $Sp(6, 2)$ by considering the $(7A, 7A, 7A)$-generations.

Proposition 4.13. *The group G is $(7A, 7A, 7A)$-generated.*

Proof. By Proposition 4.4, G is $(2C, 7A, 7A)$-generated. By the application of Theorem 2.6, it follows that G is $(7A, 7A, (7A)^2)$-generated. Since $(7A)^2 = 7A$, the group G becomes $(7A, 7A, 7A)$-generated. □

The following tables, namely Tables 6 to 13 give the partial structure constants of $Sp(6, 2)$ computed by Gap [20] that will be used in our calculations.

TABLE 6. Structure constants $\Delta_{Sp(6,2)}(2A, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2A, 2A, pX)$	0	0	2	0	3	0	0	0	0
$\Delta_G(2A, 2B, pX)$	0	0	1	1	0	0	0	0	0
$\Delta_G(2A, 2C, pX)$	30	3	0	3	0	0	0	0	0
$\Delta_G(2A, 2D, pX)$	0	12	12	3	0	0	0	0	0
$\Delta_G(2A, 3A, pX)$	32	0	0	0	0	0	0	0	0
$\Delta_G(2A, 3B, pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A, 3C, pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A, 5A, pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A, 7A, pX)$	0	0	0	0	0	0	0	0	14
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

TABLE 7. Structure constants $\Delta_{Sp(6,2)}(2B, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2B, 2B, pX)$	0	18	8	0	0	0	3	0	0
$\Delta_G(2B, 2C, pX)$	15	24	6	3	0	0	0	0	0
$\Delta_G(2B, 2D, pX)$	60	0	12	15	0	0	0	0	0
$\Delta_G(2B, 3A, pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2B, 3B, pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2B, 3C, pX)$	0	128	0	0	0	0	27	0	7
$\Delta_G(2B, 5A, pX)$	0	0	0	0	0	0	0	15	7
$\Delta_G(2B, 7A, pX)$	0	0	0	0	0	0	108	30	70
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

TABLE 8. Structure constants $\Delta_{Sp(6,2)}(2C, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2C, 2C, pX)$	0	18	34	6	45	0	9	5	0
$\Delta_G(2C, 2D, pX)$	180	36	24	21	0	0	0	0	0
$\Delta_G(2C, 3A, pX)$	0	0	32	0	45	0	0	5	0
$\Delta_G(2C, 3B, pX)$	0	0	0	0	0	27	0	0	0
$\Delta_G(2C, 3C, pX)$	0	0	128	0	0	0	54	20	14
$\Delta_G(2C, 5A, pX)$	0	0	256	0	360	0	72	140	14
$\Delta_G(2C, 7A, pX)$	0	0	0	0	0	0	216	60	210
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

TABLE 9. Structure constants $\Delta_{Sp(6,2)}(2D, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2D, 2D, pX)$	180	180	84	12	180	108	36	15	7
$\Delta_G(2D, 3A, pX)$	0	0	0	32	0	0	0	0	0
$\Delta_G(2D, 3B, pX)$	0	0	0	64	0	0	0	0	7
$\Delta_G(2D, 3C, pX)$	0	0	0	128	0	0	54	30	28
$\Delta_G(2D, 5A, pX)$	0	0	0	192	0	0	108	90	98
$\Delta_G(2D, 7A, pX)$	0	0	0	384	0	648	432	420	560
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

TABLE 10. Structure constants $\Delta_{Sp(6,2)}(3A, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3A, 3A, pX)$	0	0	32	0	46	0	2	5	0
$\Delta_G(3A, 3B, pX)$	0	0	0	0	0	12	2	0	0
$\Delta_G(3A, 3C, pX)$	0	0	0	0	40	12	20	20	7
$\Delta_G(3A, 5A, pX)$	0	0	256	0	360	0	72	120	7
$\Delta_G(3A, 7A, pX)$	0	0	0	0	0	0	108	30	133
$\lvert C_G(pX)\rvert$	23040	4608	1536	384	2160	648	108	30	7

TABLE 11. Structure constants $\Delta_{Sp(6,2)}(3B, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3B, 3B, pX)$	0	0	64	0	40	28	20	10	7
$\Delta_G(3B, 3C, pX)$	0	0	0	0	40	120	4	10	7
$\Delta_G(3B, 5A, pX)$	0	0	0	0	0	216	36	30	77
$\Delta_G(3B, 7A, pX)$	0	0	0	384	0	648	108	330	245
$\lvert C_G(pX)\rvert$	23040	4608	1536	384	2160	648	108	30	7

TABLE 12. Structure constants $\Delta_{Sp(6,2)}(3C, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3C, 3C, pX)$	0	1152	768	192	400	24	500	150	203
$\Delta_G(3C, 5A, pX)$	0	0	1024	384	1440	216	540	690	441
$\Delta_G(3C, 7A, pX)$	0	4608	3072	1536	2160	648	3132	1890	2289
$\lvert C_G(pX)\rvert$	23040	4608	1536	384	2160	648	108	30	7

TABLE 13. Structure constants $\Delta_{Sp(6,2)}(5A, qY, rZ)$ and $\Delta_{Sp(6,2)}(7A, 7A, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(5A, 5A, pX)$	0	2304	7168	1152	8640	648	2484	3998	1379
$\Delta_G(5A, 7A, pX)$	0	4608	3072	5376	2160	7128	6804	5910	7483
$\Delta_G(7A, 7A, pX)$	46080	46080	46080	30720	41040	22680	35316	32070	30595
$\lvert C_G(pX)\rvert$	23040	4608	1536	384	2160	648	108	30	7

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References

[1] F. Ali, M. A. F. Ibrahim and A. Woldar, *(3, q, r)-Generations of Fischer sporadic group $Fi_{24}^{'}$*, J. of Group Theory, **22** (2019) 453-489.

[2] F. Ali, M. A. F. Ibrahim and A. Woldar, *On (2, 3)-generations of Fischer largest sporadic simple group $Fi_{24}^{'}$*, C. R. Acad. Sci. Paris, Ser. I, **357** (2019) 401-412.

[3] A. R. Ashraffi, *Generating pairs for the Held group He*, J. Appl. Math. and Computing, **10** (2002) 167-174.

[4] A. R. Ashraffi, *(p,q,r)-generations of the sporadic group HN*, Taiwanese J. Math, **10** No. 3 (2006) 613-629.

[5] A. B. M. Basheer and T. T. Seretlo, *The (p, q, r)-generations of the alternating group A_{10}*. Quaestiones Mathematicae, **43** No. 3 (2020) 395-408.

[6] A. B. M. Basheer and T. T. Seretlo, *On two generation methods for the simple linear group $PSL(3, 5)$* Khayyam J. Math. **5** No. 1 (2019) 125-139.

[7] A. B. M. Basheer and J. Moori, *A survey on some methods of generating finite simple groups.* London Mathematical Society Lecture Note Series, published as a chapter of the book of Groups St Andrews 2017 by Cambridge University Press, **455** (2019) 106-118.

[8] A. B. M. Basheer and T. T. Seretlo, *(p,q,r)-generations of the Mathieu group M_{22}* Southeast Asian Bull. Math, (accepted) (2020).

[9] A. B. M. Basheer, *The ranks of the classes of A_{10}*. Bulletin of the Iranian Mathematical Society, **43** No. 7 (2017) 2125-2135.

[10] A. B. M. Basheer and J. Moori, *On the ranks of finite simple groups*, Khayyam J. Math., **2** No. 1 (2016) 18-24.

[11] A. B. M. Basheer, J. Motalane and T. T. Seretlo, *The (p, q, r)-generations of the alternating group A_{11}*. Accepted by the Khayyam Journal of Mathematics.

[12] A. B. M. Basheer, J. Motalane and T. T. Seretlo, *The (p,q,r)-generations of the Mathieu group M_{23}*, submitted.

[13] M. D. E. Conder, *Some results on quotients of triangle groups*, Bull. Austr Math Soc., **30** (1984) 167-174.

[14] J. H. Conway et al., *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[15] M. R. Darafsheh, A. R. Ashrafi, *Generating pairs for the sporadic group Ru*, J. Appl. Math. Comput., **12** No. 1-2 (2003) 143-154.

[16] M. R. Darafsheh, A. R. Ashrafi and G.A. Moghani, *(p,q,r)-generations and nX-complementary generations of the sporadic group Ly*, Kumamoto J. Math., **16** (2003) 13-25.

[17] M. R. Darafsheh, A. R. Ashrafi and G.A. Moghani, *(p, q, r)-generations of the Conway group Co_1, for odd p*, Kumamoto J. Math., **14** (2001) 1-20.

[18] M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, *(p,q,r)-generations of the sporadic group O'N*, Groups St. Andrews 2001 in Oxford. Vol. I

[19] S. Ganief, *2-Generations of the Sporadic Simple Groups*, PhD thesis, University of KwaZulu-Natal, Pietermaritzburg, 1997.

[20] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.10.2;* 2019. (http://www.gap-system.org)

[21] S. Ganief and J. Moori, *2-generations of the fourth Janko group J_4*, J. Algebra, **212** No. 1 (1999) 305-322.

[22] S. Ganief and J. Moori, *2-generations of the smallest Fischer group Fi_{22}*, Nova J. Math. Game Theory Algebra, **6** No. 2-3 (1997) 127-145.

[23] S. Ganief and J. Moori, *Generating pairs for the Conway groups Co_2 and Co_3*, J. Group Theory, **1** No. 3 (1998) 237-256.

[24] S. Ganeif and J. Moori, *(p,q,r)-generations and nX-complementary generations of the sporadic groups HS and McL.*, J. Algebra **188** No. 2 (1997) 531-546.

[25] S. Ganief and J. Moori, *(p,q,r)-generations of the smallest Conway group* Co_3, J. Algebra, **188** No. 2 (1997) 516-530.

[26] L. Di Martino, M. Tamburini and A. Zalesskii, *On Hurwitz groups of low rank*, J. Algebra, **28** (2000) 5383-5404.

[27] L. Di Martino, M. Pellegrini and A. Zalesski, *On generators and representations of the sporadic simple groups*, Comm. Algebra, **42** (2014) 880-908.

[28] J. Moori, $(2,3,p)$-generations for the Fischer group F_{22}, Comm. Algebra, **2** No. 11 (1994) 4597-4610.

[29] J. Moori, (p,q,r)-generations for the Janko groups J_1 and J_2, Nova J. Algebra and Geometry, **2** No. 3 (1993) 277-285.

[30] R. Ree, *A theorem on permutations*, J. Comb. Theory A, **10** (1971) 174-175.

[31] L. L. Scott, *Matrices and cohomolgy*, Ann. Math, **105** No. 3 (1977) 67-76.

[32] J. Ward, *Generation of Simple Groups by Conjugate Involutions*, PhD Thesis, University of London, 2009.

[33] R. A. Wilson et al., *Atlas of finite group representations*, (http://brauer.maths.qmul.ac.uk/Atlas/v3/)

[34] A. J. Wodlar, *Sporadic simple groups which are Hurwitz*, J. Algebra, **144** (1991) 443-450.

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