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Research Paper

# DIRECTED PRIME GRAPH OF NON-COMMUTATIVE RING

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ABSTRACT. Prime graph of a ring R is a graph whose vertex set is the whole set R any any two elements $x$ and $y$ of $R$ are adjacent in the graph if and only if $xRy = 0$ or $yRx = 0$. Prime graph of a ring is denoted by $PG(R)$. Directed prime graphs for non-commutative rings and connectivity in the graph are studied in the present paper. The diameter and girth of this graph are also studied in the paper.

## 1. INTRODUCTION

The study of rings with the help of graphs began when Beck [6] defined a graph of a commutative ring. It has drawn more attention when it was modified by Anderson and Livingston[3] and named as zero divisor graph. S. P. Redmond [13] extended this concept to non-commutative rings. Zero divisor graphs of non-commutative rings are further studied

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in [2, 5, 11, 12]. The concept of prime graph, introduced by Satyanarayana et al[14], is another graph associated with rings. Prime graph of a ring R is defined as a graph whose vertex set is the whole ring and any two distinct vertices $x$ and $y$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. Prime graph defined in [14] is a simple graph corresponding to any ring. In their paper, they investigated some basic properties of the prime graph.

Before moving to the next sections we mention definitions of some basic terms used in this paper which are also available in standard literature. A *simple graph (directed graph) G* consists of a non-empty finite set $V(G)$ of objects called vertices together with a set $E(G)$, possibly empty, of *unordered (ordered)* pairs of distinct vertices of $G$ called *edges (arcs)*. If for some ring $R$ the vertex set of $PG(R)$ becomes empty we termed the graph as an empty graph.

A graph G whose edge set $E(G) = \varphi$ is called a null graph or totally disconnected graph. A simple graph $G$ is said to be connected if for any two vertices there is a path joining them otherwise $G$ is said to be disconnected.

In a directed graph a vertex $v$ is said to be reachable from $u$ if there exists a directed path from $u$ to $v$.

A digraph is said to be *strongly* connected or *strong* if any two distinct vertices $u$ and $v$ are mutually reachable from each other. A digraph is said to be *unilaterally* connected or *unilateral* if for any two distinct vertices $u$ and $v$ at least one of them is reachable from the other. A digraph is said to be *weakly* connected or *weak* if there is at least one *semipath* (a path in which the arcs are not in the same direction) joining every pair of distinct vertices. For a simple graph, the number of edges incident with a vertex $v$ is called the *degree* of the vertex and it is denoted by *deg(v)*. For a digraph, two types of degrees are defined. For a vertex $v$ of a digraph $G$, the number of arcs incident from it is called *outdegree* of $v$ denoted by *od(v)*, and the number arcs incident to it is called *indegree* of $v$ denoted by *id(v)*. If for a vertex $v$ in a directed graph, $id(v) = 0$ and $od(v) \neq 0$, $v$ is called a *source* and if $od(v) = 0$ and $id(v) \neq 0$, $v$ is called a *sink*. The shortest $u - v$ path in a graph $G$ is called a *geodesic*. The *diameter* of a graph $G$ is the length of a longest geodesic in the graph. It is denoted by $d(G)$ or $Diam(G)$. The *girth* of a graph $G$ is the length of the smallest cycle(if it exists) in $G$. It is denoted by $g(G)$ or $girth(G)$. Girth is undefined if G has no cycle.

**Example 1.1.** Let $R = \{0, a, b, c\}$. If addition and multiplication are defined as follows

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $.$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $b$ | $0$ | $b$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $b$ | $0$ | $b$ |

then $R$ is a commutative ring. The prime graph of $R$ is given in Fig 1

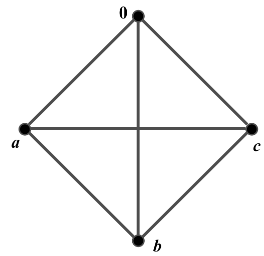



Fig 1 : Prime graph of commutative ring $R$

**Example 1.2.** Let $R = \{0, a, b, c\}$. If addition and multiplication are defined as follows

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | 0 | 0 | 0 |
| c | 0 | a | b | c |

then $R$ is a non-commutative ring. The prime graph of $R$ is given in Fig 2

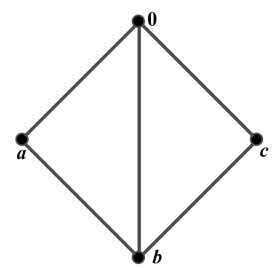



Fig 2 : Prime graph of non-commutative ring $R$

**Theorem 1.3.** [14] *For any ring R, $Diam(PG(R)) \leq 2$.*

*Proof.* By definition $0 \in R$ is adjacent to all $x \in R$. Therefore for any two distinct non zero elements $x,\ y \in R$, if $xRy \neq 0$ then $x - 0 - y$ is a path. So $Diam(PG(R)) = 2$. But if for all $x,\ y \in R$, $xRy = 0$ or $yRx = 0$ then $PG(R)$ is a complete graph and so $Diam(PG(R)) = 1$. Hence $Diam(PG(R)) \leq 2$. □

**Theorem 1.4.** *If R is a prime ring then $girth(PG(R)) = \infty$.*

*Proof.* Since $R$ is a prime ring, $PG(R)$ is a star graph *[14]*. So $PG(R)$ does not contain any cycle and so $girth(PG(R)) = \infty$. □

**Theorem 1.5.** *For a ring R, $girth(PG(R)) = 3$ or $\infty$ if R is not prime ring.*

*Proof.* Since $R$ is a prime ring, $PG(R)$ is a star graph *[14]*. So $PG(R)$ does not contain any cycle and so $girth(PG(R)) = \infty$. □

For the ring $R = \mathbb{Z}_4$, $2 \in \mathbb{Z}_4$ is the only element such that $2R2 = 0$ so $PG(\mathbb{Z}_4)$ is a star graph but $\mathbb{Z}_4$ is not prime ring.

**Corollary 1.6.** *Let $R$ be ring. If for all distinct elements $a, b \in R$, $aRb = 0$ or $bRa = 0$, then $Diam(PG(R)) = 1$ and $girth(PG(R)) = 3$.*

In the next section, we give a modified definition of prime graph of any ring $R$. In section three directed prime graphs of non-commutative rings are defined and some properties of these graphs are studied.

## 2. Prime Graph of Rings

In this section the earlier definition of prime graph is recast in new grab. In the new definition, for any ring $R$, the vertices of prime graph are the non-zero annihilators of $xR$ or $Rx$ for all $x \in R$ which exclude 0 element and all the elements which were adjacent to 0 in $PG(R)$. Here the diameter and girth are also determined for this graph.

**Definition 2.1.** Let $X$ be an element or a subset of a ring $R$, the right annihilator of $X$ in $R$ is defined by $Ann_r(X) = \{r \in R \ : \ Xr = (0)\}$. Similarly the left annihilator of $X$ in $R$ is defined by $Ann_l(X) = \{r \in R \ : \ rX = (0)\}$.

If the ring $R$ is commutative then $Ann_l(X) = Ann_r(X)$

**Definition 2.2.** Prime graph of a ring is defined as a graph whose vertex set is $V_0 = \{x \in R - \{0\} : x \in Ann_l(Ry) \ or \ x \in Ann_r(yR) \ for \ some \ y \in R - \{0\}\}$ and its edge set is $E_0 = \{(x, y) : x \in Ann_l(Ry) \ or \ x \in Ann_r(yR), x \neq y, x, y \in V\}$. We denote this graph as $PG_0(R)$.

**Observations:**

(i) $PG_0(R)$ is always a simple graph.

(ii) Vertex set of $PG_0(R)$ is empty if and only if $R$ is a prime ring.
If $R$ is a prime ring then there is no non-zero elements $x, y$ satisfying $x \in Ann_l(Ry)$ or $x \in Ann_r(yR)$. So the vertex set is empty.

(iii) If $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $PG_0(R)$ is a trivial graph.
For the ring $R = \mathbb{Z}_4$, $2 \in \mathbb{Z}_4$ is the only element such that $2 \in Ann_l(R2)$ and for the ring $R = \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $x \in R$ is the only element such that $x \in Ann_l(Rx)$. So the vertex sets contain only one element. Thus $PG_0(R)$ is a trivial graph.

(iv) If $R \cong \mathbb{Z}_6$ and $R \cong \mathbb{Z}_8$, then $PG_0(R)$ is a path.

(v) $R \cong \mathbb{Z}_{2p}$, where $p$ is an odd prime, then $PG_0(R)$ is a star graph.

(vi) If $R \cong \mathbb{Z}_{pq}$, where $p$ and $q$ are distinct odd primes, then $PG_0(R)$ is a complete bipartite graph.

(vii) If $R \cong \mathbb{Z}_{p^2}$, where $p$ is an odd prime, then $PG_0(R)$ is a complete graph.

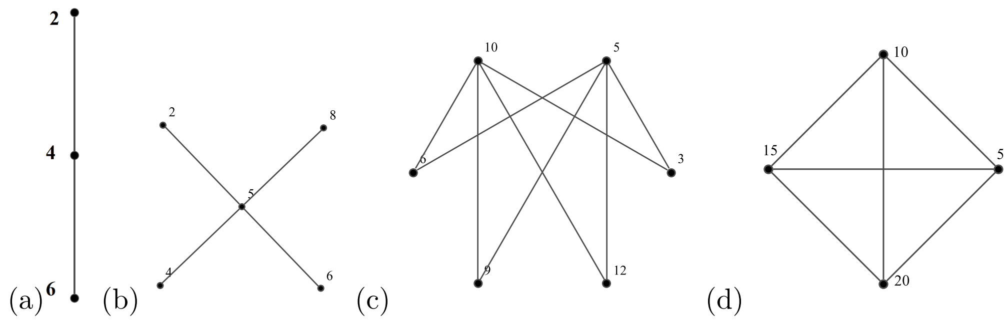



Fig 3: (a) $PG_0(\mathbb{Z}_8)$ (b) $PG_0(\mathbb{Z}_{10})$ (c) $PG_0(\mathbb{Z}_{15})$ (d) $PG_0(\mathbb{Z}_{25})$

**Theorem 2.3.** *For any ring $R$, the following holds for the graph $PG_0(R)$*

(i) *$PG_0(R)$ is connected.*

(ii) *$Diam(PG_0(R)) \leq 3$.*

(iii) *$girth(PG_0(R)) \leq 4$ or $\infty$.*

*Proof. (i)* Let $a,\ b \in V_0$ such that $(a,\ b) \notin E_0$ *i.e.* neither $a \notin Ann_l(Rx)$ nor $a \in Ann_r(xR)$, then there exist $a,\ b \in V_0$ such that $a \in Ann_l(Rx)$ or $a \in Ann_r(xR)$ and $b \in Ann_l(Ry)$ or $b \in Ann_r(yR)$. If $a \in Ann_l(Ry)$, $a \in Ann_r(yR)$, $b \in Ann_l(Rx)$ or $b \in Ann_r(xR)$ then $a - y - b$ or $a - x - b$ is a path. If not, then we have the following case: If $x = y$, then $a - x - b$ is a path. If $x \neq y$ and $(x,\ y) \in E_0$ then $a - x - y - b$ is a path. If $(x,\ y) \notin E_0$, then for some $r \in R$, $xry \neq 0$ and $yrx \neq 0$. Then $a - xry - b$, $a - yrx - b$, $a - xr'b - y - b$ or $a - x - ar'y - b$ is a path for some $r,\ r' \in R$. So $PG_0(R)$ is connected.

*(ii) $Diam(PG_0(R))$* $\geq 1$ as $PG_0(R)$ is connected.

Now let $a - b - c - d - e$ be a path of length 4 for distinct elements $a, b, c, d, e \in R$ with $a \notin Ann_l(Re)$ and $a \notin Ann_r(eR)$. We need to show that there exists a path from $a$ to $e$ of length $\leq 3$.

(a) If $a \in Ann_l(Rc)$ or $a \in Ann_r(cR)$ then $a - c - d - e$ is a path from $a$ to $e$ of length 3. If or $b \in Ann_r(dR)$ then $a - b - d - e$ is a path from $a$ to $e$ of length 3.

If $a \in Ann_l(Rd)$ or $a \in Ann_r(dR)$ then $a - d - e$ is a path from $a$ to $e$ of length 2.

If $b \in Ann_l(Re)$ or $b \in Ann_r(eR)$ then $a - b - e$ is a path from $a$ to $e$ of length 2. For $c \in Ann_l(Re)$ or $c \in Ann_r(eR)$, $a - c - e$ is a path from $a$ to $e$ of length 2 if $a \in Ann_l(Rc)$ or $a \in Ann_r(cR)$, otherwise $a - b - c - e$ is a path of length 3.

(b) If $b \notin Ann_l(Rd)$ and $b \notin Ann_r(dR)$ then $a - bd - e$ or $a - db - e$ is a path of length 2 or $a - be - d - e$ or $a - b - ad - e$ is a path of length 3.

Thus $Diam(PG_0(R)) \leq 3$.

*(iii)* If $PG_0(R)$ does not contain any cycle then $girth(PG_0(R)) = \infty$ which is supported by examples given in observations (iii), (iv), (v).

If $PG_0(R)$ consist any cycle then $girth(PG_0(R)) \geq 3$. We have to show that $girth(PG_0(R)) < 5$.

Let $a-b-c-d-e-a$ be a cycle of length 5 in $PG_0(R)$. If $a \in Ann_l(Rc)$ or $a \in Ann_l(Rd)$ or $b \in Ann_l(Rd)$ or $b \in Ann_l(Re)$ or $c \in Ann_l(Re)$ ($a \in Ann_r(cR)$ or $a \in Ann_r(dR)$ or $b \in Ann_r(dR)$ or $b \in Ann_r(eR)$ or $c \in Ann_r(eR)$) then we will get a cycle of length 3. If none of these occur then following cases may occur-

(a) If $a \in Ann_l(Rb)$, $b \in Ann_l(Rc)$, $c \in Ann_l(Rd)$, $d \in Ann_l(Re)$ and $e \in Ann_l(Ra)$ then $ac-d-e-ac$ is a cycle of length 3.

(b) If $a \in Ann_l(Rb)$, $b \in Ann_l(Rc)$, $c \in Ann_l(Rd)$, $d \in Ann_l(Re)$ and $e \in Ann_r(aR)$ then $a-bd-e-a$ is a cycle of length 3.

(c) If $a \in Ann_l(Rb)$, $b \in Ann_l(Rc)$, $c \in\in Ann_l(Rd)$, $d \in Ann_r(eR)$ and $e \in Ann_r(aR)$ then $a-ec-d-e-a$ is a cycle of length 4.

(d) If $a \in Ann_l(Rb)$, $b \in Ann_r(cR)$, $c \in Ann_l(Rd)$, $d \in Ann_r(eR)$ and $e \in Ann_l(Ra)$ then $a-be-c-b-a$ is a cycle of length 4.

Rests of the possibilities are similar to these cases.

It is observed that whenever $PG_0(R)$ consist of a cycle of length 5 it always contains a cycle of length 3 or 4. Hence $girth(PG_0(R)) \leq 4 \; or \; \infty$. □

## 3. Directed prime graph of non-commutative rings

The prime graph is defined [14] as a simple graph of any ring. But the non-commutative property of the rings is not reflected explicitly in this definition of prime graph. So the definition of prime graph further modified to get a flavour of the non-commutative property of the rings. Three different definitions of directed prime graph are given in this section, which is based on the vertex set of the graph taken. Various properties of these three graphs arising from these definitions are studied here in this paper. Also, the relation between these graphs are established.

**Definition 3.1.** The directed prime graph $PG^l_D(R)$ of a non-commutative ring is defined as a graph whose vertex set is $V^l = \{x \in R - \{0\} : x \in Ann_l(Ry) \; for \; some \; y \in R - \{0\}\}$ and its edge set is $E^l = \{(x, y) : x \in Ann_l(Ry), x \neq y, x, y \in V^l\}$ i.e. $x \to y$ is an edge if and only if $x \in Ann_l(Ry)$.

**Definition 3.2.** The directed prime graph $PG^r_D(R)$ of a non-commutative ring is defined as a graph whose vertex set is $V^r = \{x \in R - \{0\} : x \in Ann_r(yR) \; for \; some \; y \in R - \{0\}\}$ and its edge set is $E^r = \{(x, y) : x \in Ann_l(Ry), x \neq y, x, y \in V^r\}$ i.e. $x \to y$ is an edge if and only if $x \in Ann_l(Ry)$.

**Definition 3.3.** The directed prime graph $PG_D(R)$ of a non-commutative ring is defined as a graph whose vertex set is $V = \{x \in R - \{0\} : x \in Ann_l(Ry)\ or\ x \in Ann_r(yR)\ for\ some\ y \in R - \{0\}\}$ and $E = \{(x, y) : x \in Ann_l(Ry),\ x \neq y,\ x,\ y \in V\}$ i.e. $x \to y$ is an edge if and only if $x \in Ann_l(Ry)$.

It can easily be checked that $PG_D(R) = PG^l_D(R) \cup PG^r_D(R)$.

**Example 3.4.** Let $R = \{0,\ a,\ b,\ c\}$. We define addition and multiplication as follows

| $+$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 0 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 0 |

| . | 0 | $a$ | $b$ | $b$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $b$ | $c$ |
| $b$ | 0 | 0 | 0 | 0 |
| $c$ | 0 | $a$ | $b$ | $c$ |

Then $R$ is a non-commutative ring and the prime graph of $R$, $PG_D(R)$ is given in Fig 4.

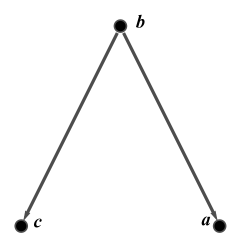



Fig 5 : Directed prime graph of commutative ring R

**Theorem 3.5.** *Let $R$ be a non-commutative ring. If $V^l \subseteq V^r$ then $PG^l_D(R)$ is strongly connected.*

*Proof.* Let $x, y \in V^l$, there exist $a, b \in R$ such that $x \in Ann_l(Ra)$ and $y \in Ann_l(Rb)$. For $x,\ y \in V^l$, let $(x,\ y) \in E^l$ i.e. $x \in Ann_l(Ry)$. Since $V^l \subseteq V^r$, for $x \in V^l \subseteq V^r$ there exist $c \in R$ such that $x \in Ann_r(cR)$, which implies that $c \in Ann_l(Rx)$ and $c \in V^l$. If $|V^l| = 2$ then $c = x$ or $c = y$. If $c = y$ then $(y,\ x) \in E^l$. If $c = x$ then $bc = bx \in R$ such that $bx \in Ann_l(Rx)$, that is $bx \in V^l$ and $y \to bx \to x$ is a path, but $bx = x$ or $bx = y$ i.e. $(y,\ x) \in E^l$. Let $|V^l| > 2$ and $c \neq x, y$. Then $y \to c \to x$ is a path if $y \in Ann_l(Rc)$ or $y \to bc \to x$ is a path. If $b \in V^l$, $y \to b \to x$ or $y \to b \to c \to x$ is a path according to $b \in Ann_l(Rx)$ or $b \in Ann_l(Rc)$.

Let $(x,\ y),\ (y,\ x) \notin E^l$. As $V^l \subseteq V^r$, there exist $c,\ d \in R$ such that $x \in Ann_r(cR)$ and $y \in Ann_r(dR)$ and so $c \in Ann_l(Rx)$ and $d \in Ann_l(Ry)$. So $x,\ y,\ c,\ d \in V^l$ and $|V^l| \geq 4$ such that $c \to x$ and $d \to y$ are edges in $PG^l_D(R)$. Then $x \to ad \to y$ and $y \to bc \to x$ are two paths between $x$ and $y$. Hence $PG^l_D(R)$ is strongly connected. □

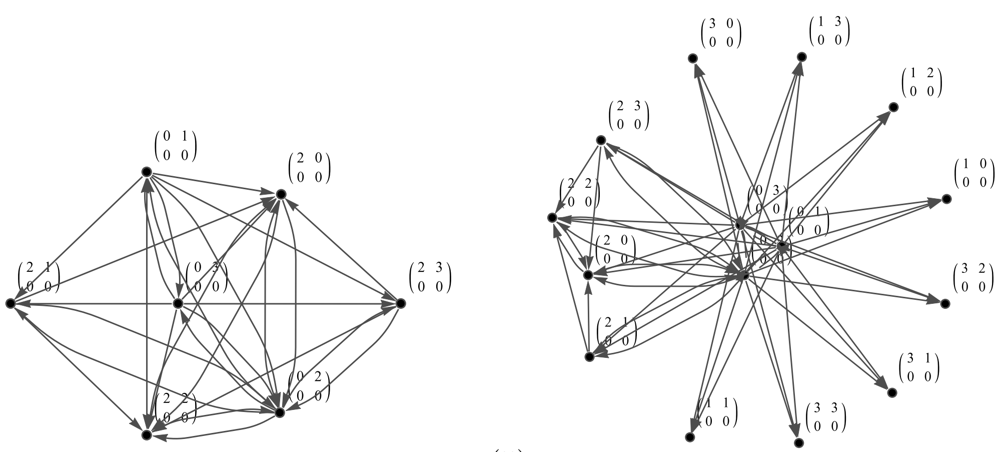
**Theorem 3.6.** *Let $R$ be a non-commutative ring. If $V^r \subseteq V^l$ then $PG_D^r(R)$ is strongly connected.*

Proof is similar to Theorem 3.5.

**Example 3.7.** Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,\ b \in \mathbb{Z}_p \right\}$ be the matrix ring over $\mathbb{Z}_p$. Then the $PG_D^l(M)$ is complete and $PG_D^l(M)$ is subgraph of $PG_D^r(M)$.

**Example 3.8.** Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a,\ b \in \mathbb{Z}_p \right\}$ be the matrix ring over $\mathbb{Z}_p$. Then the $PG_D^l(M)$ is complete and $PG_D^l(M)$ is subgraph of $PG_D^r(M)$.

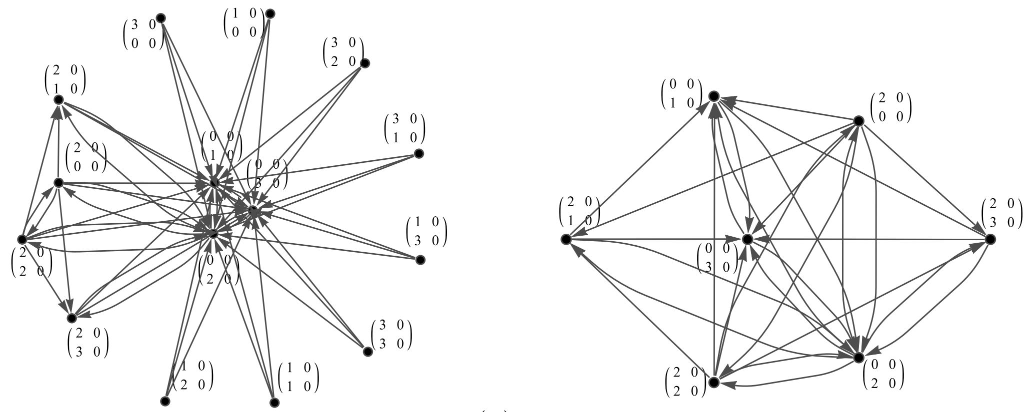
**Example 3.9.** Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,\ b \in \mathbb{Z}_4 \right\}$ be the matrix ring over $\mathbb{Z}_4$. Then the $PG_D^r(M)$ and $PG_D^l(M)$ are given in Fig 5.



(i) (ii)

Fig 6 : (i) $PG_D^r(M)$ (ii) $PG_D^l(M)$

**Example 3.10.** Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a,\ b \in \mathbb{Z}_4 \right\}$ be the matrix ring over $\mathbb{Z}_4$. Then the $PG_D^r(M)$ and $PG_D^l(M)$ are given in Fig 6.




(i) (ii)

Fig 6 : (i) $PG^r_D(M)$ (ii) $PG^l_D(M)$

**Theorem 3.11.** *Let $R$ be a non-commutative ring. If $V^l \cap V^r \neq \varphi$ then any two elements of $V^l \cap V^r$ are mutually reachable in $PG_D(R)$.*

*Proof.* Let $V^l \cap V^r = X \neq \varphi$. Let $x$ and $y$ be two distinct vertices of $PG_D(R)$ such that $x,\ y \in X$. Therefore there exists $a,\ b,\ c,\ d \in R$ such that $x \in Ann_l(Ra)$, $y \in Ann_r(bR)$, $x \in Ann_r(cR)$ and $y \in Ann_l(Rd)$. We have the following cases:

(a) If $x \in Ann_l(Ry)$, $y \in Ann_l(Rx)$ then $x \to y$ and $y \to x$ are two edges.

(b) If $y \in Ann_l(Rx)$, $x \notin Ann_l(Ry)$, $x \in Ann_l(Rx)$ and $y \in Ann_l(Ry)$ then $xy \in R$ such that $x \to xy \to y$ is a path.

(c) If $y \in Ann_l(Rx)$, $x \notin Ann_l(Ry)$, $x \in Ann_l(Rx)$ and $y \notin Ann_l(Ry)$ then $b \in R$ such that $y \in Ann_r(bR)$. If $x \in Ann_l(Rb)$ then $x \to b \to y$ is a path. If $x \notin Ann_l(Rb)$ then $xb \in R$ such that $x \to xb \to y$. Similarly we can show that for $x \notin Ann_l(Ry)$, $x \notin Ann_l(Rx)$ and $y \in Ann_l(Ry)$ there exists a path from $x$ to $y$.

(d) If $y \in Ann_l(Rx)$, $x \notin Ann_l(Ry)$, $x \notin Ann_l(Rx)$ and $y \notin Ann_l(Ry)$ then $a, b \in R$ such that $x \in Ann_l(Ra)$ and $y \in Ann_r(bR)$. If $a \in Ann_l(Rb)$ then $x \to a \to b \to y$ is a path. If $a \notin Ann_l(Rb)$ then $ab \in R$ such that $x \to ab \to y$ is a path in $PG_D(R)$.

Similarly taking $x \in Ann_l(Ry)$ and $y \notin Ann_l(Rx)$ we can show that there exist paths from $x$ to $y$ and $y$ to $x$.

(e) Let $x \notin Ann_l(Ry)$, $y \notin Ann_l(Rx)$, $x \notin Ann_l(Rx)$ and $y \notin Ann_l(Ry)$. If $a \in Ann_l(Rb)$ and $d \in Ann_l(Rc)$ then $x \to a \to b \to y$ and $y \to d \to c \to x$ are two paths. If $a \in Ann_l(Rb)$ and $d \notin Ann_l(Rc)$ then $x \to a \to b \to y$ and $y \to dc \to x$ are two paths. If $a \notin Ann_l(Rb)$ and $d \in Ann_l(Rc)$ then $x \to ab \to y$ and $y \to d \to c \to x$ are two paths. If $a \notin Ann_l(Rb)$ and $d \notin Ann_l(Rc)$ then $x \to ab \to y$ and $y \to dc \to x$ are two paths.

Hence any two elements of $V^l \cap V^r$ are mutually reachable in $PG_D(R)$. □

**Theorem 3.12.** *Let $R$ be a non-commutative ring. If $|V^l - V^r| = 1 = |V^r - V^l|$ then $PG_D(R)$ is unilaterally connected.*

*Proof.* Let $V^l \cap V^r = X \neq \varphi$. Then any two elements in $X$ are mutually reachable by *Theorem* 3.11. Let $x \in X$, then there exists $a, b \in R$ such that $x \in Ann_r(aR)$ and $x \in Ann_l(Rb)$. Let $u \in V^l - V^r$ and $v \in V^r - V^l$ such that $u$ and $v$ are not adjacent to any of $a, b, x$. If $(u, v) \in E$ then $u \to va \to x$ and $x \to bu \to v$ are two paths. Thus every vertex in $PG_D(R)$ is reachable from $u$ but no vertex is reachable to $u$. Similarly every vertex in $PG_D(R)$ is reachable to $v$ but no vertex is reachable from $v$. Again if $(u, v) \notin E$ then there exists $c, d \in R$ such that $u \in Ann_l(Rc)$ and $v \in Ann_r(dR)$. Since $|V^l - V^r| = 1 = |V^r - V^l|$, so $c, d \in X$ and are reachable from each other. So, for any two vertices of $PG_D(R)$ at least one vertex is reachable from the other through a directed path if $|V^l - V^r| = 1 = |V^r - V^l|$. Hence $PG_D(R)$ is unilaterally connected. □

**Corollary 3.13.** *Let $R$ be a non-commutative ring and $\varphi \neq V^l \cap V^r \neq V$. If $|V^l - V^r| > 1$ or $|V^r - V^l| > 1$ then $PG_D(R)$ is weakly connected.*

*Proof.* Let $V^l \cap V^r = X \neq \varphi$. Then by Theorem 3.11 any two elements in $X$ are mutually reachable. If $|V^l - V^r| > 1$ by Theorem 3.12, every vertex of $V^l - V^r$ is reachable to all vertices of $X$ but no two vertices of the set $V^l - V^r$ are reachable from each other. Similarly, if $|V^r - V^l| > 1$ every vertex of $V^r - V^l$ is reachable from all vertices of $X$ but no two vertices the set $V^r - V^l$ are reachable from each other. So there exist vertices in $PG_D(R)$ which are not reachable through any directed path but connected as a simple graph. Hence $PG_D(R)$ is weakly connected. □

**Corollary 3.14.** *Let $R$ be a non-commutative ring. If $V^l = V^r$ then $PG^r_D(R) \cong PG^l_D(R) \cong PG_D(R)$ and $PG_D(R)$ is strongly connected.*

*Proof.* Since $V^l = V^r = V$ so $E^l = E^r$ and hence $PG^r_D(R) \cong PG^l_D(R) \cong PG_D(R)$.

The strong connectivity follows from *Theorem* 3.11. □

**Corollary 3.15.** *Let $R$ be a non-commutative ring. Then $PG_D(R)$ does not contain sink and source together if any one of $V^l$ or $V^r$ is a subset of the other.*

*Proof.* The result follows from *Theorem* 3.11 and *Theorem* 3.12. □

**Theorem 3.16.** *Let $M_n(R)$ be the ring of all $n \times n$ matrices over a commutative ring. Then the vertex set of $PG_D(M_n(R))$ is empty if $R$ is a prime ring.*

*Proof.* Since $R$ is a prime ring, for any two non-zero elements $a, b \in R$, $aRb \neq 0$. Let $A(\neq 0), B(\neq 0) \in M_n(R)$, then $A$ and $B$ are adjacent in $PG_D(M_n(R))$ if and only if $a_{ij} \in Ann(Rb_{kl})$ for all entries $a_{ij}$ of $A$ and $b_{kl}$ of $B$. But it is not possible as $R$ is a prime ring. Therefore for any two elements $A(\neq 0), B(\neq 0) \in M_n(R), A \notin Ann_l(M_n(R)B)$. Similarly we can show that $B \notin Ann_l(M_n(R)A)$. Therefore the vertex set of $PG_D(M_n(R))$ is empty. □

**Theorem 3.17.** *If* $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\ b,\ c,\ d \in \mathbb{Z}_{p^2} \right\}$ *be the matrix ring, where $p$ is a prime, then $PG_D(M)$ is strongly connected and complete graph.*

*Proof.* For any $A(\neq 0) \in M, A \in V^l$ if and only if there is at least one $B(\neq 0) \in M$ such that $A \in Ann_l(MB)$. Then $B \in V^r$. It is possible if and only if all entries of both $A$ and $B$ are divisible by $p$. But for this we also have $B \in Ann_l(MA)$ i.e. $B \in V^l$ and $A \in V^r$. So we have $V^l = V^r$. By Corollary 3.14, $PG_D(M)$ is strongly connected. Now any two elements of $PG_D(M)$ are adjacent from each other and hence $PG_D(M)$ is a complete digraph. □

**Example 3.18.** Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\ b,\ c,\ d \in \mathbb{Z}_4 \right\}$ be the matrix ring over $\mathbb{Z}_4$. Then $V^l = V^r$ and $PG_D^r(M) \cong PG_D^l(M) \cong PG_D(M)$.

**Theorem 3.19.** *If* $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\ b,\ c,\ d \in \mathbb{Z}_{pq} \right\}$ *be the matrix ring, where $p$ and $q$ are distinct primes, then $PG_D(M)$ is strongly connected and complete bipartite graph.*

*Proof.* For any $A(\neq 0) \in M, A \in V^l$ if and only if there is at least one $B(\neq 0) \in M$ such that $A \in Ann_l(MB)$. Then $B \in V^r$. It is possible if and only if all entries of either $A$ or $B$ are divisible by $p$ and the entries of the other by $q$. But for this we also have $B \in Ann_l(MA)$ i.e. $B \in V^r$ and $A \in V^r$. So we have $V^l = V^r$. But By Corollary 3.14, $PG_D(M)$ is strongly connected. The vertex set of $M$ can be partitioned into two sets one containing the matrices with entries divisible by $p$ and other containing those with entries divisible by $q$. It can easily be show that $PG_D(R)$ is a complete bipartite digraph. □

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