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Research Paper

A NEW APPROACH TO SMALLNESS IN HYPERMODULES

ALI REZA MONIRI HAMZEKOLAEE*, MORTEZA NOROUZI AND VIOLETA LEOREANU-FOTEA

Abstract. In this paper, we extend the concept of small subhypermodules to all types of hypermodules and give nontrivial examples for this concept. As an application, we define and study lifting hypermodules via small subhypermodules.

1. Introduction

Let R denote an arbitrary associative ring with identity and consider all modules to be unitary right R-modules. For terminology in general module theory we refer to [12] and [15]. A submodule N of a module M is said to be *small* in M (denoted by $N \ll M$) if $N + K \neq M$ for all proper submodule K of M. In [10], Leonard defined a module M to be small provided there is another module L such that $M \ll L$. It is well-known that M is a small module if and only if M is small in its injective envelope. The concept of small submodules as one of

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*Corresponding author

the most applicable part of module theory was leading to define many classes of modules and rings such as (semi)perfect and supplemented rings and modules.

In recent decades, lifting modules have been studied extensively by many researchers as they play a key role in module theory. General properties of lifting modules such as direct summands, homomorphic images, (finite) direct sums, some special submodules and their relations with other known classes of modules have been investigated (see [4, 9]).

According to [12], a module M is called *lifting* if every submodule of M lies above a direct summand of M, i.e. for every submodule N of M, there exists a direct summand D of M such that $N/D \ll M/D$. A submodule N of M is called a *supplement* of a submodule K of M if $M = N + K$ and $N \cap K \ll N$. A module M is called *supplemented* if every submodule of M has a supplement in M. Also M is called *amply supplemented*, if $M = A+B$, then A contains a supplement of B in M. A lifting module is amply supplemented and hence supplemented.

Let H be a nonempty set and set " $\circ$ " : $H \times H \longrightarrow \mathcal{P}^*(H)$ where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. Then the mapping " $\circ$ " is called a hyperoperation on H and the algebraic hyperstructures theory is based on it. This theory was first introduced by Marty in [11]. Till now, many interesting developments have been accomplished in this context and very attractive applications of this theory in mathematics and other sciences have been characterized. To prove this claim, we refer readers to [5], [6], [7], [8] and [16]. In this paper, we intend to try for extension of some of well-known concepts in classical algebraic structures (specially, modules theory) to hyperstructures (hypermodules). We investigate conditions for which some results of modules theory are valid for hypermodules and in order to do this we introduce some concepts. Our results are based on concepts such as small submodules, direct summands and lifting modules in module theory.

In [14], Talaee introduced and analyzed the concept of small subhypermodules in hypermodules in the same manner as in general module theory. We extend here this study to a more general context.

In what follows, we give some basic concepts about hypergroups, hyperrings and hypermodules which we need in this paper.

Let " $\circ$ " be a hyperoperation on H. Then $(H, \circ)$ is said to be a *hypergroupoid*. For $x \in H$ and $A, B \in \mathcal{P}^*(H)$, we set $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$. We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, $x \circ H = H \circ x = H$. A nonempty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup*, if for all $k \in K$, we have $k \circ K = K \circ k = K$.

Definition 1.1. ([8]) A commutative hypergroup $(H, \circ)$ is said to be *canonical*, if

(1) there exists a unique $0 \in H$, such that for all $x \in H$, $x \circ 0 = \{x\}$;

(2) for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $0 \in x \circ x^{-1}$;

(3) if $x \in y \circ z$, then $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$, for all $x, y, z \in H$.

Definition 1.2. ([8]) The triple $(R, \uplus, \circ)$ is a *hyperring*, if

(1) $(R, \uplus)$ is a hypergroup;

(2) $(R, \circ)$ is a semihypergroup;

(3) " $\circ$ " is distributive over " $\uplus$ ".

A hyperring $(R, \uplus, \circ)$ is said to be *Krasner* ([8]), if $(R, \uplus)$ is a canonical hypergroup and $(R, \circ)$ is a semigroup such that 0 is a zero element (called also *absorbing element*), i.e. for all $x \in R$, we have $x \circ 0 = 0 = 0 \circ x$.

Definition 1.3. Let $(R, \uplus, \circ)$ be a hyperring and $(M, +)$ a hypergroup. If there exists an external hyperoperation $\cdot : R \times M \longrightarrow \mathcal{P}^*(M)$ such that for all $a, b \in M$ and $r, s \in R$ we have

(i) $r \cdot (a + b) = (r \cdot a) + (r \cdot b)$;

(ii) $(r \uplus s) \cdot a = (r \cdot a) + (s \cdot a)$;

(iii) $(r \circ s) \cdot a = r \cdot (s \cdot a)$,

then $(M, +, \cdot)$ is called a *left hypermodule* over R (or a left R-hypermodule).

Similarly, a right hypermodule over R is defined. We say that M is a hypermodule over R, if it is a right and left hypermodule over R. An R-hypermodule satisfying statements in Definition 1.3 is called a general R-hypermodule. Also, if $(M, +)$ is a canonical hypergroup and $(R, \uplus, \circ)$ is a Krasner hyperring in Definition 1.3, then M is said to be a canonical R-hypermodule. Moreover, we say M is a Krasner R-hypermodule, if it is a canonical R-hypermodule where "." is an external operation, that is $. : R \times M \to M$ by $(r, m) \mapsto r.m \in M$, and $r.0 = 0$.

A nonempty subset N of an R-hypermodule M is called a *subhypermodule*, denoted by $N \leq M$, if N itself is a hypermodule over R with (hyper)operation defined on $R \times M$.

Note that a general R-hypermodule may not contain an element like 0. Moreover, in canonical R-hypermodules, $\{0\}$ is not a subhypermodule in general, while for Krasner R-hypermodules there are no such problems.

2. **Several non-trivial examples of small subhypermodules**

In this section, we provide non-trivial examples of hypermodules with small subhypermodules for all types of hypermodules. Also, we study some concepts such as local, hollow and $Rad(M)$ for hypermodules.

Definition 2.1. Let M be an R-hypermodule. A subhypermodule N is small in M (denoted by $N \ll M$), if $K + N = M$ or $M = N + K$ implies $M = K$, where $K \leq M$. Equivalently, if K is a proper subhypermodule of M, then $N + K \neq M$ and $M \neq K + N$.

According to definition, the concept of small subhypermodules in canonical and Krasner hypermodules is analogues to small submodules in modules theory. The next example shows that it has a different behavior in general hypermodules:

Example 2.2. Consider the following hyperoperation on $R = \{a, b, c, d, e, f, g\}$:

$+$	a	b	c	d	e	f	g
a	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$
b	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$
c	$\{c\}$	$\{c\}$	$\{a, b\}$	$\{f\}$	$\{g\}$	$\{d\}$	$\{e\}$
d	$\{d\}$	$\{d\}$	$\{g\}$	$\{a, b\}$	$\{f\}$	$\{e\}$	$\{c\}$
e	$\{e\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{a, b\}$	$\{c\}$	$\{d\}$
f	$\{f\}$	$\{f\}$	$\{e\}$	$\{c\}$	$\{e\}$	$\{g\}$	$\{a, b\}$
g	$\{g\}$	$\{g\}$	$\{d\}$	$\{e\}$	$\{c\}$	$\{a, b\}$	$\{f\}$

and $x \cdot y = \{a, b\}$ for all $x, y \in R$. Then $(R, +, \cdot)$ is a hyperring, where $+$ is not commutative on R, $(R, +, \cdot)$ is also a general R-hypermodule ([1]). By [13], the only subhypermodules of R are $R_1 = \{a, b\}$, $R_2 = \{a, b, c\}$, $R_3 = \{a, b, d\}$, $R_4 = \{a, b, e\}$ and $R_5 = \{a, b, f, g\}$. Moreover, we have $R_2 + R_5 \neq R = R_5 + R_2$, $R_3 + R_5 = R = R_5 + R_3$ and $R_4 + R_5 = R \neq R_5 + R_4$. But, $R_1 + R_i \neq R \neq R_i + R_1$ for all $2 \leq i \leq 5$. Hence, the only proper small subhypermodule in R is R_1.

Hence, there are hypermodules with just one small subhypermodule. Also, it is worth to say that in module theory, $\{0\}$ is always a small submodule, but in hypermodules it is not valid in general.

Example 2.3. For the set of all integers, $\mathbb{Z}$, define $x \oplus y = \{x, y, x + y\}$ and $x \otimes y = \{x \cdot y\}$, for all $x, y \in \mathbb{Z}$, where "$+$" and "$\cdot$" are ordinary addition and multiplication. Then, $(\mathbb{Z}, \oplus, \otimes)$ is a general $\mathbb{Z}$-hypermodule. The subhypermodules of $\mathbb{Z}$ are $\{0\}$ and $n\mathbb{Z}$ for $n \in \mathbb{N}$. It is clear that for all $m, n \in \mathbb{N}$ with $gcd(m, n) = 1$ we have $n\mathbb{Z} \oplus m\mathbb{Z} = \mathbb{Z}$. Therefore, the only small subhypermodule of $\mathbb{Z}$ is $\{0\}$.

We next provide an R-hypermodule M such that all proper subhypermodules of M are small in M.

Example 2.4. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \times \mathbb{Z}_4$. Define $(a, b) * (c, d) = \{(a, b), (c, d)\}$ and $n \diamond (a, b) = \{n(a, b)\}$ for all $(a, b), (c, d) \in M$ and $n \in \mathbb{Z}$. Also, for all $n, m \in \mathbb{Z}$ define $n \oplus m = \{n, m\}$ and $n \odot m = \{nm\}$. Then, $(M, *, \diamond)$ is a general hypermodule over $(\mathbb{Z}, \oplus, \odot)$. All proper subhypemodules of M are:

(1) $M_1 = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$

(2) $M_2 = \{(0,0),(0,2)\}$

(3) $M_3 = \{(0,0),(1,0)\}$

(4) $M_4 = \{(0,0),(1,1),(0,2),(1,3)\}$

(5) $M_5 = \{(0,0),(1,2)\}$

(6) $M_6 = \{(0,0)\}$.

An easy verification yields that for $i, j \in \{1,\ldots,6\}$ with $i \neq j$, we have $M_i * M_j \neq M$, so that $M_i \ll M$ for all $i \in \{1,\ldots,6\}$.

The following one is an example which is a rich source of general hypermodules with different small subhypermodules. Recall that a module M is local if it has a proper submodule which contains all proper submodules of M (see[15]). The largest submodule of a local module must be its radical, i.e. $Rad(M)$ which is the intersection of all maximal submodules of M.

Example 2.5. Let $(M,+,.)$ be an R-module over a ring $(R,+',.')$. Consider a submodule N of M and define $x \oplus y = \{x+y\}$, $r \odot x = r.x + N$, $r \uplus s = \{r +' s\}$ and $r \circ s = \{r.'s\}$ $(r.'s)$ for all $x, y \in M$ and $r, s \in R$. Then $(M,\oplus,\odot)$ is a general hypermodule (canonical hypermodule) over $(R,\uplus,\circ)$ (see [1, Example 2.3]). Note that every subhypermodule of M is a submodule of M. On the other hand, every submodule of M containing N, is a subhypermodule of M. We consider some special cases for N:

(1) Set $N = \{0\}$. Then the set of subhypermodules of M coincide with the set of submodules of M.

(2) Let N be a maximal submodule of M. Then the only subhypermodules of M are N and M. As an example, we can consider a local module M and $N = Rad(M)$.

(3) If N is a small submodule of M, then N is a small subhypermodule of M. Also, every small submodule of M containing N is a small subhypermodule of M.

(4) If $N = M$, then the R-hypermodule M has just one subhypermodule, namely M.

In general module theory, a module M is *hollow* if every proper submodule of M is small in M. Let us introduce a correspondent concept in hypermodules.

Definition 2.6. Let M be a hypermodule.

(1) We say that M is *hollow* if every proper subhypermodule of M is small in M.

(2) We call M *local* if M has a proper subhypermodule that contains all proper subhypermodules of M.

In module theory, one of characterizations for hollow modules is obtained by the sum of two proper submodules of a module. Indeed, a module M is hollow if and only if the sum of any two proper submodules of it, is a proper submodule. But, the sum of two subhypermodules of a general hypermodule is not generally a subhypermodule (in Example 2.2,

$R_2 + R_3 = \{a, b, c, d, f\}$ and $R_3 + R_2 = \{a, b, c, d, g\}$ are not subhypermodules of R) while in canonical and Krasner hypermodules the sum of each two subhypermodules is a subhypermodule. Hence, similar to what happens in module theory, it can be seen that a canonical (Krasner) hypermodule M is hollow if and only if the sum of any two of its proper subhypermodules is a proper subhypermodule.

Moreover, if M is a hypermodule such that its set of subhypermodules is a chain, then M is hollow. It is clear that a local hypermodule is hollow.

Definition 2.7. Let M be a hypermodule, I an indexed set and $\mathcal{A} = \{N_i \mid i \in I\}$ be a set of subhypermodules of M. We say that M satisfies CPS (commutative property for sums) on $\mathcal{A}$ if for every subset J of I we have $\sum_{j \in J} N_j = \sum_{\gamma \in \Gamma} N_\gamma$ where Γ is a permutation of J.

Note that a Krasner R-hypermodule always satisfies CPS on the set of its all subhypermodules.

We next present a similar concept to $Rad(M)$ in hypermodules. It is well-known that $Rad(M)$ is the sum of all small submodules of a module M.

Definition 2.8. Let M be a hypermodule such that M satisfies CPS on the set of its all small subhypermodules. We define $Rad(M)$ to be the sum of all small subhypermodules of M, i.e. $Rad(M) = \sum_{N \ll M} N$. In case M has no small subhypermodules, then we set $Rad(M) = M$.

Note that we may be not able to define $Rad(M)$ for a hypermodule M that does not satisfy the property $N + K = K + N$ where $N, K \ll M$. According to the above statements, $Rad(M)$ is not generally a subhypermodule in a general hypermodule, while it is always a subhypermodule in canonical and Krasner hypermodules.

Also, in Example 2.2, we can see that the sum of two subhypermodules is not a subhypermodule, but $Rad(R)$ is a subhypermodule since $Rad(R) = R_1$.

Moreover, we must point out that there are hypermodules with at least one small subhypermodule such that $Rad(M) = M$ (see Example 2.11).

Proposition 2.9. *Let M be a Krasner R-hypermodule. Then M is local if and only if M is hollow and $Rad(M) \neq M$.*

Proof. Suppose that M is local. It is obvious that M is hollow. Let N be the largest proper subhypermodule of M. Since every proper subhypermodule of M is small in M, it follows that $Rad(M) \subseteq N$. Hence $Rad(M) \neq M$. For the converse, let M be hollow and K be a proper subhypermodule of M. Then $K \ll M$. Hence $K \subseteq Rad(M)$. It follows that $Rad(M)$ is the largest proper subhypermodule of M. Note that if $Rad(M) = M$, then the hypermodule M can not be local. □

Example 2.10. Let M be an R-module such that its chain of submodules is $\{0\} \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = M$ (for example the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^n}$ for a prime number p). Define an $\mathbb{R}$-hypermodule by choosing $N = H_1$ according to Example 2.5. Then each H_i for $i \in \{1, \ldots, n\}$ is a subhypermodule of the R-hypermodule M and clearly each of them is small in M. It is easy to see that M as an R-hypermodule is local and $Rad(M) = \sum_{i=1}^{n-1} H_i = H_{n-1}$.

There are hollow hypermodules which are not local, as we can see from the following example.

Example 2.11. Let p be a prime number and consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} = \{\frac{a}{p^n} + \mathbb{Z} \mid a \in \mathbb{Z}, n \geq 0, 0 \leq \frac{a}{p^n} < 1, p \nmid a\}$ and $N =< \frac{1}{p} + \mathbb{Z} >$. Construct a $\mathbb{Z}$-hypermodule M with respect to the hyperoperations of Example 2.5. It is clear that every subhypermodule of M is small in M. Note that every subhypermodule of M has the form $L_m =< \frac{1}{p^m} + \mathbb{Z} >$ for $m \geq 1$ and $Rad(M) = \sum_{i=1}^{\infty} L_i = M$ which shows that M is not local.

Using Example 2.5 we can obtain more examples of small subhypermodules of hypermodules.

Example 2.12. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. Consider the submodules $H_0 = \{0\}$, $H_1 =< 2 >$, $H_2 =< 3 >$, $H_3 =< 4 >$, $H_4 =< 6 >$ and $H_5 = M$. Then $H_2 + H_3 = M$, $H_2 \cap H_3 = \{0\}$, $H_1 + H_2 = M$ and $H_1 \cap H_2 = H_4 \ll H_2$. So the only direct summands of M are H_2 and H_3. Also note that the only supplement submodules of M are H_2 and H_3 which are direct summands. It is well-known that every Artinian module is amply supplemented as well as M. Therefore, M is a lifting $\mathbb{Z}$-module by [12, Lemma 4.8]. We analyze some particular cases in which M can be a $\mathbb{Z}$-hypermodule with respect to the hyperoperations of Example 2.5:

(1) Choose $N = H_4$. In this case all nontrivial subhypermodules of M are H_1, H_2 and H_4. It can be seen that $H_1 + H_2 = M$ and $H_1 \cap H_2 = H_4$ is small in both H_1 and H_2. The only small subhypermodule of M is H_4. Hence $Rad(M) = H_4$.

(2) Choose $N = H_3$. The nontrivial subhypermodules of M are H_1 and H_3 which are small in M. Therefore, M is local (hollow) and $Rad(M) = H_1$ is a proper subhypermodule of M.

(3) Set $N = H_2$. In this case, the only nontrivial subhypermodule of M is H_2. It is obvious that H_2 is small in M. Hence $Rad(M) = H_2$.

(4) Set $N = H_1$. Then the only proper subhypermodule of M is H_1. Clearly H_1 is small in M and $Rad(M) = H_1$.

Example 2.13. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_4$. Suppose that $N = \{0, 2\}$ which is a submodule of M. Consider the $\mathbb{Z}$-hypermodule $(M, \oplus, \odot)$ constructed by N according to Example 2.5. Now, set $M' = M \cup \{a, b, c\}$ such that $\{a, b, c\} \cap M = \emptyset$ and consider the following hyperoperations:

$+'$	0	1	2	3	a	b	c
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{a\}$	$\{b\}$	$\{c\}$
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
3	$\{3\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	M	$\{c\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$	M	$\{a\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{b\}$	$\{a\}$	M

and $$n \cdot' x = \begin{cases} n \odot x, & x \in M \\ \{a\}, & x = a \\ \{b\}, & x = b \\ \{c\}, & x = c \end{cases}$$

for all $n \in \mathbb{Z}$. Then $(M', +', \cdot')$ is a general $\mathbb{Z}$-hypermodule by [2, Page 46]. It can be seen that $M'_1 = M \cup \{a\}$, $M'_2 = M \cup \{b\}$, $M'_3 = M \cup \{c\}$, $M'_4 = M$ and $M'_5 = N$ are the only proper subhypermodules of M'. Since $M'_i +' M'_j = M'$ for $i \neq j \in \{1, 2, 3\}$, none of them is a small subhypermodule of M'. It is easy to see that $M'_4 +' M'_i \neq M'$ and $M'_5 +' M'_i \neq M'$ for $i \in \{1, 2, 3, 4, 5\}$. So, M'_4 and M'_5 are small subhypermodules of M'.

We end this section by presenting some R-hypermodules with no small subhypermodules.

Example 2.14. (1) Let $(M_1 = \mathbb{Z}_4, +_1, ._1)$, $N_1 = \{0, 2\}$, $(M_2 = K_4 = \{e, a, b, c\}, +_2, ._2)$ and $N_2 = \{e, a\}$. By K_4 we mean Klein's four-group. Clearly $N_1 \leq M_1$ and $N_2 \leq M_2$. Construct $\mathbb{Z}$-hypermodules $(M_1, \oplus_1, \odot_1)$ and $(M_2, \oplus_2, \odot_2)$ by N_1 and N_2, respectively, based on Example 2.5. Set $L = M_1 \cup M_2 \cup \{d\}$ where $d \notin M_1 \cup M_2$. By [2, Page 47], $(L, \boxplus, \boxdot)$ is a general $\mathbb{Z}$-hypermodule where

$\boxplus$	0	1	2	3	e	a	b	c	d
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	L	L	L	L	$\{d\}$
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{0\}$	L	L	L	L	$\{d\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$	L	L	L	L	$\{d\}$
3	$\{3\}$	$\{0\}$	$\{1\}$	$\{2\}$	L	L	L	L	$\{d\}$
e	L	L	L	L	$\{e\}$	$\{a\}$	$\{b\}$	$\{c\}$	L
a	L	L	L	L	$\{a\}$	$\{e\}$	$\{c\}$	$\{b\}$	L
b	L	L	L	L	$\{b\}$	$\{c\}$	$\{e\}$	$\{a\}$	L
c	L	L	L	L	$\{c\}$	$\{b\}$	$\{a\}$	$\{e\}$	L
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	L	L	L	L	M_1

and $n \boxdot x = n \odot_1 x$, $n \boxdot y = n \odot_2 y$ and $n \boxdot d = \{d\}$ for all $x \in M_1$, $y \in M_2$ and $n \in \mathbb{Z}$. The proper subhypermodules of L are $L_1 = M_1 \cup \{d\}$, $L_2 = M_1$, $L_3 = M_2$, $L_4 = N_1$ and

$L_5 = N_2$. Also, we have $L_1 \boxplus L_3 = L_2 \boxplus L_3 = L_4 \boxplus L_5 = L$ and hence none of them is a small subhypermodule of L.

(2) Let M_1 and M_2 be two general R-hypermodules with the external hyperoperations $\cdot_1$ and $\cdot_2$, respectively. Set $M = M_1 \cup M_2 \cup \{a\}$ ($a \notin M_1 \cup M_2$) with the following hyperoperation for $x_1, x_2 \in M_1$ and $y_1, y_2 \in M_2$:

$+'$	x_1	a	y_1
x_2	$x_1 + x_2$	$\{a\}$	M
a	$\{a\}$	M_1	M
y_2	M	M	$y_1 + y_2$

and for all $x \in M_1$, $y \in M_2$ and $r \in R$, the scalar multiplication is defined by $r \cdot' x = r \cdot_1 x$, $r \cdot' y = r \cdot_2 y$ and $r \cdot' a = \{a\}$. It can be easily seen that $(M, +', \cdot')$ is a general R-hypermodule. Note that every subhypermodule of M must be one of $M_f = N_i$ or $M_g = M_1 \cup \{a\}$ or $M_h = K_i$, where N_i is a subhypermodule of M_1 and K_i is a subhypermodule of M_2. We can easily check that $M_f +' M_h = M$ and $M_g +' M_h = M$. Hence, M does not have a small subhypermodule.

3. **Lifting hypermodules**

In this section, we introduce a new class of hypermodules namely, lifting hypermodules. We present equivalent conditions for a lifting hypermodule and we show that a direct summand of a lifting Krasner hypermodule is lifting, too.

Before dealing with lifting hypermodules, we present some results related to small subhypermodules of factor hypermodules.

Lemma 3.1. *Let M be a Krasner hypermodule. Let $K, L, N \leq M$, $K + L = M$ and $(K \cap L) + N = M$. Then $K + (L \cap N) = L + (K \cap N) = M$.*

Proof. We prove that $K + (L \cap N) = M$. The other equality is similar. Clearly, $K + (L \cap N) \subseteq M$. Let $x \in M$. Since $K + L = M$, there exist $k \in K$ and $l \in L$ such that $x \in k + l$. Also, we have $l \in L \subseteq M = (L \cap K) + N$ which implies that $l \in t + n$ for $t \in L \cap K$ and $n \in N$. Hence, $n \in l - t \subseteq L - (L \cap K) \subseteq L$, and so $n \in N \cap L$. Then, $l \in t + n \subseteq K + (L \cap N)$ and thus $x \in k + l \subseteq K + K + (L \cap N) \subseteq K + (L \cap N)$. This completes the proof. □

As an example for Lemma 3.1, we can consider the trivial Krasner $\mathbb{Z}$-hypermodule $M = \mathbb{Z}$ with $K = 2\mathbb{Z}$, $L = 3\mathbb{Z}$ and $N = 5\mathbb{Z}$.

Let $(M_1, +_1, \cdot_1)$ and $(M_2, +_2, \cdot_2)$ be two hypermodules over R. A map $f : M_1 \longrightarrow M_2$ is called a *strong homomorphism*, if $f(x +_1 y) = f(x) +_2 f(y)$ and $f(r \cdot_1 x) = r \cdot_2 f(x)$ for all $x, y \in M_1$ and $r \in R$.

Let $f : M_1 \to M_2$ be a strong homomorphism of Krasner hypermodules. It is easy to see that the set $Kerf = \{x \in M_1 \mid f(x) = 0_{M_2}\}$ is a subhypermodule of M_1. Similar to representation theorem for module theory, we can obtain the following theorem for Krasner R-hypermodules.

Theorem 3.2. *Let M and N be two Krasner hypermodules over R and $f : M \longrightarrow N$ be a strong epimorphism. Then there exists an one to one correspondence $h : \mathcal{A} \longrightarrow \mathcal{B}$, by $L \longmapsto f(L)$, where $\mathcal{A} = \{L \mid Kerh \subseteq L \leq M\}$ and $\mathcal{B} = \{T \mid T \leq N\}$.*

Each subhypermodule L of an R-hypermodule $(M, +, \cdot)$ leads to the quotient $M/L = \{x+L \mid x \in M\}$ which is an R-hypermodule by $(x + L) \oplus (y + L) = \{z + L \mid z \in (x + L) + (y + L)\}$ and $r \odot (x + L) = \{c + L \mid c \in r \cdot (x + L)\}$ for all $x + L, y + L \in M/L$ and $r \in R$.

If M is a Krasner R-hypermodule, then we have $(x + L) \oplus (y + L) = \{z + L \mid z \in x + y\}$ and $r \odot (x + L) = \{c + L \mid c \in r \cdot x\}$. In this case, $\pi : M \longrightarrow M/L$, defined by $\pi(x) = x + L$, is a strong epimorphism where $Ker\pi = L$. Therefore, by Theorem 3.2, it follows that every subhypermodule of M/L has the form K/L where $L \subseteq K \leq M$.

Proposition 3.3. *Let M be a Krasner R-hypermodule and $L \subseteq N$ be subhypermodules of M. Then $N/L \ll M/L$ if and only if for all $K \leq M$ the equality $N + K = M$ implies $L + K = M$.*

Proof. Suppose that $N/L \ll M/L$ and $N + K = M$. Then, for all $x + L \in M/L$ there exist $n \in N$ and $k \in K$ such that $x \in n+k$. Hence, $x+L \in (n+L)\oplus(k+L) = (n+L)\oplus(k+l+L) \subseteq N/L \oplus (K + L)/L$, which implies that $M/L = N/L \oplus (K + L)/L$. Since $N/L \ll M/L$, we conclude that $(K + L)/L = M/L$. Now, for all $x \in M$ we have $x + L = t + L$ for some $t \in K + L$. Thus, $x \in x + 0 \subseteq x + L = t + L \subseteq K + L + L = K + L$ which implies that $K + L = M$.

For the converse let $N/L \oplus T/L = M/L$, for $T/L \leq M/L$. Then, for $x \in M$ there exist $n \in N$ and $t \in T$ such that $x + L \in (n + L) \oplus (t + L)$. Hence, $x + L = q + L$ for $q \in n + t$, and so $x \in x + L \subseteq n + t + L \subseteq N + T$. Then $N + T = M$. Now, the assumption implies that $L + T = M$. Since L is contained in T, we can conclude by definitions that $T = M$. It follows that $T/L = M/L$. □

Before presenting definitions of lifting ((amply) supplemented) hypermodules, we discuss about the intersection of two subhypermodules of a hypermodule. From ([8]), recall that a subhypergroup N of a hypergroup $(M, +)$ is called *closed* on the left (on the right), if for all $x, y \in N$ and $m \in M$, $x \in m + y$ ($x \in y + m$) implies that $m \in N$. A subhypergroup N is said to be closed, if it is closed on the left and right. Moreover, a subhypergroup N of $(M, +)$ is closed if and only if $N + (M \setminus N) = M \setminus N$. Now, let $(M, +, \cdot)$ be a hypermodule over R

and N be a subhypermodule of M. We say that N is a *closed subhypermodule* if N is a closed subhypergroup of $(M, +)$. For example:

(1) Set $R = \{0, 1, 2, 3\}$ and consider the following hyperoperation and operation on R:

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{3\}$	$\{2,3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{2,3\}$	$\{1\}$	$\{0,1\}$

and

$$r \cdot s = \begin{cases} 2, & \text{if } r, s \in \{2,3\} \\ 0, & \text{else} \end{cases}$$

Then $(R, +, \cdot)$ is a Krasner hyperring ([3]). Moreover, if $M = R$, then $(M, +, \cdot)$ is a Krasner R-hypermodule. The subhypermodules $\{0\}$, $K = \{0, 1\}$ and $L = \{0, 2\}$ are closed in M, since

$$\{0\} + M \setminus \{0\} = \{0\} + \{1, 2, 3\} = \{1, 2, 3\} = M \setminus \{0\}$$

$$K + (M \setminus K) = \{0, 1\} + \{2, 3\} = \{2, 3\} = M \setminus K,$$

$$L + (M \setminus L) = \{0, 2\} + \{1, 3\} = \{1, 3\} = M \setminus L.$$

(2) Consider the hypermodule $(M', +', \cdot')$ in Example 2.13. All proper subhypermodules of M', which are M'_1, M'_2, M'_3, M'_4 and M'_5, are closed in M'.

(3) None of subhypermodules of the hypermodules $(L, \boxplus, \boxdot)$ defined in Example 2.14 ($L_1 = M_1 \cup \{d\}$, $L_2 = M_1$, $L_3 = M_2$, $L_4 = N_1$ and $L_5 = N_2$), is not closed in L.

The intersection of two subhypermodules of a canonical/Krasner hypermodule is a subhypermodule, while it is not true in general hypermodules. Closed subhypermodules can help us to provide one of the most fundamental conditions (intersection of two subhypermodules is a subhypermodule) for studying similar results about general R-hypermodules.

Lemma 3.4. *If N and K are closed subhypermodules of a general hypermodule $(M, +, \cdot)$ and $N \cap K \neq \emptyset$, then $N \cap K$ is a closed subhypermodule of M.*

Proof. Let $x, y \in N \cap K$ and $m \in M$ be such that $x \in m + y$. Since $x, y \in N$ and N is closed, it follows that $m \in N$, and similarly $m \in K$. Thus, $m \in N \cap K$. Then $N \cap K$ is closed on the left and similarly it is closed on the right. So, closed property is valid for $N \cap K$. Clearly, $(N \cap K, +)$ satisfies the associativity law. Now, let $t \in N \cap K$. If $x \in N \cap K$, then $x \in K = t + K$. So, there exists $k \in K$ such that $x \in t + k$. By the closed property for $N \cap K$, we have $k \in N \cap K$. Then, $x \in t + k \subseteq t + (N \cap K)$. Hence, $N \cap K \subseteq t + (N \cap K)$. It is not difficult to see that $t + (N \cap K) \subseteq N \cap K$. Therefore, the reproduction axiom is valid for $(N \cap K, +)$ and so $N \cap K$ is a subhypermodule of M. □

Definition 3.5. ([14]) Let M be a Krasner R-hypermodule. A subhypermodule N is a direct summand of M, if there is a subhypermodule L of M such that $N \cap L = \{0\}$ and $N + L = M$.

Note that if M is a general R-hypermodule with element 0 such that the intersection of each two subhypermodules is a subhypermodule, we can state that N is a direct summand of M if $N + K = M = K + N$ and $N \cap K = \{0\}$. For instance, choose $N = H_0$ in Example 2.12. Then H_2 and H_3 are direct summands of M.

Definition 3.6. (1) Let M be an R-hypermodule. We call M *lifting* if for all proper subhypermodule K of M, there is a direct summand L of M such that $L \leq K$ and K/L is a small subhypermodule of M/L.

(2) Let M be an R-hypermodule such that the intersection of each two subhypermodules of M is a subhypermodule of M. We say M is *amply supplemented* if $B + A = M = A + B$ for two proper subhypermodules A and B of M implies that B (A) contains a subhypermodule A' (B') such that $A' + A = M = A + A'$ $(B' + B = M = B + B')$ and $A \cap A' \ll A'$ $(B \cap B' \ll B')$.

(3) Let M be an R-hypermodule such that the intersection of each two subhypermodules of M is a subhypermodule of M. We say M is *supplemented* if for all proper subhypermodule N of M there is a proper subhypermodule K of M, such that $K + N = M = N + K$ and $N \cap K \ll K$.

Example 3.7. (1) Consider the Krasner R-hypermodule $(M = R = \{0, 1, 2, 3\}, +, \cdot)$ defined by

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{3\}$	$\{2,3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{2,3\}$	$\{1\}$	$\{0,1\}$

and

$$r \cdot s = \begin{cases} 2, & \text{if } r, s \in \{2,3\} \\ 0, & \text{else} \end{cases}$$

We can see that the only proper subhypermodules of M are $\{0\}$, $K = \{0, 1\}$ and $L = \{0, 2\}$. Note that $K + L = M$. Hence, $\{0\}$ is the only small subhypermodule of M. It follows that M is lifting, since all subhypermodules of M are direct summands of M and $\{0\}/\{0\} \ll M/\{0\}$, $K/K \ll M/K$, $L/L \ll M/L$. Note also that M is amply supplemented.

(2) Consider $(R,+,\cdot)$ defined in (1). Let $M_2(R)=\left\{\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix} \mid a,b\in R\right\}$. Define the following hyperoperation and operation on $M_2(R)$, for all $a,b,c,d\in R$:

$$\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\oplus\begin{pmatrix} c & d\\ 0 & 0\end{pmatrix}=\left\{\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}\mid x\in a+c,\ y\in b+d\right\}$$

$$\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} c & d\\ 0 & 0\end{pmatrix}=\begin{pmatrix} a\cdot c & a\cdot d\\ 0 & 0\end{pmatrix}$$

Then $M_2(R)$ is a Krasner $M_2(R)$-hypermodule. All proper subhypermodules of $M_2(R)$ are:

$N_1=\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\}$,

$N_2=\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}\}$ and

$N_3=\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 2\\ 0 & 0\end{pmatrix},\begin{pmatrix} 2 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 2 & 2\\ 0 & 0\end{pmatrix}\}$.

It is obvious that $N_1\ll M_2(R)$ and $N_2\oplus N_3=M_2(R)$. Therefore, N_1 is the only small subhypermodule of $M_2(R)$. Similar to (1), we conclude that $M_2(R)$ is lifting.

Example 3.8. The general R-hypermodules M in Examples 2.10 and 2.11 are amply supplemented. Note also that the general R-hypermodules M in Example 2.12(1,2) are amply supplemented.

Now, we are ready to present an equivalent condition for a Krasner R-hypermodule to be lifting.

Proposition 3.9. *Let M be a Krasner R-hypermodule. Then M is lifting if and only if for all subhypermodule N of M, there are subhypermodules M_1 and M_2 such that $M=M_1+M_2$, $M_1\cap M_2=\{0\}$, $M_1\subseteq N$ and $N\cap M_2\ll M_2$.*

Proof. The proof is similar to the module case. □

The following result describes subhypermodules of lifting hypermodules.

Corollary 3.10. *Let M be a Krasner R-hypermodule. Then M is lifting if and only if all subhypermodule N of M can be written as $N=M_1+M_2$ with $M_1\cap M_2=\{0\}$ where M_1 is a direct summand of M and M_2 is small in M.*

4. Conclusion

We give non-trivial examples of small subhypermodules for all types of hypermodules and study local, hollow and $Rad(M)$ for hypermodules by them. Also, lifting hypermodules and equivalent conditions for lifting hypermodules are presented. We conclude the paper by stating this point that for a Krasner R-hypermodule the similar result to the module case do also hold. But it remains open to present an equivalent condition for a general R-hypermodule to be lifting.

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Ali Reza Moniri Hamzekolaee

Department of Mathematics, Faculty of Mathematical Sciences

University of Mazandaran

Babolsar, Iran.

a.monirih@umz.ac.ir

Morteza Norouzi

Department of Mathematics, Faculty of Basic Sciences

University of Bojnord

Bojnord, Iran.

m.norouzi65@yahoo.com, m.norouzi@ub.ac.ir

Violeta Leoreanu-Fotea

Faculty of Mathematics, Al. I. Cuza University of Iaşi,

Romania.

violeta.fotea@uaic.ro