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Research Paper

GENERALIZATION OF REDUCTION AND CLOSURE OF IDEALS

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Abstract. Throughout this paper, all rings are commutative with identity and all modules are unital. Let R be a ring and M be an R-module. Then M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$. Also M is said to be a comultiplication module if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. In this paper, we introduce the notions of reduction and coreduction of submodules, integral dependence, integral codependence, integral closure and Δ-closure over multiplication and comultiplication modules.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. Let R be a ring and M be an R-module. Then M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$. Note that $I \subseteq (N : M)$ and hence $N = IM \subseteq (N : M)M \subseteq N$, so that $N = (N : M)M$. Moreover, a

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submodule N of R-module M is said to be a multiplication submodule, if for each submodule K of $M, N \cap K = (K : N)N$. In this paper $S(M)$ is the multiplicative semigroup of all finitely generated faithful multiplication submodule of M (See section 2 for details on the definition of the product of two submodules of M.). Let N be a submodule of an R-module M. Following [10], we call the intersection of all prime submodules of M containing N the M-radical of N, denoted by $M - \mathrm{rad}\, N$ or $\sqrt{N}$. In [10] it is shown that if N is a submodule of a finitely generated multiplication R-module M, then $M - \mathrm{rad}\, N = \sqrt{(N : M)}M$. This result has been generalized to an arbitrary multiplication module by El-Bast and Smith [1]. Let N be a submodule of M, for each ideal I of R, the residual submodule of N by I is $(N :_M I) = \{x \in M : xI \subseteq N\}$. If M is a multiplication module, then by [3], $(N :_M I) = (N : IM)M$, for each ideal I of R. Also if M is finitely generated faithful multiplication, then for ideals I, J of $R, (IM : JM) = (I : J)$. Since if $r \in (IM : JM)$, then $rJM \subseteq IM$ and by [1], M is cancellation, so $rJ \subseteq I$ hence $r \in (I : J)$. The other inclusion is clear.

Reduction of ideals of local rings was first considered by D. G. Northcott and D. Rees in [12]. Suppose that I, J are ideals of R, I is a reduction of J, if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $IJ^s = J^{s+1}$. Also $r \in R$ is integrally dependent on I if there exist $n \in \mathbb{N}$ and $c_1, c_2, ..., c_n \in R$ with $c_i \in I^i$, for $i = 1, 2, ..., n$ such that $r^n + c_1 r^{n-1} + ... + c_{n-1} r + c_n = 0$. In this paper we will define these concepts for submodules and we present a similar results as hold for ideals. For any ideal $\mathfrak{b}$ of R, the radical of $\mathfrak{b}$, denoted by $\sqrt{\mathfrak{b}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [8], [9], and [12].

The concepts of integral closure of an ideal relative to a module and Δ-closure of an ideal were introduced in [13]. In section 3 we define the concept of Δ-closure of a submodule of a noetherian module and we present some results about this notion as in [14]. Let M be a noetherian R-module and Δ be a multiplicatively closed set of submodules of M. The maximum element of $\{(NK :_M K) \mid K \in \Delta\}$ is said to be Δ-closure of a submodule N of M. We show some results about the Δ-closure of a submodule of a multiplication module. For instance it is shown that if M is a faithful multiplication module and Δ is a multiplicatively closed set of submodules of M, such that each $N \in \Delta$ is not contained in a minimal prime submodule of M, then the integral closure of submodule N is equal to the Δ-closure of N.

An R-module M is said to be a comultiplication module if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. Also M is a comultiplication module if and only if $N = (0 :_M (0 :_R N))$ for each submodule N of M(see [5]). In section 4 we introduce the concepts of coreduction of submodules and integral codependence. We give some results on coreduction over comultiplication modules.

2. REDUCTION AND CLOSURE OVER MULTIPLICATION MODULES

Definition 2.1. Let M be an R-module and N, K be submodules of M. The product of N and K is defined as $NK = (N : M)(K : M)M$. (see [4]). Note that if $N = IM, K = JM$ for some ideals I, J of R, then $NK = (IM)(JM) = (IM : M)(JM : M)M = IJM$. Therefore, if M is multiplication, then $I(NK) = (IN)K = N(IK)$ for each ideal I of R.

Definition 2.2. Let M be an R-module and N, K be submodules of M. We say that K is a reduction of N, if $K \subseteq N$ and there exists $s \in \mathbb{N}$ such that $KN^s = N^{s+1}$.

Definition 2.3. Let M be an R-module and N, K be submodules of M such that K is a reduction of N. The least integer s such that $KN^s = N^{s+1}$ is said to be the reduction number of N with respect to K and is denoted by $r_K(N)$.

Proposition 2.4. *Let M be a multiplication R-module, N, K be submodules of M and I, J be ideals of R. Then:*
1) If K is a reduction of N, then IK is a reduction of IN.
2) If I is a reduction of J, then IN is a reduction of JN.
3) If I is a reduction of J and K is a reduction of N, then IK is a reduction of JN.
4) If $K \subseteq N$ and $(K : M)$ is a reduction of $(N : M)$, then K is a reduction of N.
5) If M is finitely generated faithful and K is a reduction of N, then $(K : M)$ is a reduction of $(N : M)$.
6) If $N \in S(M)$ and K is a reduction of N, then $N = K$.

Proof. 1,2,3 and 4 are easy.
5. Since M is finitely generated faithful multiplication, it is cancellation by [1]. So the result is obtained.
6. Let $N = IM, K = JM$ for some ideals I, J of R. Since $N \in S(M)$, for each $n \in \mathbb{N}, N^n \in S(M)$. Assume that $KN^s = N^{s+1}$, for some $s \in \mathbb{N}$, then $JN^s = JI^sM = I^{s+1}M = II^sM = IN^s$. Therefore by [11, theorem 6.1], $J = I$,and so $K = JM = IM = N$. □

Proposition 2.5. *Let M be an R-module and N, K, L be submodules of M. Then:*
1) If K is a reduction of N, then for every $m \in \mathbb{N}$ and for each $j \geq r_K(N)$ we have $K^mN^j = N^{m+j}$.
2) If M is multiplication and K is a reduction of N, then $\sqrt{K} = \sqrt{N}$.
3) If K is a reduction of N and N is a reduction of L, then K is a reduction of L.

Proof. 1. Suppose that K is a reduction of N and $s = r_K(N)$. Set $I = (K : M)$ and $J = (N : M)$. Then we have the following relations:

$$\begin{aligned} K^m N^j &= I^m J^j M = I^{m-1} J^{j-s} I J^s M = I^{m-1} J^{j-s} J^{s+1} M \\ &= I^{m-1} J^{j+1} M = \ldots = I J^{j+m-1} M = J^{j+m-s-1} I J^s M \\ &= J^{j+m-s-1} J^{s+1} M = J^{j+m} M = N^{m+j}. \end{aligned}$$

2. Let $K = IM$ and $N = JM$, so by [1, theorem 2.12],

$$\sqrt{N} = \sqrt{J}M = \sqrt{J^s}M = \sqrt{N^s} = \sqrt{KN^{s-1}}$$

$$\sqrt{IM \cdot J^{s-1}M} = \sqrt{IJ^{s-1}M} \subseteq \sqrt{IM} = \sqrt{K}.$$

The other inclusion is clear.

3. Let $KN^s = N^{s+1}$ and $NL^t = L^{t+1}$, for some $s, t \in \mathbb{N}$. Therefore, by 1 we have the following:

$$KL^{ts+s} = K(L^{t+1})^s = K(NL^t)^s = KN^s L^{ts} = N^{s+1} L^{ts} = L^{ts+s+1}.$$

□

Lemma 2.6. *Let M be an R-module and K, N, L be submodules of M, such that K is a reduction of N and L. Then K is a reduction of $N + L$.*

Proof. Let $I = (K : M)$, $J = (N : M)$ and $l = (L : M)$. By assumption there exist natural numbers s, t, such that $KN^s = N^{s+1}$ and $KL^t = L^{t+1}$. Consequently we have the following relations:

$$\begin{aligned} K(N+L)^{t+s} &= I(J+l)^{t+s} M = I \sum_{i=0}^{t+s} J^i l^{t+s-i} M \\ &= \sum_{i=0}^{s} J^i l^{t+s-i} IM + \sum_{i=s}^{t+s} J^i l^{t+s-i} IM \\ &= \sum_{i=0}^{s} J^i l^{t+s-i+1} M + \sum_{i=s}^{t+s} l^{t+s-i} J^{i+1} M \\ &= \sum_{i=0}^{t+s+1} J^i l^{t+s+1-i} M = (J+l)^{t+s+1} M = (N+L)^{t+s+1}. \end{aligned}$$

□

Definition 2.7. Let M be an R-module and K be a submodule of M. If φ is the set of all submodules of M such that K is a reduction of them, then by Lemma 2.6, φ has a unique maximal element, that we say K^*. Indeed $K^* = \sum_{k_i \in \varphi} K_i$.

Proposition 2.8. *Let $(R, \mathfrak{m})$ be a local ring, M be a faithful multiplication R-module and N, K be submodules of M such that $K \subseteq N$. Then K is a reduction of N, if and only if $K + NM'$ is a reduction of N, where $M' = \mathfrak{m}M$.*

Proof. We know that $M' = \mathfrak{m}M$ is the only maximal submodule of M, also by [1, 2.8, 3.1], M is cyclic and finitely generated cancellation R-module. First we assume that $K + NM'$ is a reduction of N. Then there exists $s \in \mathbb{N}$, such that $(K + NM')N^s = N^{s+1}$. Set $K = IM$ and $N = JM$. Hence we have the following:

$$(IM + J\mathfrak{m}M)J^sM = J^{s+1}M$$

$$IJ^sM + J^{s+1}\mathfrak{m}M = J^{s+1}M$$

$$IJ^s + J^{s+1}\mathfrak{m} = J^{s+1}.$$

Now from [8, lemma 18.1.8], we deduce that $IJ^s = J^{s+1}$.

Consequently, $KN^s = IJ^sM = J^{s+1}M = N^{s+1}$. Now let $KN^s = N^{s+1}$, for some $s \in \mathbb{N}$. So we have the following:

$$(K + NM')N^s = KN^s + N^{s+1}M' = N^{s+1} + N^{s+1}M' = N^{s+1}.$$

□

Definition 2.9. Let M be an R-module and N be a submodule of M. An element $x \in R$ is said to be integrally dependent on N, if there exists $n \in \mathbb{N}$ such that $x^nM \subseteq \sum_{i=1}^{n} x^{n-i}N^i$.

Theorem 2.10. *Let M be a multiplication R-module and $N = IM$ be a submodule of M. An element x of R is integrally dependent on N, if and only if N is a reduction of $K = N + xM$.*

Proof. Let $x \in R$ be integrally dependent on N and so $x^nM \subseteq \sum_{i=1}^{n} x^{n-i}N^i$ for some $n \in \mathbb{N}$. We claim that $NK^{n-1} = K^n$, because

$$NK^{n-1} = I(I + Rx)^{n-1}M \subseteq (I + Rx)^nM = K^n$$

and also

$$K^n = (I + Rx)^n M = \sum_{i=0}^{n} I^i (Rx)^{n-i} M = (Rx)^n M + I(Rx)^{n-1} M + \cdots + I^n M$$
$$= I(Rx^{n-1}M + I^1 Rx^{n-2}M + I^2 Rx^{n-3} + \cdots + I^{n-1}M) + (Rx)^n M$$
$$\subseteq I(I + Rx)^{n-1}M + \sum_{i=1}^{n} x^{n-i} N^i$$
$$= I(I + Rx)^{n-1}M + I(Rx^{n-1} + IRx^{n-2} + \cdots + I^{n-1})M$$
$$= I(I + Rx)^{n-1}M + I(I + Rx)^{n-1}M$$
$$= I(I + Rx)^{n-1}M = NK^{n-1}.$$

For the converse, suppose that $N = IM$ be a reduction of $N + xM$, so there exists a natural number n such that $I(I + Rx)^n M = (I + Rx)^{n+1}M$. So $I(\sum_{i=0}^{n} I^i x^{n-i})M = (\sum_{i=0}^{n+1} I^i x^{n+1-i})M$. Therefore, $(\sum_{i=0}^{n} I^{i+1} x^{n-i})M = (\sum_{i=0}^{n+1} I^i x^{n+1-i})M$, and it means $x^{n+1}M \subseteq (\sum_{i=1}^{n+1} I^i x^{n+1-i})M$. □

Theorem 2.11. *Let M be a multiplication R-module and $N = IM$, $K = JM$ be submodules of M such that $N \subseteq K$. If each element of J is integrally dependent on N, then N is a reduction of K*

Proof. Suppose that each element of J is integrally dependent on N. Then by 2.10, for every $x \in J$, the R-module N is a reduction of $(I + Rx)M$. So there exists a number $n_x \in \mathbb{N}$, such that $I(I + Rx)^{n_x}M = (I + Rx)^{n_x+1}M$. Set $n = \sum_{x \in J} n_x$ where x is a generator of J. Hence $I(I + J)^n M = (I + J)^{n+1}M$ and so $NK^n = K^{n+1}$. □

Corollary 2.12. *Let M be a finitely generated faithful multiplication R-module, $N = IM$ be a submodule of M and $J = \{x \in R$: x is integrally dependent on $N\}$. Set $\bar{N} = JM$, then $\bar{I}M = \bar{N} \subseteq N^*$, where $\bar{I} = \{x \in R$: x is integrally dependent on $I\}$.*

Proof. It is clear from 2.11, that $\bar{N} = JM \subseteq N^*$.

Assume that $x \in \bar{I}$, so there exist $c_i \in I^i$ for $i = 0, 1, ..., n$ such that $\sum_{i=0}^{n} c_i x^{n-i} = 0$, then $x^n \in \sum_{i=1}^{n} x^{n-i} I^i$ and so $x^n M \subseteq \sum_{i=1}^{n} x^{n-i} I^i M = \sum_{i=1}^{n} x^{n-i} N^i$. Therefore, $x \in J$. For the reverse inclusion suppose that $x \in J$ so there exists $n \in \mathbb{N}$ such that $x^n M \subseteq \sum_{i=1}^{n} x^{n-i} N^i = \sum_{i=1}^{n} x^{n-i} I^i M$, by [1], M is cancellation and so $x^n \in \sum_{i=1}^{n} x^{n-i} I^i$. Therefore, there exist $c_i \in I^i$ for $i = 1, ..., n$ such that $x^n = \sum_{i=1}^{n} c_i x^{n-i}$, hence $\sum_{i=0}^{n} c_i x^{n-i} = 0$, so $x \in \bar{I}$. We show that $\bar{I} = J$ and so $\bar{N} = JM = \bar{I}M$. □

3. Δ-closure of a submodule of multiplication modules

In this section M is a noetherian R-module. If M is a multiplication R-module and N is a submodule of M, then by [3], for each ideal I of R, $(N :_M I) = (N : IM)M$. Moreover, for each submodule $K = JM$ of M, by considering the definition of product of submodules $(N :_M K) = (N : K)M$. Since $(N :_M K) = (N :_M JM) = (N :_M J) = (N : JM)M = (N : K)M$.

Definition 3.1. Let M be a noetherian R-module, and Δ be a multiplicatively closed set of submodules of M. Then the set $\{(NK :_M K) \mid K \in \Delta\}$ has a maximum element. Since $(NK :_M K) \subseteq (NLK :_M LK)$ and $(NL :_M L) \subseteq (NLK :_M LK)$, it follows that the maximum element of this set is unique. We show this maximum element with N_Δ and we call it the Δ-closure of N.

Lemma 3.2. *Let M be an R-module, N, K be submodules of M and Δ be a multiplicatively closed set of submodules of M. Then:*
1) $N \subseteq N_\Delta$.
2) If $N \subseteq K$, then $N_\Delta \subseteq K_\Delta$.
3) $N_\Delta K_\Delta \subseteq (NK)_\Delta$.
4) $IN_\Delta \subseteq (IN)_\Delta$, for each ideal I of R.

Proof. Proofs of 1 and 2 are easy.
3. Let $m = xy \in N_\Delta K_\Delta$. Then there exist submodules L_1 and L_2 of M such that $xL_1 \subseteq NL_1$ and $yL_2 \subseteq KL_2$. Consequently $xyL_1L_2 \subseteq KNL_1L_2$ and so $xy \in (KNL_1L_2 :_M L_1L_2) \subseteq (KN)_\Delta$.
4. $IN_\Delta = I(NK :_M K)$, for suitable submodule K of M. So $IN_\Delta \subseteq (INK :_M K) \subseteq (IN)_\Delta$. □

Lemma 3.3. *Suppose that M is an R-module, N is a submodule of M and Δ is a multiplicatively closed set of submodules of M. Then for each $K \in \Delta$, $N_\Delta = (N_\Delta K :_M K)$.*

Proof. Obviously $N_\Delta \subseteq (N_\Delta K :_M K)$. Let $m \in (N_\Delta K :_M K)$ and $N_\Delta = (NL :_M L)$, for suitable $L \in \Delta$. So $mK \subseteq N_\Delta K = (NL :_M L)K \subseteq (NLK :_M L)$. Therefore, $m \in ((NLK :_M L) :_M K) = (NLK :_M LK) \subseteq N_\Delta$. □

Proposition 3.4. *Let M be an R-module, Δ be a multiplicatively closed set of submodules of M and $\{N_i\}_{i\in I}$ be a family of submodules of M. Then:*
1)$(\cap_{i\in I} N_i)_\Delta \subseteq \cap_{i\in I} (N_i)_\Delta$.
2)$(N_\Delta)_\Delta = N_\Delta$.
3)$\sum_{i\in I} (N_i)_\Delta \subseteq (\sum_{i\in I} N_i)_\Delta$.

Proof. 1. For each $i \in I, \cap_{i\in I} N_i \subseteq N_i$, so for each $i \in I, (\cap_{i\in I} N_i)_\Delta \subseteq (N_i)_\Delta$ and hence $(\cap_{i\in I} N_i)_\Delta \subseteq \cap_{i\in I}(N_i)_\Delta$.
2. It is clear by 3.3.
3. $N_i \subseteq \sum_{i\in I} N_i$, for each $i \in I$, so $\sum_{i\in I}(N_i)_\Delta \subseteq (\sum_{i\in I} N_i)_\Delta$. □

Corollary 3.5. *Let M be an R-module, N be a submodule of M and Δ be a multiplicatively closed subset of submodules of M. Then:*
1)$((N : I)M)_\Delta \subseteq (N_\Delta :_M I)$, for each ideal I of R.
2) $(IN)_\Delta = (IN_\Delta)_\Delta$, for each ideal I of R.
3) $(N_\Delta :_M K_\Delta) = (N_\Delta :_M K)$, for each submodule $K \in \Delta$.

Proof. 1. Since $I(N : I)M \subseteq N$, so by 3.2 we have

$$I((N : I)M)_\Delta \subseteq (I(N : I)M)_\Delta \subseteq N_\Delta$$

thus $((N : I)M)_\Delta \subseteq (N_\Delta :_M I)$.
2. $IN \subseteq IN_\Delta \subseteq (IN)_\Delta$, so by 3.2 and 3.4

$$(IN)_\Delta \subseteq (IN_\Delta)_\Delta \subseteq ((IN)_\Delta)_\Delta = (IN)_\Delta.$$

3.It is clear that $(N_\Delta :_M K_\Delta) \subseteq (N_\Delta :_M K)$, since by 3.2, $K \subseteq K_\Delta$. Moreover, $K(N_\Delta :_M K) \subseteq N_\Delta$, so by 3.2 and 3.3

$$K_\Delta(N_\Delta :_M K) \subseteq (K(N_\Delta :_M K))_\Delta \subseteq (N_\Delta K :_M K)_\Delta = (N_\Delta)_\Delta = N_\Delta$$

hence $(N_\Delta :_M K) \subseteq (N_\Delta :_M K_\Delta)$. □

Lemma 3.6. *Suppose that M is an R-module, N is a submodule of M and Δ is a multiplicatively closed set of submodules of M. Then for each $K \in \Delta, N_\Delta = (N_\Delta K :_M K) = ((NK)_\Delta :_M K)$.*

Proof. $N_\Delta = (N_\Delta K :_M K) \subseteq (N_\Delta K_\Delta :_M K) \subseteq ((NK)_\Delta :_M K)$. Now let $m \in ((NK)_\Delta :_M K)$, then $mK \subseteq (NK)_\Delta = (NKL :_M L)$ for suitable submodule L of M. Hence $mKL \subseteq NKL$ and so $m \in (NKL :_M KL) \subseteq N_\Delta$. □

Theorem 3.7. *Let M be a faithful multiplication module and $N = IM$ be a submodule of M. If Δ is a multiplicatively closed set of submodules of M . Then $\bar{N} \subseteq N_\Delta$.*

Proof. Let $x \in \bar{N} = \bar{I}M$. Then $x = im$ for some $i \in \bar{I}$ and N is a reduction of $(I + Ri)M$, by 2.10 and 2.12. For some $n \in \mathbb{N}$, we have

$$I(I + Ri)^n M = (I + Ri)^{n+1} M = (I + Ri)^n (I + Ri)M \quad = (I + Ri)^n IM + (I + Ri)^n iM.$$

So $iM(I + Ri)^n \subseteq (I + Ri)^n IM$, it follows that $i \in ((I + Ri)^n IM :_R (I + Ri)^n M)$. Set $(I + Ri)^n M = K$. Therefore, $i \in (KN :_R K)$ and so $iK \subseteq KN$ and $iKM \subseteq KNM = KN$. Consequently $im \in (KN :_M K) \subseteq N_\Delta$. □

Theorem 3.8. *Let R be a noetherian ring, M be a faithful multiplication R-module, Δ be a multiplicatively closed set of submodules of M, and Λ be the set of all submodules of M that are not contained in any minimal prime submodules. If $\Delta \subseteq \Lambda$, then $\bar{N} = N_\Delta$ for each submodule N of M.*

Proof. Let $N = IM$ for some ideal I of R, Λ_1 be the set of all ideals of R that are not contained in any minimal prime ideal, and $\Delta_1 = \{J \trianglelefteq R; JM \in \Delta\}$. By [1, 2.11, 3.1], $\Delta_1 \subseteq \Lambda_1$, so by [13, 3.2], for each ideal I of $R, I_{\Delta_1} \subseteq \bar{I}$. We claim that $N_\Delta = I_{\Delta_1} M$ and therefore, by 2.12

$$N_\Delta = I_{\Delta_1} M \subseteq \bar{I}M = \bar{N}.$$

Assume that $m \in I_{\Delta_1} M$, so $m = im_1$ and $i \in I_{\Delta_1} = (IJ : J)$ for some ideal J of R. Thus $iJM \subseteq IJM$, so $m = im_1 \in (IJM : JM)M \subseteq (IJM :_M JM) = (NK :_M K) \subseteq N_\Delta$, where $K = JM \in \Delta$. On the other hand since M is finitely generated faithful multiplication, we have

$$N_\Delta = (NK :_M K) = (NK : K)M = (IJM : JM)M = (IJ : J)M \subseteq I_{\Delta_1} M$$

□

Theorem 3.9. *With the conditions of Theorem 3.8, if $L, K \in \Delta$ and $\overline{NK} = \overline{LK}$, then $\overline{N} = \overline{L}$.*

Proof. Let M be a faithful multiplication R-module and N, L, K be submodules of M such that $\overline{NK} = \overline{LK}$. Then $(NK)_\Delta = (LK)_\Delta$. Suppose that $m \in \bar{N} = \bar{I}M$. Hence $m = xm'$ for some $x \in \bar{I}$. By 2.10 and 2.12 N is a reduction of $(I + Rx)M$ and there exists $n \in \mathbb{N}$, such that $I(I + Rx)^n M = (I + Rx)^{n+1} M$. Now as we show in 3.7 $x(I + Rx)^n M \subseteq I(I + Rx)^n M$ and so for all $K \in \Delta$,

$$xK(I + Rx)^n M \subseteq K(I + Rx)^n IM.$$

Therefore,

$$x \in (NK(I+Rx)^n : K(I+Rx)^nM) \subset ((NK)_\Delta(I+Rx)^nM :_R K(I+Rx)^nM)$$
$$= ((LK)_\Delta(I+Rx)^nM :_R K(I+Rx)^nM).$$

Now by 3.6

$$m = xm' \in ((LK)_\Delta(I+Rx)^nM :_M K(I+Rx)^nM)$$
$$= (L_\Delta K(I+Rx)^nM :_M K(I+Rx)^nM)$$
$$= L_\Delta = \bar{L}.$$

Consequently $\bar{N} \subseteq \bar{L}$. In the similar way we can prove that $\bar{L} \subseteq \bar{N}$. □

Theorem 3.10. *Let Δ, N and K be as in theorem 3.8. The following are equivalent:*
1) $NL = KL$ for some $L \in \Delta$;
2) $\overline{NT} = \overline{KT}$ for every $T \in \Delta$;
3) $\bar{N} = \bar{K}$.

Proof. $1 \to 2$) Suppose that $m \in \overline{NT} = \overline{IJ}M$, where $N = IM$ and $T = JM$. Then $m = xm'$, where $x \in \overline{IJ}$. Hence NT is a reduction of $(IJ+Rx)M$ and so there exists $s \in \mathbb{N}$, such that

$$x(IJ+Rx)^sM \subseteq IJ(IJ+Rx)^sM.$$

Let $L = lM$. Then $xl(IJ+Rx)^sM \subseteq IJl(IJ+Rx)^sM$ and we have the following relations:

$$xL(IJ+Rx)^sM \subseteq TNL(IJ+Rx)^sM = TKL(IJ+Rx)^sM.$$

This shows that $x \in (TKL(IJ+Rx)^sM :_R L(IJ+Rx)^sM)$ and $m \in (TKL(IJ+Rx)^sM :_M L(IJ+Rx)^sM) \subseteq (TK)_\Delta = \overline{TK}$. Similary, we prove that $\overline{TK} \subseteq \overline{NT}$
$2 \to 3$) By 3.9.
$3 \to 1$) $\bar{N} = N_\Delta = (NT_1 :_M T_1) = K_\Delta = (KT_2 :_M T_2)$ for some $T_1, T_2 \in \Delta$. Set $L = T_1T_2$. Clearly $L \in \Delta$ and $NL = (NL :_M L)L = (KL :_M L)L = KL$ because $(NT_1T_2 :_M T_1T_2) = (NL :_M L) \subseteq N_\Delta = \bar{N} = \bar{K}$ and $N_\Delta = (NT_1 :_M T_1) \subseteq (NT_1T_2 :_M T_1T_2)$. Consequently $(NL :_M L) = N_\Delta = K_\Delta = (KL :_M L)$. □

4. Coreduction over comultiplication modules

Definition 4.1. Let M be an R-module and N, K be submodules of M. The coproduct of N and K is defined in [4] as $C(NK) = (0 :_M Ann(N)Ann(K))$. It is easy to see that if M is comultiplication and $N = (0 :_M I), K = (0 :_M J)$, then $C(NK) = (0 :_M IJ)$.

Definition 4.2. Let M be an R-module and N, K be submodules of M such that $N \subseteq K$. We say that K is a coreduction of N, if there exists a natural number s such that $(0 :_M Ann(K)Ann^s(N)) = (0 :_M Ann^{s+1}(N))$. We denote it by $C(KN^s) = C(N^{s+1})$. The least integer s such that $C(KN^s) = C(N^{s+1})$ is said to be the coreduction number of N with respect to K and is denoted by $r_K(N)$.

Lemma 4.3. *Let M be an R-module and $N \subseteq K$ be submodules of M such that K is a coreduction of N. Then for any $m \in \mathbb{N}$ and any $j \geq r_k(N)$; $C(K^m N^j) = C(N^{m+j})$.*

Proof. Let $I = Ann(K)$, $J = Ann(N)$ and $r_k(N) = s$. Then we have the following:

$$\begin{aligned} C(K^m N^j) &= (0 :_M I^m J^j) = (0 :_M IJ^s I^{m-1} J^{j-s}) \\ &= \big((0 :_M IJ^s) :_M I^{m-1} J^{j-s}\big) \\ &= \big((0 :_M J^{s+1}) :_M I^{m-1} J^{j-s}\big) \\ &= (0 :_M I^{m-1} J^{j+1}) = \ldots = (0 :_M IJ^{j+m-1}) \\ &= ((0 :_M IJ^s) :_M J^{j+m-1-s}) = (0 : J^{s+1} J^{j+m-1-s}) \\ &= (0 :_M J^{j+m}) = C(N^{m+j}). \end{aligned}$$

□

Lemma 4.4. *Let M be an R-module and $L \subseteq N \subseteq K$ be submodules of M. If K is a coreduction of N and N is a coreduction of L then K is a coreduction of L.*

Proof. Easy. □

Lemma 4.5. *Let M be an R-module and a submodule K of M be a coreduction of submodules N and L of M. Then K is a coreduction of $N \cap L$.*

Proof. Let $K = (0 :_M I), N = (0 :_M J), L = (0 :_M l)$. By assumption $N \subseteq K$ and $L \subseteq K$ and so $N \cap L \subseteq K$. Also we have $C(KL^t) = C(L^{t+1}), C(KN^s) = C(N^{s+1})$, for suitable $s, t \in \mathbb{N}$.

So we have:

$$
\begin{aligned}
C(K(N\cap L)^{t+s}) &= C((0:_M I)\big((0:_M J)\cap(0:_M l)\big)^{t+s}) \\
&= C((0:_M I)\big(0:_M (J+l)^{t+s}\big)) \\
&= \big(0:_M I(J+l)^{t+s}\big) \\
&= (0:_M \sum_{i=0}^{t+s} IJ^i l^{t+s-i}) \\
&= \left(0:_M \sum_{i=0}^{t+s+1} J^i l^{t+s+1-i}\right) \\
&= \bigcap_{i=0}^{t+s+1}(0:_M J^i l^{t+s+1-i}) = C((N\cap L)^{t+s+1}).
\end{aligned}
$$

□

Definition 4.6. Let K be a submodule of R-module M and φ be the set of all submodules of M such that K is a coreduction of them. By the previous lemma φ has a unique minimum element. We denote it by $\underline{K}$. Indeed $\underline{K} = \bigcap_{K_i\in\varphi} K_i$.

Definition 4.7. Let M be an R-module and N be a submodule of M. An element $x\in R$ is said to be integrally codependent on N, if there exists $n\in\mathbb{N}$, such that $\bigcap_{i=1}^{n}(C(N^i):_M x^{n-i})\subseteq (0:_M x^n)$.

Theorem 4.8. *Let M be a comultiplication R-module and $N=(0:_M I)$ be a submodule of M. Then an element $x\in R$ is integrally codependent on N, if and only if N is a coreduction of $K=(0:_M I+Rx)$.*

Proof. Let x be integrally codependent on N. Then there exists $n\in\mathbb{N}$ such that $\bigcap_{i=1}^{n}(C(N^i):_M x^{n-i})\subseteq(0:_M x^n)$. We prove that $C(NK^n)=C(K^{n+1})$.

$$C(K^{n+1})=(0:_M (I+Rx)^{n+1})\subseteq(0:_M I(I+Rx)^n)=C(NK^n).$$

Now suppose that $m\in C(NK^n)=(0:_M I(I+Rx)^n)$. Then $mI(I+Rx)^n=0$ and this follows that:

$$mI^{n+1}+mI^nRx+\cdots+mIRx^n=0. \tag{4.1}$$

On the other hand:

$$\begin{aligned} m \in (0 :_M I(I+Rx)^n) &= \big((0 :_M I) :_M (I+Rx)^n\big) \\ &= (N :_M (I+Rx)^n) \\ &= (N :_M \sum_{i=0}^{n} I^i (Rx)^{n-i}) \\ &= \bigcap_{i=0}^{n} \big(N :_M I^i (Rx)^{n-i}\big) \\ &= \bigcap_{i=0}^{n} \big((N :_M I^i) :_M (Rx)^{n-i}\big) \\ &= \bigcap_{i=0}^{n} \big(C(N^{i+1}) :_M (Rx)^{n-i}\big) \\ &\subseteq (0 :_M x^n) \\ &\subseteq (0 :_M x^{n+1}). \end{aligned}$$

Consequently

$$mx^{n+1} = 0. \tag{4.2}$$

Now from(4.1),(4.2), we conclude that

$$mI^{n+1} + mI^n(Rx) + \cdots + mI(Rx)^n + mx^{n+1} = 0.$$

Hence $m \in \big(0 :_M (I+Rx)^{n+1})\big) = C(K^{n+1})$.

Conversely, let N be a coreduction of K and $C(NK^{n-1}) = C(K^n)$ for some $n \in \mathbb{N}$. Therefore,

$$(0 :_M I(I+Rx)^{n-1}) = (0 :_M (I+Rx)^n).$$

We claim that

$$\bigcap_{i=1}^{n} (C(N^i) :_M x^{n-i}) \subseteq (0 :_M x^n).$$

Let $m \in \bigcap_{i=1}^{n} (C(N^i) :_M x^{n-i})$. Then for each $i = 1, \ldots, n$,

$$m \in (C(N^i) :_M x^{n-i}) = ((0 :_M I^i) :_M x^{n-i})) = (0 :_M I^i x^{n-i})$$

and so $mI^i(Rx)^{n-i} = 0$ for each $i = 1, \ldots, n$. Consequently

$$mI^n + mI^{n-1}Rx + \cdots + mI(Rx)^{n-1} = 0. \tag{4.3}$$

Hence $m \in (0 :_M I(I+Rx)^{n-1}) = \big(0 :_M (I+Rx)^n\big)$ and this follows that

$$mI^n + mI^{n-1}Rx + \cdots + mI(Rx)^{n-1} + m(Rx)^n = 0. \tag{4.4}$$

Now from (4.3) and (4.4), we conclude that $mx^n = 0$ and so $m \in (0 :_M x^n)$. □

Corollary 4.9. *Let R be a noetherian ring, M be a comultiplication R-module and $K \subseteq N$ be submodules of M. Then:*
1) If each element of $Ann(K)$ is integrally codependent on N, then N is a coreduction of K.
2) If $J = \{x \in R$: x is integrally codependent on $N\}$, then $\underline{N} \subseteq (0 :_M J)$

Proof. 1) Let each element of $Ann(K)$ be integrally codependent on N. Then $N = (0 :_M I)$ is a coreduction of $(0 :_M I + Rx)$ for every $x \in Ann(K)$ and so N is a coreduction of $(0 :_M I + Ann(K)) = (0 :_M I) \cap (0 :_M Ann(K)) = N \cap K = K$.
2) It is clear from 1 and 4.6. □

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References

[1] Z. Abd El-Bast and P. F. Smith, , *Multiplication modules* , Comm. Algebra., **16** No. 4 (1988) 755-779.

[2] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules, for associated primes of local cohomology modules*, Comm. Algebra., **36** No. 12 (2008) 4620-4642.

[3] M. M. Ali, *Residual submodules of multiplication modules*, Beitr. Algebra Geom., **46** No. 2 (2005) 405-422.

[4] H. Ansari-Toroghy and F. Farshadifar, *Product and dual product of submodules*, Far. East. J. Math. Sci., **25** No. 3 (2007) 447-455.

[5] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math., **11** No. 4 (2007) 1189-1201.

[6] R. Ameri,*On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci., No. 27 (2003) 1715-1724.

[7] A. Barnard, *Multiplication modules*, Manuscripta Math., **71** No. 1 (1981) 174-178.

[8] M.P. Brodmann and R.Y. Sharp, *Local cohomology; an algebraic introduction with geometric applications,* Cambridge University Press, Cambridge, 1998.

[9] H. Matsumura, *Commutative ring theory*, Cambridge University press, Cambridge, UK, 1986.

[10] R. L. McCasland and M. E. Moore, *On radicals of submodules of finitely generated modules*, Canad. Math. Bull., **29** (1986) 37-39.

[11] A. G. Naoum and A.S. Mijbass, *Weak cancellation modules*, Kyungpook Math. J., **37** (1997) 73-82.

[12] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc., **50** (1954) 145-158.

[13] L. J. Ratliff, Jr., Δ - *closure of ideals and rings*, Trans. Amer. Math. Soc., **313** (1989) 221-247.

[14] Y. Tiras, *Integral closure of an ideal relative to a module and* Δ *- closure*, Trans. Amer. Math. Soc., **21** (1997) 381-386.

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