

Research Paper

**ON THE SMALL INTERSECTION GRAPH OF SUBMODULES OF A  
 MODULE**

LOTF ALI MAHDAVI\* AND YAHYA TALEBI

ABSTRACT. Let  $M$  be a unitary left  $R$ -module, where  $R$  is a (not necessarily commutative) ring with identity. The small intersection graph of nontrivial submodules of  $M$ , denoted by  $\Gamma(M)$ , is an undirected simple graph whose vertices are in one-to-one correspondence with all nontrivial submodules of  $M$  and two distinct vertices are adjacent if and only if the intersection of corresponding submodules is a small submodule of  $M$ . In this paper, we investigate the fundamental properties of these graphs to relate the combinatorial properties of  $\Gamma(M)$  to the algebraic properties of the module  $M$ . We determine the diameter and the girth of  $\Gamma(M)$ . We obtain some results for connectivity and planarity of these graphs. Moreover, we study orthogonal vertex, domination number and the conditions under which the graph  $\Gamma(M)$  is complemented.

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\*Corresponding author

## 1. INTRODUCTION

Let  $S = \{S_i | i \in I\}$  be a family of sets. By the intersection graph  $\Gamma(S)$  of  $S$  we mean the graph whose vertex set is  $S$  and in which the vertices  $S_i$  and  $S_j$  are adjacent if and only if  $S_i \cap S_j \neq \emptyset$ , for  $i \neq j$ . (see [18, p. 1]). In this paper, we introduce a kind of intersection graph of an algebraic structure. The intersection graph of algebraic structures has been studied by several authors. As a pioneer, J. Bosak [8] in 1964, defined the graph of semigroups. Inspired by his work, B. Csakany and G. Pollak [11] in 1969, studied the graph of subgroups of a finite group. The intersection graphs of finite abelian groups studied by B. Zelinka [21] in 1975. In 2009, the intersection graph of ideals of a ring, was considered by I. Chakrabarty et. al. in [9]. S. H. Jafari and N. Jafari Rad in [12,13] in 2010 and 2011, studied the planarity of intersection graphs of ideals of rings and domination in the intersection graph of rings and modules, respectively. For more results on the intesection graphs of ideals see [1, 2]. In 2012, on a graph of ideals, researched by A. Amini et. al. in [4] and also, intersection graph of submodules of a module, introduced by S. Akbari et. al. in [3]. In 2013, the intersection graph of a module, was considered by E. Yaraneri in [20] and in 2016, we introduce the dual graph of the intersection graph of submodules of a module in [14]. Recently, we searched some properties of the intersection graph and the co-intersection graph of submodules of a module, see [15,16,17]. Motivated by previous studies on the intersection graph of algebraic structures, in this paper, we define the small intersection graph of submodules of a module. Our main goal is to study the connection between the algebraic properties of a module and the graph theoretic properties of the graph associated with it.

Throughout this paper,  $R$  is a (not necessarily commutative) ring with identity and  $M$  is a unitary left  $R$ -module. We mean from a nontrivial submodule of  $M$  is a nonzero proper left submodule of  $M$ . A submodule  $N$  of an  $R$ -module  $M$  is called *small* or *superfluous* in  $M$  (we write  $N \ll M$ ), if for every submodule  $X \subseteq M$ , with  $N + X = M$  implies that  $X = M$ . The *radical* of an  $R$ -module  $M$ , denoted by  $Rad(M)$ , is the sum of all small submodules of  $M$ , and also, is the intersection of all maximal submodules of  $M$ , by [5, Proposition 9.13]. A submodule  $K$  of a nonzero module  $M$  is said to be *large* or *essential* (we write  $K \trianglelefteq M$ ), if  $K \cap L \neq (0)$ , for every nonzero proper submodule  $L$  of  $M$ . If every nonzero submodules of  $M$  is essential in  $M$ , then  $M$  is called a *uniform* module. A nonzero  $R$ -module  $M$  is called *hollow*, if every proper submodule of  $M$  is small in  $M$ . An  $R$ -module  $M$  is called *uniserial*, if any two submodules are comparable. A nonzero  $R$ -module  $M$  is called *local*, if it has a unique maximal submodule, i.e., a proper submodule which contains all other proper submodules. A nonzero  $R$ -module  $M$  is said to be *simple*, if it has no nontrivial submodule. The *socle* of an  $R$ -module  $M$ , denoted by  $Soc(M)$ , is the sum of all simple submodules of  $M$ , and also, is the intersection of all essential submodules of  $M$ , by [5, Proposition 9.7]. The module  $M$  is called

*semisimple* if it is a direct sum of simple submodules.

Let  $\Gamma$  be a graph with the vertex set  $V(\Gamma)$ . By order of  $\Gamma$ , we mean the number of vertices of  $\Gamma$  and we denoted it by  $|\Gamma|$ . A graph  $\Gamma$  is finite, if  $|\Gamma| < \infty$ , otherwise,  $\Gamma$  is infinite. If  $x$  and  $y$  are two adjacent vertices of  $\Gamma$ , then we write  $x \leftrightarrow y$ . The degree of a vertex  $\nu$  in a graph  $\Gamma$ , denoted by  $deg(\nu)$ , is the number of edges incident with  $\nu$ . The minimum degree of  $\Gamma$  is  $\delta(\Gamma)$ . Let  $x$  and  $y$  be two distinct vertices of  $\Gamma$ . An  $x, y$ -path is a path with starting vertex  $x$  and ending vertex  $y$ . For distinct vertices  $x$  and  $y$ ,  $d(x, y)$  is the least length of an  $x, y$ -path. If  $\Gamma$  has no such a path, then  $d(x, y) = \infty$ . The *diameter* of  $\Gamma$ , denoted by  $diam(\Gamma)$ , is the supremum of the set  $\{d(x, y): x \text{ and } y \text{ are distinct vertices of } \Gamma\}$ . A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. The *girth* of a graph is the length of its shortest cycle. A graph with no-cycle has infinite girth. A graph is said to be *connected*, if there is a path between every pair of vertices of the graph. A *tree* is a connected graph which does not contain a cycle. A *star graph* is a tree consisting of one vertex adjacent to all the others. A complete graph with  $n$  distinct vertices, denoted by  $K_n$ . A *complete bipartite graph* with two part sizes  $m$  and  $n$  is denoted by  $K_{m,n}$ . By a *clique* in a graph  $\Gamma$ , we mean a complete subgraph of  $\Gamma$ . A graph is said to be *planar*, if it has a drawing in a plane without crossings. We say that two distinct vertices  $u$  and  $v$  of the graph  $\Gamma(M)$  are *orthogonal*, denoted by  $u \perp v$ , if  $u$  and  $v$  are adjacent in  $\Gamma(M)$  and there is no vertex  $w \in \Gamma(M)$  which be adjacent to both  $u$  and  $v$ . A graph  $\Gamma$  is called *complemented*, if for each vertex  $v$  of  $\Gamma$ , there is a vertex  $w$  of  $\Gamma$  (called a complement of  $v$ ) such that  $v \perp w$ . By a *dominating set*(DS) in a graph  $\Gamma$ , we mean a subset  $D$  of vertex set  $V(\Gamma)$  such that every vertex in  $\Gamma \setminus D$  is adjacent to at least one vertex in  $D$ . A DS is called *minimal dominating set*, denoted by mDS, if for any subset  $S$  of DS with  $S \neq DS$ ,  $S$  is not a DS. The *domination number* of  $\Gamma$ , written  $\gamma(\Gamma)$ , is the smallest of the cardinalities of the minimal dominating sets of  $\Gamma$ .

## 2. Connectivity, Diameter, Girth and Planarity of $\Gamma(M)$

In this section, we characterize some modules whose small intersection graphs of nontrivial submodules are connected, complete and planar. Also, the diameter and the girth of  $\Gamma(M)$  are determined.

**Definition 2.1.** The small intersection graph of nontrivial submodules of an  $R$ -module  $M$ , denoted by  $\Gamma(M)$ , is an undirected simple graph whose vertices are in one-to-one correspondence with all nontrivial submodules of  $M$  and two distinct vertices  $X$  and  $Y$  are adjacent if and only if  $X \cap Y \ll M$ .

**Proposition 2.2.** *Let  $M$  be an  $R$ -module with the graph  $\Gamma(M)$ . Then  $\Gamma(M)$  is complete, if one of the following holds.*

- (1) *If  $M = M_1 \oplus M_2$ , where  $M_1$  and  $M_2$  are two simple  $R$ -modules.*
- (2) *If  $M$  is hollow.*

*Proof.* (1) Assume that  $M = M_1 \oplus M_2$  such that  $M_1$  and  $M_2$  are two simple  $R$ -modules. Then every nontrivial submodule of  $M$  is simple. Let  $N$  and  $K$  be two distinct vertices of the graph  $\Gamma(M)$ . Then they are the nontrivial submodules of  $M$  which are simple and minimal. Moreover,  $N \cap K \subseteq N, K$  and if  $N \cap K \neq (0)$ , then minimality of  $N$  and  $K$  implies that  $N \cap K = N = K$ , which is a contradiction. Therefore,  $N \cap K = (0) \ll M$ , thus  $\Gamma(M)$  is complete.

(2) Let  $M$  be a hollow  $R$ -module. Suppose that  $N_1$  and  $N_2$  are two distinct vertices of the graph  $\Gamma(M)$ . Hence  $N_1$  and  $N_2$  are two nonzero small submodule of  $M$ . As  $N_1 \cap N_2 \subseteq N_i$ , for  $i = 1, 2$ , by [5, Proposition 5.17(1)],  $N_1 \cap N_2 \ll M$ . Therefore,  $\Gamma(M)$  is a complete graph.  $\square$

**Corollary 2.3.** *Let  $M$  be an  $R$ -module. Then the following hold:*

- (1) *If  $M$  is a uniserial or local  $R$ -module, then the graph  $\Gamma(M)$  is complete.*
- (2) *Every nonzero small submodule of  $M$  is adjacent to all other vertices of  $\Gamma(M)$  and the induced subgraphs on the sets of small submodules of  $M$  are cliques.*

*Proof.* (1) Suppose that  $M$  is a uniserial  $R$ -module. Then each two nontrivial submodules of  $M$  are comparable. Clearly, every nontrivial submodule of  $M$  is a small submodule. Hence  $M$  is a hollow  $R$ -module. Also by [19, 41.4(2)], local  $R$ -modules are hollow and the rest follows from Part 2 of Proposition 2.2.

(2) Obvious.  $\square$

**Example 2.4.** (1) For every prime number  $p$  and for all  $n \in \mathbb{N}$  with  $n \geq 2$ , the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^n}$  is local, then it is hollow. Also since each two submodules of  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty}$  are comparable, then every proper submodule of  $\mathbb{Z}_{p^\infty}$  is small in  $\mathbb{Z}_{p^\infty}$ . Hence for every prime number  $p$ , the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty}$  is hollow. Thus by Part 2 of Proposition 2.2,  $\Gamma(\mathbb{Z}_{p^n})$  and  $\Gamma(\mathbb{Z}_{p^\infty})$  are two complete graphs. (Clearly,  $\Gamma(\mathbb{Z}_{p^n}) \cong K_{n-1}$ .)

(2) The induced subgraph on the set of finitely generated submodules of the left  $\mathbb{Z}$ -module  $\mathbb{Q}$  are cliques in the graph  $\Gamma(\mathbb{Q})$ . To see this, assume that  $X \subseteq \mathbb{Q}$  and  $X$  is finitely generated. Let  $X = \langle q_1, q_2, \dots, q_n \rangle$ , where  $q_i \in \mathbb{Q}$ , for  $1 \leq i \leq n$ . Hence  $X = q_1\mathbb{Z} + q_2\mathbb{Z} + \dots + q_n\mathbb{Z}$ . Now, for  $Y \subseteq \mathbb{Q}$ , if  $X + Y = \mathbb{Q}$ , then as  $X$  has a spanning set  $\{q_1, q_2, \dots, q_n\}$ , thus  $\{q_1, q_2, \dots, q_n\} \cup X$  is a spanning set of  $\mathbb{Q}$  and it can be possible if  $X$  is a spanning set of  $\mathbb{Q}$ . Hence  $Y = \mathbb{Q}$ .

Consequently, every finitely generated submodule of  $\mathbb{Q}$  as a  $\mathbb{Z}$ -module, is a small submodule. Then it follows from Part 2 of Corollary 2.3.

**Lemma 2.5.** *Let  $M$  be an  $R$ -module. If  $K$  is a finitely generated or a semisimple submodule of  $M$  contained in  $\text{Rad}(M)$ , then  $K \ll M$ .*

*Proof.* First assume that  $K$  is a finitely generated submodule of  $M$ . Then see Part(a) of [6, Proposition 3.2.9]. Now, suppose that  $K$  is a semisimple submodule of  $M$ . Then  $\text{Soc}(K) = K$  and since  $K \subseteq \text{Rad}(M)$ ,  $\text{Soc}(K) \subseteq \text{Soc}(\text{Rad}(M))$ . Also by [10, 2.8(9)],  $\text{Soc}(\text{Rad}(M)) \ll M$ . Therefore, by [5, Proposition 5.17(1)],  $K \ll M$ .  $\square$

Let  $\Omega$  be a non-empty set (class) of  $R$ -modules. An  $R$ -module  $M$  is said to be *finitely cogenerated by  $\Omega$* , or *finitely  $\Omega$ -cogenerated*, if there is a monomorphism  $M \rightarrow \prod_{i \leq k} \omega_i = \bigoplus_{i \leq k} \omega_i$  with finitely many  $\omega_i \in \Omega$  and  $k \in \mathbb{N}$ .

**Proposition 2.6.** *Let  $M$  be an  $R$ -module with the graph  $\Gamma(M)$  and  $\text{Rad}(M) \neq (0)$ . Then the following conditions hold:*

- (1) *If  $X$  is a nontrivial submodule of  $M$  which is direct summand of  $M$  with  $(0) \neq \text{Rad}(X) \ll M$ , then  $\Gamma(M)$  contains at least one cycle of length 3.*
- (2) *If  $K$  is a nontrivial finitely generated or semisimple submodule of  $M$  contained in  $\text{Rad}(M)$ , then  $d(K, \text{Rad}(M)) = 1$  and  $d(K, N) = 1$  for every nontrivial submodule  $N$  of  $M$ .*
- (3) *If  $M$  is a uniform  $R$ -module with  $\text{Soc}(M) \neq (0)$  or a finitely cogenerated by a set, then  $d(\text{Rad}(M), \text{Soc}(M)) = 1$ .*

*Proof.* (1) Since  $X$  is a direct summand of  $M$ , there exists a submodule  $Y$  of  $M$  such that  $X \oplus Y = M$ . Then  $\text{Rad}(X) \oplus \text{Rad}(Y) = \text{Rad}(M)$ . Since  $\text{Rad}(X) \subseteq X$  and  $X \cap \text{Rad}(Y) \subseteq X \cap Y = (0)$ , by the modularity condition,  $X \cap \text{Rad}(M) = \text{Rad}(X)$ . Then  $X \cap \text{Rad}(M) \ll M$ . Also,  $\text{Rad}(X) = X \cap \text{Rad}(X) \ll M$  and  $\text{Rad}(X) = \text{Rad}(X) \cap \text{Rad}(M) \ll M$  and we have  $d(X, \text{Rad}(M)) = 1$ ,  $d(X, \text{Rad}(X)) = 1$  and  $d(\text{Rad}(X), \text{Rad}(M)) = 1$ . Hence  $(X, \text{Rad}(X), \text{Rad}(M))$  is a cycle. Consequently, the graph  $\Gamma(M)$  contains at least one cycle of length 3.

(2) Suppose that  $K$  is a nontrivial finitely generated or semisimple submodule of  $M$ . Then by Lemma 2.5,  $K \ll M$ . Since  $K \subseteq \text{Rad}(M)$ ,  $K = K \cap \text{Rad}(M) \ll M$  and since  $K \cap N \subseteq K$ ,  $K \cap N \ll M$  for every other nontrivial submodule  $N$  of  $M$ . Hence  $d(K, \text{Rad}(M)) = 1$  and  $d(K, N) = 1$ .

(3) We consider the nontrivial submodule  $\text{Soc}(\text{Rad}(M))$  of  $M$ . Suppose that  $M$  is a uniform  $R$ -module, then  $\text{Rad}(M) \trianglelefteq M$  and  $\text{Soc}(M) \trianglelefteq M$ . Hence by [5, Corollary 9.9],  $\text{Soc}(\text{Rad}(M)) = \text{Rad}(M) \cap \text{Soc}(M) \neq (0)$ . Also, if  $M$  is a finitely cogenerated  $R$ -module, then by [19, 21.3],  $\text{Soc}(M)$  is finitely generated and essential in  $M$  and again  $\text{Soc}(\text{Rad}(M)) \neq (0)$ .

Thus  $Soc(Rad(M))$  is a vertex of the graph  $\Gamma(M)$ . Since by [10, 2.8 (9)],  $Soc(Rad(M)) \ll M$ , we have  $Rad(M) \cap Soc(M) \ll M$ . Therefore,  $d(Rad(M), Soc(M)) = 1$ .  $\square$

**Corollary 2.7.** *Let  $M$  be an  $R$ -module with the graph  $\Gamma(M)$ . If  $M$  has at least one nonzero small submodule, then  $\Gamma(M)$  is a connected graph and  $diam(\Gamma(M)) \leq 2$ .*

**Corollary 2.8.** *Let  $M$  be an  $R$ -module with  $Rad(M) \neq (0)$ . Then  $\Gamma(M)$  is a connected graph, if one of the following holds.*

- (1) *The module  $M$  is finitely generated.*
- (2) *There exists a nontrivial finitely generated or semisimple submodule of  $M$  contained in  $Rad(M)$ .*

*Proof.* Part 1 is obvious and Part 2 follows from Lemma 2.5 and Corollary 2.7.  $\square$

**Theorem 2.9.** *Let  $M$  be an  $R$ -module with graph  $\Gamma(M)$ . If  $\delta(\Gamma(M)) \geq 1$ , then  $\Gamma(M)$  is connected and  $diam(\Gamma(M)) \leq 3$ .*

*Proof.* Let  $A$  and  $B$  be two non-adjacent vertices of  $\Gamma(M)$ . Since  $\delta(\Gamma(M)) \geq 1$ , there exist submodules  $A_1$  and  $B_1$  such that  $A \cap A_1 \ll M$  and  $B \cap B_1 \ll M$ . Now, if  $A_1 \cap B_1 \ll M$ , then  $A \leftrightarrow A_1 \leftrightarrow B_1 \leftrightarrow B$  is a path of length 3. Otherwise  $A \leftrightarrow A_1 \cap B_1 \leftrightarrow B$  is a path of length 2. It follows that  $\Gamma(M)$  is a connected graph and  $diam(\Gamma(M)) \leq 3$ .  $\square$

**Proposition 2.10.** *Let  $M$  be a semisimple  $R$ -module such that it is not simple. Then the following statements hold:*

- (1) *The graph  $\Gamma(M)$  has no isolated vertex.*
- (2) *The graph  $\Gamma(M)$  is connected and  $diam(\Gamma(M)) \leq 3$ .*

*Proof.* (1) Suppose that  $M$  is semisimple. Let  $X$  be a vertex of the graph  $\Gamma(M)$ . Since  $M$  is semisimple, then by Theorem 9.6 Part (e) of [5, p. 117], every submodule of  $M$  is a direct summand of  $M$ . Thus there exists a submodule  $Y$  of  $M$  such that  $M = X \oplus Y$ . Hence  $X \cap Y = (0) \ll M$  and thus there exists an edge between vertex  $X$  of  $\Gamma(M)$  and another vertex of this graph. Then  $X$  is not an isolated vertex. Consequently,  $\Gamma(M)$  has no isolated vertex.

(2) By Theorem 2.9 and Part 1.  $\square$

**Example 2.11.** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_{30}$ , which is semisimple. Then the graph  $\Gamma(\mathbb{Z}_{30})$  is connected and  $diam(\Gamma(\mathbb{Z}_{30})) = 3$ . (see Fig.1)

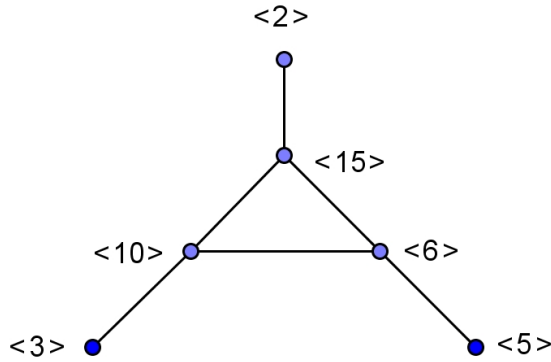


Fig. 1.  $\Gamma(\mathbb{Z}_{30})$

For a module  $M$ , we use  $S(M)$  which denotes the set of all the nonzero small submodules of  $M$ .

**Theorem 2.12.** *Let  $M$  be an  $R$ -module and  $|S(M)| \geq 1$ . If  $\Gamma(M)$  is a tree, then  $\Gamma(M) \cong K_1$  or  $\Gamma(M)$  is a star graph.*

*Proof.* Assume that  $\Gamma(M)$  is a tree. Then  $|S(M)| < 2$ . Otherwise, let  $X$  and  $Y$  be two nonzero small submodules of  $M$ . Thus  $(X, X \cap Y, Y)$  is a cycle of length 3, a contradiction. Since  $|S(M)| \geq 1$ , then  $|S(M)| = 1$ . Hence  $M$  has only one nonzero small submodule. Let  $S \in S(M)$ . For every vertex  $V$  of  $\Gamma(M)$ , if  $V = S$ , then  $\Gamma(M) \cong K_1$  and if  $V \neq S$ , as  $V \cap S \ll M$ , we deduce  $\Gamma(M) \cong K_2$ . Now, let  $\Omega = \{v_i : v_i \neq s, i \in I\}$ . Then every two arbitrary distinct vertices  $v_i$  and  $v_j$ ,  $i \neq j$ , are not adjacent and for  $i \neq j$ ,  $v_i \leftrightarrow S \leftrightarrow v_j$  is a path and thus  $\Gamma(M)$  is star graph, and the proof is complete.  $\square$

**Theorem 2.13.** *Let  $M$  be an  $R$ -module with the graph  $\Gamma(M)$ . If  $|S(M)| \geq 2$ , then  $\Gamma(M)$  contains at least one cycle and  $grith(\Gamma(M)) = 3$ .*

*Proof.* Suppose that  $|S(M)| \geq 2$ . Then  $M$  has at least two different nonzero small submodules, say  $M_1$  and  $M_2$ . Since  $M_1 \cap M_2 \subseteq M_i$ , for  $i = 1, 2$ , by [5, Proposition 5.17(1)],  $M_1 \cap M_2 \ll M$ . Moreover  $M_1 \cap (M_1 \cap M_2) \ll M$  and  $M_2 \cap (M_1 \cap M_2) \ll M$ . We consider two possible cases for  $M_1 \cap M_2$ .

Case 1: If  $M_1 \cap M_2 \neq (0)$ , then  $d(M_1, M_2) = 1$ ,  $d(M_1, M_1 \cap M_2) = 1$  and  $d(M_2, M_1 \cap M_2) = 1$ . Hence  $(M_1, M_1 \cap M_2, M_2)$  is a cycle of length 3. Also by [5, Proposition 5.17(2)],  $M_1 + M_2 \ll M$  and since  $M_1 \cap (M_1 + M_2) \ll M$  and  $M_2 \cap (M_1 + M_2) \ll M$ ,  $(M_1, M_1 + M_2, M_2)$  is a cycle of length 3. Similarly,  $(M_1 \cap M_2, M_1, M_1 + M_2)$  and  $(M_1 \cap M_2, M_2, M_1 + M_2)$  are cycles of length 3 and  $M_1 \leftrightarrow M_1 + M_2 \leftrightarrow M_2 \leftrightarrow M_1 \cap M_2 \leftrightarrow M_1$  is a cycle of length 4.

Case 2: If  $M_1 \cap M_2 = (0)$ , then  $(M_1, M_1 + M_2, M_2)$  is a cycle of length 3 in the graph  $\Gamma(M)$ . Therefore,  $\Gamma(M)$  contains at least one cycle and so  $girth(\Gamma(M)) = 3$ .  $\square$

To prove the next results, we need a celebrated theorem due to Kuratowski.

**Theorem 2.14.** ([5, Theorem 10.30]) *A graph is planar if and only if it contains no subdivision of either  $K_5$  or  $K_{3,3}$ .*

**Theorem 2.15.** *Let  $M$  be an  $R$ -module with the graph  $\Gamma(M)$ . Then the following conditions hold:*

(1) *If  $|S(M)| = 1$  or  $|S(M)| = 2$ , and the intersection of every pair of non-small submodules of  $M$  is a non-small submodule, then  $\Gamma(M)$  is a planar graph.*

(2) *If  $|S(M)| \geq 3$ , then  $\Gamma(M)$  is not a planar graph.*

*Proof.* (1) According to assumption, if  $|S(M)| = 1$ , then  $\Gamma(M)$  is a star graph which is planar and if  $|S(M)| = 2$ , then by the definition of planar graph,  $\Gamma(M)$  is planar.

(2) Suppose that  $|S(M)| \geq 3$ . Then  $M$  has at least three different nonzero small submodules, say  $M_1, M_2$  and  $M_3$ . A similar argument in the proof of Theorem 2.13 shows that  $M_1, M_2, M_3, M_1 + M_2, M_1 + M_3$  and  $M_2 + M_3$  are adjacent vertices in the graph  $\Gamma(M)$ . Therefore,  $\Gamma(M)$  contains at least one subgraph of the complete graph  $K_5$  such as the subgraph induced on the set  $\{M_1, M_2, M_3, M_1 + M_2, M_1 + M_3\}$  which is a complete graph  $K_5$ . Hence, by Theorem 2.13,  $\Gamma(M)$  is not a planar graph.  $\square$

**Example 2.16.** We Consider  $\mathbb{Z}_{20}$  as a  $\mathbb{Z}_{20}$ -module. The vertex  $\langle 10 \rangle$  of the graph  $\Gamma(\mathbb{Z}_{20})$  is the only nonzero small submodule of  $\mathbb{Z}_{20}$ . The graph  $\Gamma(\mathbb{Z}_{20})$  is connected and planar with  $diam(\Gamma(\mathbb{Z}_{20})) = 2$  and  $girth(\Gamma(\mathbb{Z}_{20})) = 3$ . (see Fig.2)

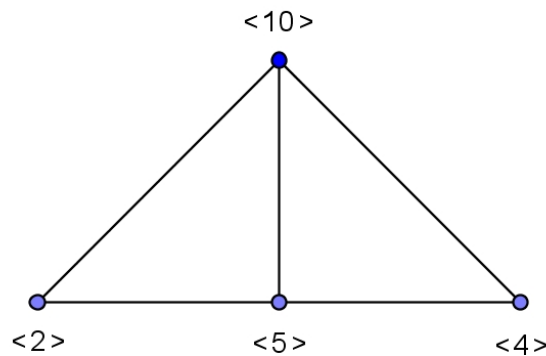


Fig. 2.  $\Gamma(\mathbb{Z}_{20})$



### 3. Orthogonal Vertices and Domination of $\Gamma(M)$

In this section, we study the orthogonal vertices of  $\Gamma(M)$  and their relationship with the nonzero small submodules of  $M$  and also we introduce some modules whose small intersection graphs are complemented. We obtain some results on the domination of  $\Gamma(M)$  and study the condition under which the domination number of  $\Gamma(M)$  is finite.

Two distinct vertices  $X$  and  $Y$  of the graph  $\Gamma(M)$  are orthogonal, denoted by  $X \perp Y$ , if  $X \cap Y \ll M$  and there is no vertex  $Z \in \Gamma(M)$  such that  $X \cap Z \ll M$  and  $Y \cap Z \ll M$ .

**Proposition 3.1.** *Let  $M$  be an  $R$ -module with  $|S(M)| \geq 2$  and  $|\Gamma(M)| \geq 3$ . Then the following conditions hold:*

- (1) *Every two small submodules of  $M$  cannot be orthogonal vertices to each other.*
- (2) *The graph  $\Gamma(M)$  has at least one triangle.*
- (3) *For  $|\Gamma(M)| = n$  and  $|S(M)| = 2$ , if we take  $m$  equal to the number of the triangles of this graph, then  $(n - 2) \leq m \leq \frac{1}{6}n(n - 1)(n - 2)$ .*

*Proof.* (1) Suppose that  $N$  and  $K$  are two nonzero small submodules of  $M$ , then  $N \cap K \ll M$  and for any vertex  $L \in \Gamma(M)$ , since  $N \cap L \subseteq N$  and  $K \cap L \subseteq K$ ,  $N \cap L \ll M$  and  $K \cap L \ll M$ . Thus  $L$  is adjacent to both  $N$  and  $K$ . Therefore, they cannot be orthogonal vertices to each other.

(2) It is obvious.

(3) Let  $|\Gamma(M)| = n$ , where  $n \geq 3$  and  $|S(M)| = 2$ . Then the number of the submodules of  $M$ , which are non-small in  $M$ , is equal to  $n - 2$ . Now we consider  $C(k, r)$  for the number of possible combinations of  $r$  objects from a set of  $k$  objects. Hence, if we choose two vertices of the triangles are small submodules, then the number of the triangles is at least equal to  $C(n - 2, 1) = n - 2$ . Also, the number of the triangles in this graph is at most equal to  $C(n - 2, 1) + 2C(n - 2, 2) + C(n - 2, 3) = \frac{1}{6}n(n - 1)(n - 2)$ . Therefore,  $(n - 2) \leq m \leq \frac{1}{6}n(n - 1)(n - 2)$ .  $\square$

**Theorem 3.2.** *Let  $M$  be an  $R$ -module. Then the following are equivalent:*

- (1) *The graph  $\Gamma(M)$  has no triangle.*
- (2) *Every two adjacent vertices of the graph  $\Gamma(M)$  are orthogonal vertices.*
- (3) *The module  $M$  has at most one nonzero small submodule such that the intersection of every pair of the non-small nontrivial submodules of  $M$  is non-small.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $A$  and  $B$  be two adjacent vertices of the graph  $\Gamma(M)$  which are not orthogonal vertices. Then by the definition of orthogonality, there exists another vertex  $C$  of the graph  $\Gamma(M)$  which is adjacent to both  $A$  and  $B$ , then there exists a cycle of the form

$(A, B, C)$ . This means that there is a triangle in the graph  $\Gamma(M)$ , a contradiction.

(2)  $\Rightarrow$  (3) Assume that every two adjacent vertices of the graph  $\Gamma(M)$  are orthogonal vertices. Let there exist at least two nonzero small submodules  $S_1$  and  $S_2$  of  $M$ . Since  $S_1 \cap S_2 \ll M$ , they are adjacent vertices of the graph  $\Gamma(M)$  and also  $S_1 \perp S_2$ . However, by Proposition 3.1(1), they cannot be orthogonal to each other, a contradiction.

(3)  $\Rightarrow$  (1) Suppose that  $M$  has no nonzero small submodule. Since the intersection of every pair of the non-small nontrivial submodules of  $M$  is non-small,  $\Gamma(M)$  has no triangle. Moreover, Let  $S$  be the only nonzero small submodule of  $M$ . Then for every three arbitrary vertices  $M_1, M_2$  and  $M_3$  of the graph  $\Gamma(M)$ , at least two of them are non-small. Let  $S = M_1$ . As  $M_2 \cap M_3$  is non-small submodules of  $M$ , then  $M_2 \leftrightarrow S \leftrightarrow M_3$  is a path. Also if  $S \neq M_i$ , for  $i = 1, 2, 3$ . Since  $M_i \cap M_j$  is non-small submodules of  $M$ , for  $i, j = 1, 2, 3$  and  $i \neq j$ , then  $M_1, M_2$  and  $M_3$  are not adjacent vertices in the graph  $\Gamma(M)$ . Therefore, there is no triangle in the graph  $\Gamma(M)$ .  $\square$

**Corollary 3.3.** *Let  $M$  be an  $R$ -module and  $V, W$  be two submodules of  $M$  such that  $M = V + W$ . If  $\text{Rad}(M) = (0)$  and  $V \perp W$  in  $\Gamma(M)$ , then  $M = V \oplus W$ .*

*Proof.* Suppose that  $V$  and  $W$  are two submodules of  $M$  such that  $M = V + W$ . Since  $V \perp W$  in the graph  $\Gamma(M)$ , then  $V \cap W \ll M$ . Hence  $V \cap W \subseteq \text{Rad}(M)$ . Thus  $V \cap W = (0)$ . Consequently,  $M = V \oplus W$ .  $\square$

**Theorem 3.4.** *Let  $M$  be an  $R$ -module, then the following statements hold:*

- (1) *If  $M$  is a semisimple  $R$ -module which is not simple, then the graph  $\Gamma(M)$  is complemented.*
- (2) *If  $M$  is a finitely generated  $R$ -module with  $\text{Rad}(M) \neq (0)$ , then the graph  $\Gamma(M)$  is not complemented.*

*Proof.* (1) Suppose that  $M$  is a semisimple  $R$ -module. Let  $N$  be a vertex of the graph  $\Gamma(M)$ . Then, by part 1 of proposition 2.10,  $\Gamma(M)$  has no isolated vertex. Hence there exists a vertex  $K$  of  $\Gamma(M)$  such that  $d(N, K) = 1$ . Since  $M$  has no nonzero small submodule, there is no vertex  $L$  of  $\Gamma(M)$  such that  $L \cap N \ll M$  and  $L \cap K \ll M$ . Therefore,  $\Gamma(M)$  is a complemented graph.

(2) Since  $M$  is a finitely generated  $R$ -module,  $(0) \neq \text{Rad}(M) \ll M$ . We consider two possible cases for  $\text{Rad}(M)$ .

Case 1: If  $\text{Rad}(M)$  is a simple submodule of  $M$ , since  $\text{Rad}(M) = \bigcap_{i \in I} M_i$ , where  $M_i$  is the maximal submodules of  $M$ , for all  $i \in I$ , we choose  $N = \bigcap_{i \in I - \{1\}} M_i$ . Then  $(M_1, N, \text{Rad}(M))$  is a triangle in the graph  $\Gamma(M)$ . Therefore, the graph  $\Gamma(M)$  is not complemented.

Case 2: If  $\text{Rad}(M)$  is not a simple submodule of  $M$ , then there exists a nontrivial submodule

$X$  of  $M$  which  $X \subset \text{Rad}(M)$ , then by [5, Proposition 5.17(1)],  $X \ll M$ . Thus for each vertex  $Y$  of  $\Gamma(M)$ ,  $(X, Y, \text{Rad}(M))$  is a triangle of this graph. Consequently, the graph  $\Gamma(M)$  is not complemented.  $\square$

**Example 3.5.** Let  $n = p_1 p_2 \dots p_k$ , where  $p_i$  is a prime number, for all  $i = 1, 2, \dots, k$  and  $k \in \mathbb{N}$ . Then  $\mathbb{Z}_n = \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \dots \oplus \mathbb{Z}_{p_k}$  is a semisimple  $\mathbb{Z}$ -module and  $\Gamma(\mathbb{Z}_n)$  is a complemented graph.

**Proposition 3.6.** Let  $M = V \oplus W$  be a finitely generated  $R$ -module such that  $\text{Rad}(V) \neq (0)$  and  $\text{Rad}(W) \neq (0)$ . Then the following conditions hold:

- (1) The vertices  $V$  and  $W$  are not orthogonal in the graph  $\Gamma(M)$ .
- (2) The graph  $\Gamma(M)$  is not planar.
- (3) The graph  $\Gamma(M)$  is not complemented.

*Proof.* (1) Suppose that  $M = V \oplus W$ . Then  $V \cap W = (0) \ll M$  and since  $M$  is a finitely generated  $R$ -module,  $\text{Rad}(M) \ll M$ . Hence by [5, Proposition 5.17(1)],  $V \cap \text{Rad}(M) \ll M$  and  $W \cap \text{Rad}(M) \ll M$ . Thus  $\Gamma(M)$  has a triangle of the form  $(V, \text{Rad}(M), W)$ . Therefore, by Theorem 3.2(2), the vertices  $V$  and  $W$  are not orthogonal in the graph  $\Gamma(M)$ .

(2) Let  $M = V \oplus W$ . Then  $\text{Rad}(M) = \text{Rad}(V) \oplus \text{Rad}(W)$  and  $\text{Rad}(V) \cap \text{Rad}(W) = (0) \ll M$ . Also  $\text{Rad}(V) \subseteq V$  and  $V \cap \text{Rad}(W) \subseteq V \cap W = (0)$ , and so the modularity condition, implies that  $V \cap \text{Rad}(M) = \text{Rad}(V)$  and similarly,  $W \cap \text{Rad}(M) = \text{Rad}(W)$ . Moreover,  $V \cap \text{Rad}(V) = \text{Rad}(V) \ll M$ ,  $W \cap \text{Rad}(W) = \text{Rad}(W) \ll M$ ,  $V \cap \text{Rad}(M) = \text{Rad}(V) \ll M$ ,  $W \cap \text{Rad}(M) = \text{Rad}(W) \ll M$ ,  $\text{Rad}(V) \cap \text{Rad}(M) = \text{Rad}(V) \ll M$  and  $\text{Rad}(W) \cap \text{Rad}(M) = \text{Rad}(W) \ll M$ . Hence  $V, W, \text{Rad}(V), \text{Rad}(W)$  and  $\text{Rad}(M)$  are adjacent vertices in the graph  $\Gamma(M)$ . Therefore, the set  $\{V, W, \text{Rad}(V), \text{Rad}(W), \text{Rad}(M)\}$  induces a complete subgraph  $K_5$  in  $\Gamma(M)$ . Hence, by Theorem 2.13,  $\Gamma(M)$  is not a planar graph.

(3) Take  $M = V \oplus W$  in Theorem 3.4(2).  $\square$

Let  $\Gamma$  be a graph with the vertex set  $V(\Gamma)$  and DS be a dominating set of the graph  $\Gamma$ . For a ring  $R$ , we define  $\gamma(\Gamma(R)) = 0$ , if  $R$  is a field and also for an  $R$ -module  $M$ ,  $\gamma(\Gamma(M)) = 0$ , if  $M$  is simple. In this paper, a subset  $S$  of the vertex set of the graph  $\Gamma(M)$  is a DS if and only if for any nontrivial submodule  $N$  of  $M$  there is a  $K$  in  $S$  such that  $N \cap K \ll M$ .

**Lemma 3.7.** Let  $M$  be an  $R$ -module with  $|\Gamma(M)| \geq 2$ , then the following hold:

- (1) If  $S$  is a subset of the vertex set of the graph  $\Gamma(M)$  such that  $S$  either contains at least one small submodule of  $M$  or there exists a vertex  $X \in S$  which  $X \cap Y = (0)$ , for every vertex  $Y \in V(\Gamma(M)) \setminus S$ . Then  $S$  is a DS in  $\Gamma(M)$ .
- (2) If  $M$  has at least one nonzero small submodule, then for each nonzero small submodules  $X$  of  $M$ ,  $\{X\}$  is a mDS and  $\gamma(\Gamma(M)) = 1$ .

**Theorem 3.8.** *Let  $M$  be an  $R$ -module and  $M = N \oplus K$ , where  $N$  and  $K$  are two simple  $R$ -modules. Then  $\gamma(\Gamma(M)) = 1$ .*

*Proof.* Suppose that  $M = N \oplus K$ , such that  $N$  and  $K$  are two simple  $R$ -modules. Then by Part 1 of Proposition 2.2,  $\Gamma(M)$  is a complete graph. Let  $X$  be an arbitrary vertex of the graph  $\Gamma(M)$ . Then for any distinct vertex  $Y$  of  $\Gamma(M)$ ,  $X \cap Y \ll M$ , thus  $\{X\}$  is a  $mDS$  and  $\gamma(\Gamma(M)) = 1$ .  $\square$

The following example shows that the Theorem 3.8 may not be hold for any semisimple module with three simple direct summand.

**Example 3.9.** We can see easily that in the graph  $\Gamma(\mathbb{Z}_{30})$ , two minimal dominating sets  $\{< 2 >, < 3 >, < 5 >\}$  and  $\{< 6 >, < 10 >, < 15 >\}$  have the smallest of the cardinalities, hence  $\gamma(\Gamma(\mathbb{Z}_{30})) = 3$ .

**Corollary 3.10.** *Let  $M$  be an  $R$ -module with nontrivial submodule  $\text{Rad}(M)$  and  $|\Gamma(M)| \geq 2$ . If  $M$  is a uniserial, hollow or local  $R$ -module, then every subset of the vertex set of the graph  $\Gamma(M)$  is a  $DS$  in  $\Gamma(M)$  and  $\gamma(\Gamma(M)) = 1$ .*

**Example 3.11.** Let  $R = F[x, y]/(x, y)^2$ , where  $F$  is an infinite field and  $x$  and  $y$  are indeterminates. Then  $S = \overline{(x, y)}$  is the only maximal ideal of  $R$ . Thus  $S = J(R)$ , which  $J(R)$  is the Jacobson radical of the ring  $R$ , and since  $R$  is a finitely generated,  $J(R) \ll R$ . Hence  $I \ll R$  for every proper ideal  $I$  of  $R$ . Thus  $R$  is a hollow. Therefore, the set  $S = \overline{(x, y)}$  is a  $mDS$  in  $\Gamma(R)$  and  $\gamma(\Gamma(R)) = 1$ .

**Corollary 3.12.** *Let  $M$  be an  $R$ -module with  $\text{Rad}(M) \neq (0)$  and  $S$  be a subset of the vertex set of the graph  $\Gamma(M)$ . Then  $S = DS(\Gamma(M))$  and  $\gamma(\Gamma(M)) = 1$ , if one of the following condition holds:*

- (1) *The module  $M$  is finitely generated with  $\text{Rad}(M) \in S$ .*
- (2) *There exists a nontrivial submodule  $N$  of  $M$  which is direct summand of  $M$  with  $d(N, \text{Rad}(M)) = 1$  and  $\text{Rad}(N) \in S$ .*
- (3) *If  $K$  is a nontrivial finitely generated or semisimple submodule of  $M$  contained in  $\text{Rad}(M)$ .*
- (4) *If  $M$  is a uniform  $R$ -module with  $\text{Soc}(M) \neq (0)$  and  $\text{Soc}(\text{Rad}(M)) \in S$ .*

*Proof.* (1) Since  $M$  is a finitely generated  $R$ -module,  $\text{Rad}(M) \ll M$ , and  $\text{Rad}(M) \in S$ . Then, by Part 2 of Lemma 3.7,  $S = DS(\Gamma(M))$  and  $\gamma(\Gamma(M)) = 1$ .

(2) Since  $N$  is a direct summand of  $M$ , there exists a submodule  $K$  of  $M$  such that  $N \oplus K = M$ . Then  $\text{Rad}(N) \oplus \text{Rad}(K) = \text{Rad}(M)$ . Since  $\text{Rad}(N) \subseteq N$  and  $N \cap \text{Rad}(K) \subseteq N \cap K = (0)$ , by the modularity condition,  $N \cap \text{Rad}(M) = \text{Rad}(N)$ . Since  $d(N, \text{Rad}(M)) = 1$ ,  $N \cap \text{Rad}(M) \ll M$ . Then  $\text{Rad}(N) \ll M$  and since  $\text{Rad}(N) \in S$ , it follows from Part 2 of Lemma 3.7.

- (3) By Lemma 2.5 and Part 2 of Lemma 3.7.  
 (4) By [10, 2.8(9)] and Part 2 of Lemma 3.7.  $\square$

#### 4. ACKNOWLEDGMENTS

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**Lotf Ali Mahdavi**

Department of Mathematics, Faculty of Mathematical Sciences,

University of Mazandaran

Babolsar, Iran.

[l.a.mahdavi154@gmail.com](mailto:l.a.mahdavi154@gmail.com)

**Yahya Talebi**

Department of Mathematics, Faculty of Mathematical Sciences,

University of Mazandaran

Babolsar, Iran.

[talebi@umz.ac.ir](mailto:talebi@umz.ac.ir)