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ON GENERALIZATIONS OF VECTOR AND BANACH SPACES BY HYPERSTRUCTRES

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Abstract. In this paper, we generalize the vector space by considering the group as a canonical m-ary hypergroup, the field as a Krasner (m, n)-hyperfield. Moreover, we define the m-ary hyper Banach spaces and investigate some of their related properties.

1. Introduction

It is well known that the concept of hypergroups were first introduced in 1934 by Marty [14] in the 8^{th} Congress of Scandinavian Mathematicians. Since then, many papers and monographs have been written on this topic. Nowadays, hyperstructures have an extensive applications in several domains of mathematics and computer science, see [1, 2, 3, 17]. It is noteworth that the algebraic theory of hyperstructures is a suitable generalization of classical algebraic structures. In the classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if V is a non-empty set and $\mathcal{P}^*(V)$ is the set of all non-empty subsets of V, then we consider

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the maps $*: V \times V \longrightarrow \mathcal{P}^*(V)$. We notice that these maps are called the *(binary) hyperoperations*. In some cases, the external hyperoperations are considered, which are the maps $*: R \times V \longrightarrow \mathcal{P}^*(V)$, where $R \neq V$. An example of a hyperstructure endowed both with an internal hyperoperation and an external hyperoperation is the so-called *hypermodule*.

The notion of an n-ary generalization of an algebraic structure is the most natural way for further development and deeper understanding of their fundamental properties. The notion of an n-ary group was first introduced by Dörnte [7]. Since then many papers concerning various n-ary algebras have been published, for example see [8, 9, 13, 18]. The concept of an n-ary hypergroup is defined by Davvaz and Vougiouklis in [4], which is a generalization of hypergroup in the sense of Marty as well as a generalization of an n-ary group. Then this concept was further studied by Ghadiri and Waphare [10], Leoreanu-Fotea and Davvaz [11, 12], Davvaz et. al. [5, 6] and others. Also Leoreanu-Fotea and Davvaz have introduced and studied the partial n-hypergroupoid, associated with a binary relation and some important results, concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-hypergroupoids

Recently, the notation of (m,n)-hyperrings was defined by Mirvakili and Davvaz [15] and they established the (m,n)-rings from the (m,n)-hyperrings by using some fundamental relations. Moreover, they defined a certain class of (m,n)-hyperrings namely, the Krasner (m,n)-hyperrings. In fact, the Krasner (m,n)-hyperrings are generalizations of both (m,n)-rings and Krasner hyperrings. In this paper, several properties normed of Krasner (m,n)-hyperrings are presented. We recall here that Ostadhadi-Dehkordi [16] has recently generalized and developed the theory of a normed vector space. He also explored some basic properties of an m-ary hypervector space and obtained several important properties of such m-ary hypervector space.

The main purpose of this paper is to generalize and develop a number of basic properties of the hypervector spaces and normed vector spaces. Moreover, our aim is to explore some additional properties of an m-ary hypervector space. Also, we introduce the notions of a m-ary hyper Banach space,a countable subadditive prenorm and the linear transformation. Finally,we establish a necessary condition for a countable subadditive prenorm to be continuous.

2. m-ary Hypervector Space

Let R be a non-empty set and $n \in \mathbb{N}$, $n \geq 2$ and $f: R^n \longrightarrow \mathcal{P}^*(R)$, where $\mathcal{P}^*(R)$ is the set of all non-empty subsets of R. Then, we call f an *n-ary hyperoperation* on R and the pair (R, f) is called an *n-ary hypergroupoid*. If $R_1, ..., R_n$ are non-empty subsets of R, then we define

$$f(R_1, R_2, ..., R_n) = \bigcup\{f(x_1, x_2, ..., x_n) : x_i \in R_i,\ 1 \leq i \leq n\}.$$

The sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. An n-ary hypergroupoid (R, f) will be called an *n-ary semihypergroup* if we have:

$$f\left(\overset{(i-1)}{x_1}, f\left(\overset{(n+i-1)}{x_i}\right), x_{n+i}^{(2n-1)}\right) = f\left(\overset{(j-1)}{x_1}, f\left(\overset{(n+j-1)}{x_j}\right), \overset{(2n-1)}{x_{n+j}}\right),$$

for every $i, j \in \{1, 2, ..., n\}$ and $x_1, x_2, ...x_{2n-1} \in R$. Let the equation

$$y \in f\left(\overset{(i-1)}{x_1}, z_i, x_{i+1}^n\right),$$

has a solution $z_i \in R$ for every $x_1, x_2, ..., x_{i-1}, x_{i+1}, .., x_n, y \in R$. Then, R is called *n-ary hypergroup*. An n-ary hypergroupoid (R, f) is *commutative* if for all $\sigma \in S_n$, $f(x_1, x_2, ..., x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$. A commutative n-ary hypergroupoid (R, f) is called a *canonical n-ary hypergroup* if the following axioms hold for all $1 \leq i, j \leq n$ and $x, x_i \in R$:

(i) There exists a unique element $0 \in R$ such that $x = f\left(\overset{(i-1)}{0}, x, \overset{(n-i)}{0}\right)$,

(ii) There exists a unique operation $-$ on R such that $x \in f(x_1^n)$ implies that $x_i \in f(-x_{i-1}, -x_{i-2}, ... - x_1, x, -x_n, ..., -x_{i+1})$.

We now give the following crucial definition.

Definition 2.1. A Krasner hyperring is a hyperstructure $(R, +, \cdot)$, which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,
(i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,
(ii) for every $x, y \in R$, $x + y = y + x$,
(iii) there exists $0 \in R$ such that $0 + x = x$, for every $x \in R$,
(iv) for every $x \in R$ there exists a unique element $x^{'} \in R$ such that $0 \in x + x^{'}$. (We shall write $-x$ for $x^{'}$ and we call it the opposite of x.)
(iv) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$.
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $0 \cdot x = x \cdot 0 = 0$
(3) The multiplication is distributive with respect to the hyperoperation $+$.

Definition 2.2. [16] A Krasner (m, n)-hyperfield is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

1. (R, f) is a canonical m-ary hypergroup,
2. (R, g) is an n-ary semigroup,
3. The n-ary operation is distributive with respect to the m-ary hyperoperation f, i.e., for every a_i^{i-1}, a_{i+1}^n, x_1^m, $1 \leq i \leq n$

$$g\left(a_1^{i-1}, f\left(x_1^m\right), a_{i+1}^m\right) = f\left(g\left(a_1^{i-1}, x_1, a_{i+1}^n\right), ..., g\left(a_1^{i-1}, x_m, a_{i+1}^n\right)\right),$$

4. 0 is a zero element (absorbing element) of the n-ary operation g, i.e., for every $x_2^n \in R$ we have
$$g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0) = 0,$$
5. There exists an element $e \in R$, called the identity element such that
$$g(a, \underbrace{e, ..., e}_{n-1}) = g(e, a, \underbrace{e, ..., e}_{n-2}) = ... = g(\underbrace{e, ..., e}_{n-1}, a) = a,$$
for every $a \in R$,
6. For each non-zero element $a \in R$, there exists an element a^{-1} such that
$$g(a, a^{-1}, ..., a^{-1}) = g(a^{-1}, a, ..., a^{-1}) = ... = g(a^{-1}, ..., a^{-1}, a) = e,$$
7. g is a commutative operation, i.e., the value of $g(x_1^n)$ is independent on the permutation of elements $x_1, x_2, ..., x_n$.

We now describe a Krasner (m, n)-hyperring in the following example.

Example 2.3. Let $\mathbb{R}$ be the set of all real numbers and G be a subgroup of $(\mathbb{R}, \cdot)$. Then, we define $(a, b) \in \rho$ if and only if there exists $g \in G$ such that $a = bg^{-1}$. This is an equivalence relation on $\mathbb{R}$. Set $[\mathbb{R} : \rho] = \{\rho(a) : a \in \mathbb{R}\}$, where $\rho(a)$ is an equivalence class $a \in \mathbb{R}$, and define the m-ary hyper operation f and n-ary multiplication g as follows:
$$\begin{aligned} f(\rho(a_1), \rho(a_2), ..., \rho(a_m)) &= \{\rho(x) : \rho(x) \subseteq \rho(a_1) + \rho(a_2) + ... + \rho(a_m)\}, \\ g(\rho(a_1), \rho(a_2), ..., \rho(a_n)) &= \rho(a_1 a_2 ... a_n), \end{aligned}$$
Equiped with the above equalities, $\mathbb{R}$ clearly forms a Krasner (m, n)-hyperring.

We cite the following useful definitions.

Definition 2.4. [16] Let $\mathbb{R}$ be the set of all real numbers. Then, the Krasner (m, n)-hyperfield on real numbers $\mathbb{R}$ is called a *real Krasner* (m, n)*-hyperfield*.

Definition 2.5. [16] Let (F, f, g) and (V, h) be a Krasner (m_1, n_1)-ary hyperfield and a canonical m-ary hypergroup, respectively. Then, V is said to be a *m-ary hypervector space* over the Krasner (m_1, n_1)-hyperfield F, if there exists a hypermultiplication $* : F \times V \longrightarrow \mathcal{P}^*(V)$(image to be denoted by $x.v$ for $x \in F$ and $v \in V$) such that for all $x, x_1^{m_1}, x_1^{n_1} \in F$ and $v, v_1^m \in V$ satisfies the following axioms:

1. $x.(h(v_1^m)) = h(x.v_1, ..., x.v_m)$,

2. $f(x_1^{m_1}).v = h(x_1.v, x_2.v, ..., x_{m_1}.v)$,

3. $g(x_1^{n_1}).v = x_1.(x_2.(x_3 \cdot (...x_{n_1}.v),$

4. $(-x).v = x.(-v),$

5. $v \in 1_F.v,\ 0 = 0.v,$

where 1_F is the identity element of F and $\mathcal{P}^*(V)$ is the set of all non-empty subsets of V.

Throughout this paper, by an m-ary hypervector space V, we mean a hypervector space $(V, h, *)$ and by a Krasner (m, n)-hyperfield F, we mean a hyperfield (F, f, g).

In the definition 2.5, if h is an m-ary operation, then we call V an *additive m-ary hypervector space.*

We give below another example of an m-ary hypervector space.

Example 2.6. Let (F, f, g) be a Krasner $(m, 2)$-hyperfield and $V = F \times F$. Then, we define an m-ary hyperoperation h on V as follows:

$$h((a_1, b_1), (a_2, b_2), ..., (a_m, b_m)) = \{(x, y) : x \in f(a_1, a_2, ..., a_m), y \in f(b_1, b_2, ..., b_m)\},$$

Now, we call (V, h) a canonical m-ary hypergroup. In the above m-ary hypergroup, we define a scalar multiplication $\star : F \times V \longrightarrow \mathcal{P}^*(V)$ by

$$c \star (a, b) = \{(x, y) : x \in g(c, a), y \in g(c, b)\},$$

where $c \in F$ and $(a, b) \in V$. Then we can easily verify that V is an m-ary hypervector space.

Definition 2.7. Let R be a hyperring. An R-hypermodule M is a commutative hypergroup $(M, +)$ together with a map $R \times M \longrightarrow M$ defined by

$$r \cdot m \longmapsto r \cdot m,$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$ we have

1. $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_2,$
2. $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2,$
3. $(r_1 r_2) \cdot m_1 = r_1 \cdot (r_2 \cdot m_1),$
4. $0_R \cdot m_1 = 0_M$

In the following Proposition, we describe the construction of an m-ary hypermodule.

proposition 2.8. *(Construction) Let $(V, +, \cdot)$ be a hypermodule over field F and m-ary hyperoperation h on V defined by $h(v_1^m) = \sum_{i=1}^{m} v_i$. Then, V is an m-ary hypermodule.*

Proof. We first prove that V is a canonical m-ary hypergroup. Since $+$ is well-defined so that h is also well-defined. Let 0 be the zero element of $(V,+)$. Then, 0is a zero element of (V,h). Now, let $v, v_1^m \in V$ and $1 \leq j \leq m$ such that $v \in h(v_1, v_2, ..., v_{j-1}, v_j, v_{j+1}, ..., v_m)$. Then, $v \in \sum_{i=1, i\neq j}^{m} v_i + v_j$. This implies that there exists $z \in \sum_{i=1, i\neq j}^{m} v_i$ such that $x \in z + v_j$. Hence, we obtain $v_j \in -z + x$. But $-z \in -\left(\sum_{i=1, i\neq j}^{m} v_i\right) = \sum_{i=1, i\neq j}^{m} -v_i$. This implies that $v_j \in h(-v_{j-1}, ..., -v_1, v, -v_m, ..., v_{m+1})$. Hence, (V,h) is indeed a canonical m-ary hypergroup. Since the multiplication $\cdot$ is distributive with respect to the hyperoperation $+$, it is easy to verify that $(V, h, \cdot)$ is an m-ary hypermodule. $\square$

Recall that a subset V_1 of an m-ary hypervector space V over F is an *m-ary subhypervector space* if V_1 is an m-ary subhyper vector space over F. Hence, a subset V_1 of V is an m-ary hypervector space if and only if following statements hold:

1. For every $v_1^m \in V_1$, $h(v_1^m) \subseteq V_1$,

2. For every $x \in F$ and $v_1 \in V_1$, $x \cdot v_1 \subseteq V_1$.

For an m-ary subhypervector space, we have the following Proposition.

proposition 2.9. *Let V_1 be a non-empty subset of V. Then, V_1 is an m-ary subhypervector space if and only if $h(x_1.v_1, ..., x_m.v_m) \subseteq V_1$, for every $x_1^m \in F$ and $v_1^m \in V_1$.*

Proof. Suppose that V_1 is an m-ary subhypersubspace of V. Obviously, we have $h(x_1.v_1, x_2.v_2, ..., x_m.v_m) \subseteq V_1$.

Conversely, let $v_1^m \in V_1$. Since $1_F \in F$, we have

$$h(v_1^m) \subseteq h(1_F.v_1, 1_F.v_2, ..., 1_F.v_m) \subseteq V_1.$$

Let $x \in F$ and $v_1 \in V_1$. Then, we have

$$h(x.v_1, 0.v_1, ..., 0.v_1) = h(x.v_1, \underbrace{0, 0, ..., 0}_{m-1}) = x.v_1 \subseteq V_1.$$

This completes the proof. $\square$

In the following Proposition, we state some additional properties of an m-ary hypervector spaces.

proposition 2.10. *Let V be an m-ary hypervector space over a Krasner (m,n)-hyperfield F. Then, the following statemants hold.*

1. $x.0 = \{0\}$, for every $x \in F$,
2. $x.v = \{0\}$, implies that $x = 0$ or $v = 0$.

Proof. 1. Suppose that $x \in F$. Then, by axiom (5), for every $v \in V$, we have $0.v = 0$. Consequently, we have

$$x.0 = x.(0.v) = x.(0.(0.v)) = ... = x.(\underbrace{0.(0...(0}_{n-1}.v)) = g(x, \underbrace{0, 0, ..., 0}_{n-1}).v = 0.v = 0.$$

2. Let $0 \neq x \in F$ and $v \in V$ be such that $x.v = 0$. Since $x^{-1} \in F$, we can easily prove that

$$0 = x.v = x^{-1}.(x.v) = ... = \underbrace{x^{-1}(x^{-1}(...x^{-1}}_{n-1}(x.v)) = g(x, \underbrace{x^{-1}, x^{-1}, ..., x^{-1}}_{n-1}).v = v.$$

□

3. m-ary hyper Banach space

In this section, we introduce the concept of m-ary hyper Banach space, hypernorm and prenorm. Some of their properties will be studied and investigated. Our aim is to establish a necessary condition for the countable subadditive prenorms to be continuous.

We first state the definition of hypernorm of an m-ary hyepervector space.

Definition 3.1. Let V be an m-ary hypervector space over a real Krasner (m, n)-hyperfield $\mathbb{R}$. Then, a *hypernorm* on V is a mapping $\|\cdot\| : V \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is the usual real space such that for all $x \in \mathbb{R}$ and $v, v_1, v_2, ..., v_m \in V$,the following conditions hold:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
2. $sup\|h\left(v_1^m\right)\| \leq \sum_{i=1}^{m} \|v_i\|$, where $\|h\left(v_1^m\right)\| = \{\|x\| : x \in h\left(v_1^m\right)\}$,
3. $sup\|x \cdot v\| \leq |x|\|v\|$, where $\|x.v\| = \{\|y\| : y \in x \cdot v\}$.

Suppose that $\|\cdot\|$ is a hypernorm on V. Then the couple $(V, \|\cdot\|)$ is said to be a *normed m-ary hypervector space* or a *hypernormed space.* We first consider V as a hypernormed space.

Let V be a hypernormed space and $a \in V$. Then, for every $\varepsilon > 0$ *open neighborhood(ball)* and *neighborhood(closed) ball* defined as

$$N_\varepsilon(a) = \left\{x \in V : sup\left\|h\left(x, -a, \overset{(m-2)}{0}\right)\right\| < \varepsilon\right\},$$

$$C_\varepsilon(a) = \left\{x \in V : sup\left\|h\left(x, -a, \overset{(m-2)}{0}\right)\right\| \leq \varepsilon\right\}.$$

Let V_1 be a subset of V. Then, the set of all *interior points* of V_1, is denoted by $Int(V_1)$ and the *closure* of V_1 is denoted by $Cl(V_1)$.

We call a sequence $\{x_n\}$ in V a *Cauchy sequence* if for every $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that

$$sup\left\|h\left(x_n, -x_m, \overset{(m-2)}{0}\right)\right\| < \varepsilon,$$

for every $n, m \geq k$.

A hypernormed space V is called an *m-ary hyper Banach space* if every Cauchy sequence in V is convergent. We call a subset V_1 of V *Convex* if

$$sup\left\|h\left(r \cdot x, (1-r) \cdot y, \overset{(m-2)}{0}\right)\right\| \subseteq V_1,$$

for every $x, y \in V_1$ and $r \in [0,1]$. Also the subsect V_1 of V is said to be *absorbing* if for each $a \in V$, there exists a positive number r_a such that for every $r > r_a$, $a \in r \cdot V_1$.

Let $\{a_n\}$ be a sequence in additive m-ary hypervector space V. Then, We define the following:

$$\begin{aligned}
A_1 &= h\left(a_1, \overset{(m-1)}{0}\right)\\
&\vdots\\
A_{m+1} &= h\left(A_m, a_{m+1}, \overset{(m-2)}{0}\right)\\
&\vdots\\
A_{km-(k-1)} &= h\left(A_{(k-1)m-(k-2)}, a_{(k-1)m-(k-2)+1}, ..., a_{(k-1)m-(k-2)+1+m}\right).
\end{aligned}$$

Let $T : V \longrightarrow \mathbb{R}^+$ be a map. Then, we call T *countably subadditive* if for every sequence $\{a_n\}$ in V such that $\{a_n\}$ is convergent. Thus, we have

$$T\left(lim_{n\longrightarrow\infty}\, a_n\right) \leq \sum_{n=1}^{\infty} T(a_n).$$

Let V be a hypernormed space. Then, the *prenorm* is a mapping $T : V \longrightarrow \mathbb{R}^+$ such that the following conditions are satisfied:

1. $T(x_1, x_2, ..., x_m) \leq \sum_{i=1}^{m} T(x_i)$,
2. $T\left(h\left(x, -y, \overset{(m-2)}{0}\right)\right) = T\left(h\left(y, -x, \overset{(m-2)}{0}\right)\right)$,
3. $\sup T(\lambda \cdot x) \leq \mid \lambda \mid T(x)$,
4. T(0)=0.

A prenorm $T : V \longrightarrow \mathbb{R}^+$ is said to be continuous at the point $a \in V$, if

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that } x \in N_\delta(a) \text{ implies that } \mid T(x) - T(a) \mid < \varepsilon.$$

For the additive m-ary hyper Banach space, we have the following crucial Proposition.

proposition 3.2. *Let V be an additive m-ary hyper Banach space such that $\lambda \cdot 0 = \{0\}$, for every $\lambda \in F$, $y \in \lambda^{-1} \cdot x$, for every $x \in \lambda \cdot y$, and $\lambda \cdot Cl(V_1) \subseteq Cl(\lambda \cdot V_1)$, for every $\lambda \in F$ and $V_1 \subseteq V$. Then, every countably subadditive prenorm on V is continuous.*

Proof. Suppose that T is a countably prenorm on V and $a, b \in V$. It suffices to show that the prenorm T is continuous at 0. Indeed, we have

$$\begin{aligned} T(a) = T\left(h\left(a, \overset{(m-1)}{0}\right)\right) &= T\left(h\left(a, h\left(b, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right) \\ &= T\left(h\left(b, h\left(a, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right) \\ &\leq T\left(h\left(a, -b, \overset{(m-2)}{0}\right)\right) + T(b). \end{aligned}$$

and

$$\begin{aligned} T(b) = T\left(h\left(b, \overset{(m-1)}{0}\right)\right) &= T\left(h\left(b, h\left(a, -a, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right) \\ &= T\left(h\left(a, h\left(b, -a, \overset{(m-2)}{0}\right), \overset{(m-1)}{0}\right)\right) \\ &\leq T\left(h\left(b, -a, \overset{(m-2)}{0}\right)\right) + T(a). \end{aligned}$$

Hence, we obtain the following equality.

$$|T(a) - T(b)| \leq T\left(a, -b, \overset{(m-2)}{0}\right) = \left|T\left(h\left(a, -b, \overset{(m-2)}{0}\right)\right) - T(0)\right|.$$

Let

$$\Omega = \{a \in V : T(a) < 1\},$$

and $t \in \mathbb{R}$. Then, we have

$$t \cdot \Omega = \bigcup_{x \in \Omega} t \cdot x \subseteq \{x \in V : T(x) < t\}.$$

Moreover, we have

$$\begin{aligned} sup\ T\left(h\left(t \cdot x, (1-t) \cdot y, \overset{(m-2)}{0}\right)\right) &\leq sup\ T(t \cdot x) + sup\ T((1-t) \cdot y) \\ &\leq tT(x) + (1-t)T(y) < 1, \end{aligned}$$

where $x, y \in \Omega$ and $0 \leq t \leq 1$. Hence ,$\overline{\Omega}$ is a closed, convex and absorbing subset of V.

We are now going to show that there is an open ball centered at 0 with some radius ε. Let $\Theta = Cl(\Omega) \cap (-Cl(\Omega))$. Since Θ is convex set, we have $0 \in \Theta$. By contradiction, we show that $Int(\Theta)$ is a non-empty set. Suppose on the contray,that $Int(\Theta)$ is empty. Then for every $n \in \mathbb{N}$, it is obvious that $A_n[\Theta]$ is closed and $Int(A_n[\Theta]) = \emptyset$. Let C_1 be a closed ball with radius 1. Then, $(V - A_2[\Theta]) \cap Int(C_1)$ is a non-empty open set. In a similar manner, we can see that there is a closed ball C_2 in $C_1 - A_2[\Theta]$ with radius $\frac{1}{2}$. By repeating the same process, we are able to find a sequence $\{C_n\}$ such that $C_n \cap A_n[\Theta] = \emptyset$. Clearly, the centers of these balls form a Cauchy sequence which is convergent. Let x be the limit of this Cauchy sequence. Hence, we have $x \in V - A_n[\Theta]$, for every $n \in \mathbb{N}$. Since $\overline{\Omega}$ is a absorbing set, there is $n_1 \in \mathbb{N}$ such that for every $n > n_1$, $x, -x \subseteq n_1 \cdot \overline{\Omega}$. This implies that $x \in n_1 \cdot \Theta$ and so $x \in A_n[\Theta]$, which is a contradiction. Therefore ,We have

$$N_\varepsilon(0) \subseteq h\left(\frac{1}{2} \cdot Int(\Theta), \frac{1}{2} \cdot Int(-\Theta), \overset{(m-2)}{0}\right) \subseteq \Theta.$$

Suppose that $N_\varepsilon(0) \subseteq \overline{\Omega}$.

Let $x \in V$ such that $\|x\| < \varepsilon$. Since $x \in N_\varepsilon(0) \subseteq Cl(\Omega)$, there is $x_1 \in \Omega$ such that $\left\|h\left(x, -x_1, \overset{(m-2)}{0}\right)\right\| < \frac{\varepsilon}{s}$, where $s > 2$. Since

$$h\left(x, -x_1, \overset{(m-2)}{0}\right) \subseteq s^{-1} \cdot U \subseteq s^{-1} \cdot Cl(\Omega) \subseteq Cl(s^{-1} \cdot \Omega),$$

we deduce that there is $x_2 \in s^{-1} \cdot \Omega$. In the same way, we conclude that there is a $x_3 \in s^{-2} \cdot \Omega$ such that $\left\|h\left(x, -h\left(x_1, x_2, x_3, \overset{(m-3)}{0}\right), \overset{(m-2)}{0}\right)\right\| < s^{-3}\varepsilon$. Then, we find a sequence $\{x_n\}$, such that $x_n \in s^{-n+1} \cdot \Omega$ and $\left\|h\left(x, -A_n, \overset{(m-2)}{0}\right)\right\| < s^{-n}\varepsilon$, for every $n \in \mathbb{N}$. This implies that $lim_{n \longrightarrow \infty} A_n = x$. By the countable subadditivity of T, we hence deduce that

$$T(x) = T(lim_{n \longrightarrow \infty} A_n) \leq \sum_{n=1}^{\infty} T(x_n) < s.$$

Now, if $x \in V$ is such that $\|x\| < s^{-1}t\varepsilon$, then $sup\|(st^{-1} \cdot x)\| < \varepsilon$, where $T(x) < s$. This shows that for every $y \in (st^{-1} \cdot x)$, $T(y) < s$, and therefore we deduce that $sup\ T((s^{-1}t) \cdot y) \leq (s^{-1}t \cdot T(y)) < t$. Hence, we obtain $T(x) < t$. This implies that T is continuous at 0. Thus, our proof is completed. □

We now establish the main theorem of this paper.

Theorem 3.3. *Let V_1 and V_2 be two additive m-ary hyper Banach spaces and $T : V_1 \longrightarrow V_2$ be a bounded and onto linear transformation such that for every $\lambda \in F, a, b \in V_1$ and $\Theta \subseteq V_1$ following conditions are satisfied:*

1. $\lambda \cdot 0 = 0$
2. $b \in \lambda^{-1} \cdot a$, *for every* $a \in \lambda \cdot b$,
3. $\lambda \cdot Cl(\Theta) \subseteq Cl(\lambda \cdot \Theta)$.

Then, T is an open map.

Proof. Suppose that $T : V_1 \longrightarrow V_2$ is an onto and bounded linear transformation. It suffices to show that for every $\varepsilon > 0$, $T(N_1(0))$, is an open set. Indeed, let O be an open subset of V_1

and $v \in O$. Then, there is an $r > 0$ such that $N_r(v) \subseteq O$. Hence

$$N_r(v) = \left\{ y \in V_1 : \left\| h\left(y, -v, \overset{(m-2)}{0}\right) \right\| < r \right\} \subseteq O.$$

We have shown that

$$h\left(v, r_1 \cdot N_1(0), \overset{(m-2)}{0}\right) \subseteq N_r(v),$$

where $r_1 < r$. Let $y \in h\left(v, r_1 \cdot N_1(0), \overset{(m-2)}{0}\right)$. Then, there is a $x \in r \cdot N_1(0)$ such that $y = h\left(v, x, \overset{(m-2)}{0}\right)$. Hence, we obtain the following equality.

$$\begin{aligned} \left\| h\left(y, -v, \overset{(m-2)}{0}\right) \right\| &= \left\| h\left(h\left(v, x, \overset{(m-2)}{0}\right), -v, \overset{(m-2)}{0}\right) \right\| \\ &\leq \|x\| < r. \end{aligned}$$

Let $y \in N_r(v)$ and $a = r^{-1} \cdot h\left(y, -v, \overset{(m-2)}{0}\right)$. Then, we deduce that

$$\|a\| = \left\| r^{-1} \cdot h\left(y, -v, \overset{(m-2)}{0}\right) \right\| = \frac{1}{r} \left\| h\left(y, -v, \overset{(m-2)}{0}\right) \right\| < 1,$$

and

$$y \in h\left(v, r \cdot a, \overset{(m-2)}{0}\right) \in h\left(v, r \cdot N_1(0), \overset{(m-2)}{0}\right).$$

Hence, we obtain that $h\left(v, r_1 \cdot N_1(0), \overset{(m-2)}{0}\right) = N_r(v)$. This implies that

$$h\left(T(v), r_1 \cdot T(N_1(0)), \overset{(m-2)}{0}\right),$$

so that it is contained in a neighborhood of $T(v)$.

Let

$$\Psi(b) = inf\{\|a\| : a \in V_1, T(a) = b\},$$

for every $z \in V_2$ and $\lambda \in F$. Then, we have the following equalities:

$$\begin{aligned} sup\ \Psi(\lambda \cdot b) &= sup\{inf\{\|a\| : a \in V_1, T(a) = c\} : c \in \lambda \cdot T(b)\} \\ &\leq sup\|\lambda \cdot b\| \\ &=|\ \lambda\ |\ \|b\| \\ &= |\lambda|\Psi(b). \end{aligned}$$

For every $\varepsilon > 0$, let $\{b_n\}$ be a sequence in V_2 and $\{a_n\}$ be another sequence in V_1 such that $T(a_n) = b_n$ and $lim_{n \longrightarrow \infty} A_n[b_n]$ is convergence and $\|a_n\| < \Psi(b_n) + 2^{-n}\varepsilon$. Then, we deduce that

$$\sum_{n=1}^{\infty} \|a_n\| < \sum_{n=1}^{\infty} \Psi(b_n) + \varepsilon.$$

Since V_1, is m-ary hyper Banach space, we can easily show that $lim_{n\longrightarrow\infty}A_n[a_n]$ is convergent. Moreover, we observe that

$$\Psi(lim_{n\longrightarrow\infty}A_n[b_n]) \leq \|lim_{n\longrightarrow\infty}A_n[a_n]\| \leq \sum_{n=1}^{\infty}\|a_n\| < \sum_{n=1}^{\infty}\Psi(b_n)+\varepsilon.$$

Hence, T is continuous and since $T(N_1(0)) = \Psi^{-1}(-\infty,1)$, we see immediately that $T(N_1(0))$ is an open set and hence, our proof is completed. □

4. Fundamental m-Ary Banach Space

By using a certain type of equivalence relations, we can connect an m-ary hypervector spaces to another m-ary vector space. These kind of equivalence relations are called the *strongly regular* relations. More exactly, starting with an m-ary hypervector space and using a strong regular relation, we can construct another m-ary vector space structure on the quotient set. We first cite some notations.

Let V be an m-ary hypervector space and ρ be an equivalence relation on V. If A and B are non-empty subsets of V, then, we have the following statements.:

$$A\overline{\rho}B \text{ means that } \quad \begin{array}{l}\forall a\in A, \exists b\in B \text{ such that } a\rho b,\\ \forall b^{'}\in B, \exists a^{'}\in A \text{ such that } a^{'}\rho b^{'}.\end{array}$$

$$A\overline{\overline{\rho}}B \text{ means that } \quad \forall a\in A, \forall b\in B \text{ we have } a\rho b.$$

The equivalence relation ρ is called

1. *regular* if for all $a, b, x_2, x_3, ..., x_m \in V$ and $\lambda \in F$

$$h(a, x_2, x_3, ..., x_m)\overline{\rho}\ h(b, x_2, x_3, ..., x_m) \text{ and } (\lambda\cdot a)\overline{\rho}(\lambda\cdot b).$$

2. *strongly regular* if for all $a, b, x_2, x_3, ..., x_m \in V$ and $\lambda \in F$

$$h(a, x_2, x_3, ..., x_m)\overline{\overline{\rho}}\ h(b, x_2, x_3, ..., x_m) \text{ and } (\lambda\cdot a)\overline{\overline{\rho}}(\lambda\cdot b).$$

Let ρ be an equivalence relation on an m-ary hypervector space.

The above equivalence relation plays an important role in the study of m-ary hypervector space. Let V be an m-ary hypervector space. Then, We define

$$\begin{aligned}
\beta_1 &= h\left(\lambda_1 \cdot v_1, \overset{(m-1)}{0}\right) \\
&\vdots \\
\beta_m &= h\left(\lambda_1 \cdot v_1, \lambda_2 \cdot v_2, \overset{(m-3)}{0}, \lambda_m \cdot v_m\right) \\
&\vdots \\
\beta_{2m-1} &= h\left(\beta_m, \lambda_{m+1} \cdot v_{m+1}, \overset{(m-3)}{0}, \lambda_{m+1} \cdot v_{2m-1}\right) \\
&\vdots \\
\beta_{km-(k-1)} &= h\Big(\beta_{(k-1)m-(k-2)}, \lambda_{(k-1)m-(k-2)+1} \cdot v_{(k-1)m-(k-2)+1}, \overset{(m-3)}{0}, \\
&\lambda_{(k-1)m-(k-2)+m-1} \cdot v_{(k-1)m-(k-2)+m-1}).
\end{aligned}$$

We say that

$$(v, w) \in \beta_k \text{ if there exists } , n \in \mathbb{N} \text{ such that } \{v, w\} \subseteq \beta_n.$$

Let $\beta = \bigcup_{n \geq 1} \beta_n$. Then, it is clear that the above relation β is reflexive and symmetric. Denote by β^* the transitive closure of the m-ary hypervector V.

The properties of an m-ary hypervector space are described in the following theorem.

Theorem 4.1. *Let V be an m-ary hypervector space. Then, we have the following properties*

1. *β^* is strongly regular relation on V,*
2. *if ρ is a strongly regular relation on V, then $\beta^* \subseteq \rho$.*

Proof. 1. Suppose that $a\beta^*b$ and $x_2, x_3, ..., x_m \in V$. It follows that there exists $a = v_0, v_1, v_2, ..., v_n = b$ such that for all $i \in \{0, 1, 2, ..., n-1\}$, we have $v_i \beta v_{i+1}$. Let $u_1 \in h(a, x_2, x_3, ..., x_m)$ and $u_2 \in h(b, x_2, x_3, ..., x_m)$. We now show that $u_1 \beta u_2$. From $v_i \beta v_{i+1}$, it follows that there exists a hyperproduct β_i such that $\{v_i, v_{i+1}\} \subseteq \beta_i$, and so we obtain the following results:

$$\begin{aligned}
h(v_i, x_2, x_3, ..., x_m) &\subseteq h(\beta_i, x_2, x_3, ..., x_m), \\
h(v_{i+1}, x_2, x_3, ..., x_m) &\subseteq h(\beta_i, x_2, x_3, ..., x_m),
\end{aligned}$$

which means that $h(v_i, x_2, x_3, ..., x_m)\overline{\overline{\beta}}h(v_{i+1}, x_2, x_3, ..., x_m)$. We obtain that $v_1 \beta v_2$. This implies that $u_1 \beta u_2$.

2. We have $\beta_1 = \{(v, v) : v \in V\} \subseteq \rho$, since ρ is reflexive. Suppose that $\beta_{n-1} \subseteq \rho$ and we can show that $\beta_n \subseteq \rho$. If $a\beta_n b$, there exist $x_2, x_3, ..., x_m \in V$ such that $\{a, b\} \subseteq h(\beta_{n-1}, x_2, x_3, ..., x_m)$. Hence, there exist $u, v \in \beta_{n-1}$ such that $a \in h(u, x_2, x_3, .., x_m)$ and $b \in h(v, x_2, x_2, ..., x_m)$. We have $u\beta_m v$ and according to our hypothesis, we obtain $a\rho b$. Hence , $\beta_n \subseteq \rho$.It follows that $\beta \subseteq \rho$, whence $\beta^* \subseteq \rho$. □

We have the following corollary:

Corollary 4.2. *Let V be an m-ary hypervector space. Then, $[V : \beta^*]$ is an m-ary vector space with respect to the following m-ary operation and external product.*

$$\overline{h}(\beta^*(v_1), \beta^*(v_2), ..., \beta^*(v_m)) = \{\beta^*(v) : v \in h(v_1, v_2, ..., v_m)\},$$

$$\lambda \cdot \beta^*(v) = \{\beta^*(v) : v \in \lambda \cdot v\}.$$

Let V be an m-ary hypervector space. Then, the quotient set $[V : \beta^*]$ is called a *fundamental m-ary Banach space.*

5. Conclusions

In this paper, we have studied the m-ary hyper Banach space as a generalization of the Banach space. In particular, we have established some fundamental properties of an m-ary hypervector space and also we establish a necessary condition for a countable subadditive prenorm of the m-ary hypervector space to be continuous.

In our future research, we will further consider the m-ary hyper Banach spaces and investigate the fundamental properties and the structure of the m-ary Banach spaces.

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