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Research Paper

SOME CLASSICAL THEOREMS IN STATE RESIDUATED LATTICES

MOHAMMAD TAHERI, FARHAD KHAKSAR HAGHANI*, SAEED RASOULI

Abstract. This paper, by considering the notion of a state residuated lattice morphism in the class of state residuated lattices, investigates some classical theorems namely the going up and lying over theorems. Results show that each state residuated lattice morphism fulfills these theorems. Also, some properties about prime filters of residuated lattices are obtained which are given in the paper.

1. Introduction

Flaminio and Montagna in [8, 9] presented an algebraizable logic using a probabilistic approach, which its equivalent algebraic semantics is precisely the variety of state MV-algebras. They enlarged the language of MV-algebras by adding a unary operation σ (called as an inner state or a state operator) equationally described so as to preserve the basic properties of a state in its original meaning. Reference [6] gave a stronger version of state MV-algebras, namely

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*Corresponding author

state-morphism MV-algebras, where by definition, a state-morphism MV-algebra is an idempotent endomorphism on an MV-algebra. They completely described subdirectly irreducible state-morphism MV-algebras and investigated some types of state-morphism MV-algebras [5]. The concept of a state BL-algebra was introduced by [2], as an extension of the concept of a state MV-algebra. The concept of state operators was extended by [7] to $R\ell$-monoids (not necessarily commutative). The notion of state operators on residuated lattices introduced in [12] and investigated some related properties of such operators. Recently, [18, 4] and [19] defined and studied some properties of generalized state operators on residuated lattices.

In commutative ring theory the going up and lying over theorems describe the interrelation between prime ideals of a subring and prime ideals of the over ring. More precisely, if $\mathfrak{B}$ is a subring of $\mathfrak{A}$ and $P \subseteq Q$ are prime ideals of $\mathfrak{B}$, the going up theorem states that if P' is a prime ideal of $\mathfrak{A}$ lying over P (i.e., $P' \cap B = P$), then there exists a prime ideal $P' \subseteq Q'$ of $\mathfrak{A}$ lying over Q. The lying over theorem states that if P is a prime ideal of $\mathfrak{B}$, then there exists a prime ideal P' of $\mathfrak{A}$ lying over P. More general versions of this theorem stated for arbitrary ring morphisms. Suppose $h : \mathfrak{B} \longrightarrow \mathfrak{A}$ is a ring morphism. We say that h fulfills the going up property provided, if $P \subseteq Q$ are prime ideals of $\mathfrak{B}$ and P' is a prime ideal of $\mathfrak{A}$ lying over P (i.e., $h^{\leftarrow}(P') = P$), then there exists a prime ideal $P' \subseteq Q'$ of $\mathfrak{A}$ lying over Q. We say that h fulfills the lying over property provided, if P is a prime ideal of $\mathfrak{B}$, then there exists a prime ideal P' of $\mathfrak{A}$ lying over P. The going up and lying over theorems have been studied for some algebraic structures: MV-algebras [1], BL-algebras [17], Congruence-modular Algebras [11] and residuated lattices [16]. This work is motivated by the above works and a desire to extend these investigations to state residuated lattices. Our findings show that the results obtained in [11] and [16] can also be reproduced via state residuated lattices.

This paper is organized in five sections. Section 2 recalls some definitions, properties and results relative to residuated lattices and state residuated lattices. We illustrate them by some examples which will be used in the next sections of the paper. Section 3 investigates the prime filters which are essential notions in the class of state residuated lattices. In this section the behavior of prime filters under state residuated lattice morphisms are studied and a fundamental characterization for prime filters in Theorem 3.6 is given. Section 4 shows that any state residuated lattice morphism of state residuated lattices fulfills the going up and lying over properties and in Corollary 4.10 characterizes prime filters of a subalgebra of a state residuated lattice. Section 5 introduces and investigates the prime spectrum of a state residuated lattices and obtains a topological characterizations for going up and lying over theorems by means of this notion.

2. Preliminaries

In this section we recall some definitions, properties and results relative to residuated lattices and state residuated lattices which will be used in the next sections.

2.1. **Residuated lattices.** An algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a *residuated lattice* if $\ell(\mathfrak{A}) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice, $(A; \odot, 1)$ is a commutative monoid and $(\odot, \rightarrow)$ is an adjoint pair. A residuated lattice $\mathfrak{A}$ is called *non-degenerate* if $0 \neq 1$. For a residuated lattice $\mathfrak{A}$ and $a \in A$ we put $\neg a := a \rightarrow 0$ and $a^n := a \odot \cdots \odot a$ (n times), for any positive integer n. An element $a \in A$ is called *idempotent* if $a^2 = a$ and *nilpotent* if $a^n = 0$, for some positive integer n. The set of nilpotent elements of $\mathfrak{A}$ shall be denoted by $N(\mathfrak{A})$. It is well-known that $N(\mathfrak{A})$ is an ideal of $\ell(\mathfrak{A})$. The class of residuated lattices is equational and so it forms a variety. The properties of residuated lattices were presented in [10]. For a survey of residuated lattices we refer to [13].

Remark 2.1. [12, Proposition 2.2] Let $\mathfrak{A}$ be a residuated lattice. The following conditions are satisfied for any $x, y, z \in A$:

r_1 $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;

r_2 $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Example 2.2. Let $A_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 1. Routine calculation shows that $\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation " $\odot$ " is given by Table 1 and the operation " $\rightarrow$ " is defined by $x \rightarrow y = \vee\{a \in A | x \odot a \leq y\}$, for any $x, y \in A_6$.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a		a	a	0	a	a
b			a	0	a	b
c				c	c	c
d					d	d
1						1

TABLE 1

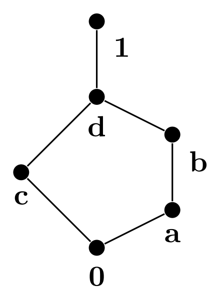

FIGURE 1

Example 2.3. Let $B_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 2. Routine calculation shows that $\mathfrak{B}_6 = (B_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation " $\odot$ " is given by Table 2 and the operation " $\rightarrow$ " is defined by $x \rightarrow y = \vee\{a \in A | x \odot a \leq y\}$, for any $x, y \in B_6$.

⊙	0	a	b	c	d	1
0	0	0	0	0	0	0
a		a	0	a	0	a
b			0	0	b	b
c				a	b	c
d					d	d
1						1

TABLE 2

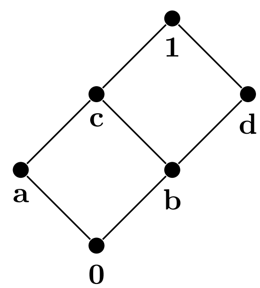

FIGURE 2

Let $\mathfrak{A}$ be a residuated lattice. A non-void subset F of A is called a *filter* of $\mathfrak{A}$ if $x, y \in F$ implies $x \odot y \in F$ and $x \vee y \in F$ for any $x \in F$ and $y \in A$. The set of filters of $\mathfrak{A}$ is denoted by $\mathcal{F}(\mathfrak{A})$. A filter F of $\mathfrak{A}$ is called *proper* if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$. For any subset X of A the *filter of* $\mathfrak{A}$ *generated by* X is denoted by $\mathcal{F}(X)$. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $\mathcal{F}(x)$ and called *principal filter*. The set of principal filters is denoted by $\mathcal{PF}(\mathfrak{A})$. Let $\mathcal{F}$ be a collection of filters of $\mathfrak{A}$. Set $\veebar \mathcal{F} = \mathcal{F}(\cup \mathcal{F})$. It is well-known that $(\mathcal{F}(\mathfrak{A}); \cap, \veebar, \mathbf{1}, A)$ is a frame and so it is a complete Heyting algebra. The following remark has a routine verification.

Remark 2.4. Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x, y \in A$:

(1) $\mathcal{F}(x) = \{a \in A | x^n \leq a, \text{ for some positive integer } n\}$;
(2) $x \leq y$ implies $\mathcal{F}(y) \subseteq \mathcal{F}(x)$.
(3) $\mathcal{F}(x) \cap \mathcal{F}(y) = \mathcal{F}(x \vee y)$;
(4) $\mathcal{F}(x) \veebar \mathcal{F}(y) = \mathcal{F}(x \odot y)$;
(5) $\mathcal{PF}(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$.

	Filters
$\mathfrak{A}_6$	$\{1\}, \{d,1\}, \{a,b,d,1\}, \{c,d,1\}, A_6$
$\mathfrak{B}_6$	$\{1\}, \{a,c,1\}, \{d,1\}, B_6$

TABLE 3

A proper filter P of $\mathfrak{A}$ is called *prime*, if for any $x, y \in A$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. The set of all prime filters of $\mathfrak{A}$ is denoted by Spec($\mathfrak{A}$). By Zorn's lemma it follows that any proper filter is contained in a maximal filter and so in a prime filter.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be residuated lattices. A mapping $h : A \longrightarrow B$ is called a residuated lattice morphism, in symbols $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, if it preserves the fundamental operations. If $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a residuated lattice morphism we put coker(h) $= h^{\leftarrow}(1)$. It is easy to check that coker(h) is a filter of $\mathfrak{A}$. Also, it is obvious that h is one to one if and only if coker(h) $= \{1\}$. The following proposition has a routine verification.

Proposition 2.5. *Let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a residuated lattice morphism.*

(1) *If F is a filter of $\mathfrak{B}$, then $h^{\leftarrow}(F)$ is a filter of $\mathfrak{A}$ and it contains coker(h);*
(2) *if h is onto and F is a filter of $\mathfrak{A}$ containing coker(h), then $h(F)$ is a filter of $\mathfrak{B}$.*

2.2. **State residuated lattices.** This section is devoted to recalling some definitions, properties, and results relative to state residuated lattices, which will be used in the following. The notion of a state operator on a residuated lattice has been introduced by [12] and some related properties of such operators are investigated. For a survey of state residuated lattices we refer to [14].

Let $\mathfrak{A}$ be a residuated lattice. Recalling that [12, Definition 3.1] a mapping $\nu : A \to A$ is called a state operator on $\mathfrak{A}$ if it satisfies the following assertions:

$\mathfrak{s}_1$ $\nu(0) = 0$;
$\mathfrak{s}_2$ ν is monotone;
$\mathfrak{s}_3$ $\nu(x \to y) = \nu(x) \to \nu(x \wedge y)$;
$\mathfrak{s}_4$ $\nu(\nu(x) \odot \nu(y)) = \nu(x) \odot \nu(y)$;

	Prime filters	
	Maximal	
$\mathfrak{A}_6$	$\{a,b,d,1\}, \{c,d,1\}$	$\{d,1\}, \{1\}$
$\mathfrak{B}_6$	$\{d,1\}, \{a,c,1\}$	

TABLE 4

$\mathfrak{s}_5$ $\nu(\nu(x) \vee \nu(y)) = \nu(x) \vee \nu(y)$;

$\mathfrak{s}_6$ $\nu(\nu(x) \wedge \nu(y)) = \nu(x) \wedge \nu(y)$.

If $\mathfrak{A}$ is a residuated lattice and ν is a state operator on $\mathfrak{A}$, then the pair $\mathfrak{A}_\nu = (\mathfrak{A}; \nu)$ is called a state residuated lattice or more precisely, a residuated lattice with internal state ν. We denote by $\mathcal{SRL}$ the class of state residuated lattices. It follows that the class $\mathcal{SRL}$ is equational and so a variety. For any state operator ν on a residuated lattice $\mathfrak{A}$ we set $\ker(\nu) = \nu^{\leftarrow}(1)$. A state operator ν is called faithful if $\ker(\nu) = \{1\}$.

Example 2.6. Let $\mathfrak{A}$ be a residuated lattice. Obviously, Id_A is a state operator on $\mathfrak{A}$. So $\mathfrak{A}_{Id_A}$ is a state residuated lattice.

Example 2.7. Consider Example 2.2. One can check that the mapping $\nu : \mathfrak{A}_6 \longrightarrow \mathfrak{A}_6$ defined by $\nu(x) = 0$ for $x \in \{0, a, c\}$ and $\nu(x) = 1$ for $x \in \{b, d, 1\}$ is a state operator. Hence, $(\mathfrak{A}_6; \nu)$ is a state residuated lattice.

Example 2.8. Consider Example 2.3. One can check that the mapping $\nu : \mathfrak{B}_6 \longrightarrow \mathfrak{B}_6$ defined by $\nu(x) = 0$ for $x \in \{0, a, b\}$ and $\nu(x) = 1$ for $x \in \{c, d, 1\}$ is a state operator. Hence, $(\mathfrak{B}_6; \nu)$ is a state residuated lattice.

Proposition 2.9. [12, Proposition 3.5] *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions hold for any $x, y \in A$:*

$\mathfrak{s}_7$ $\nu(\neg x) = \neg\nu(x)$;

$\mathfrak{s}_8$ $\nu(x) \odot \nu(y) \leq \nu(x \odot y)$;

$\mathfrak{s}_9$ $\nu^2(x) = \nu(x)$;

Let $\mathfrak{A}_\nu$ be a state residuated lattice. A subset F of A is called a filter of $\mathfrak{A}_\nu$ if F is a filter of $\mathfrak{A}$ and $\nu(F) \subseteq F$. The set of filters of $\mathfrak{A}_\nu$ is denoted by $\mathcal{F}(\mathfrak{A}_\nu)$. It is obvious that $\{1\}, \ker(\nu), A \in \mathcal{F}(\mathfrak{A}_\nu)$. Easily, one can check that if F is a filter of $\mathfrak{A}$ contains in $\ker(\nu)$, then F is a filter of $\mathfrak{A}_\nu$. For any subset X of A the *filter of $\mathfrak{A}_\nu$ generated by X* is denoted by $\mathcal{F}^{\mathfrak{A}_\nu}(X)$ and when there is no ambiguity denoted by $\mathcal{F}^\nu(X)$. For any $x \in A$, the filter of $\mathfrak{A}_\nu$ generated by $\{x\}$ is denoted by $\mathcal{F}^\nu(x)$ and called *principal filter* of $\mathfrak{A}_\nu$. The set of principal filters of $\mathfrak{A}_\nu$ is denoted by $\mathcal{PF}(\mathfrak{A}_\nu)$. By [18, Proposition 4.8] it follows that $\mathcal{F}(\mathfrak{A}_\nu)$ is a subframe of $\mathcal{F}(\mathfrak{A})$.

Proposition 2.10. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and F be a filter of $\mathfrak{A}_\nu$. The following assertions hold for any $x, y \in A$:*

(1) $\mathcal{F}^\nu(F, x) := F \veebar \mathcal{F}^\nu(x) = \{a \in A | f \odot (x \odot \nu(x))^n \leq a,\ f \in F,\ n \geq 1\}$;

(2) $x \leq y$ *implies* $\mathcal{F}^\nu(F, y) \subseteq \mathcal{F}^\nu(F, x)$;

(3) $\mathcal{F}^\nu(F, x) \cap \mathcal{F}^\nu(F, y) = \mathcal{F}^\nu(F, (x \odot \nu(x)) \vee (y \odot \nu(y)))$;

(4) $\mathcal{F}^\nu(F, x) \veebar \mathcal{F}^\nu(F, y) = \mathcal{F}^\nu(F, x \odot y)$;

(5) *$\mathcal{PF}(\mathfrak{A}_\nu)$ is a sublattice of $\mathcal{F}(\mathfrak{A}_\nu)$.*

Proof. It follows by [18, Corollary 4.5]. See also [12], Proposition 3.17 and Proposition 4.1 for special case. □

Definition 2.11. Let $\mathfrak{A}_\nu$ and $\mathfrak{B}_\upsilon$ be state residuated lattices. A mapping $h : A \longrightarrow B$ is called a state residuated lattice morphism, in symbols $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$, if it preserves the fundamental operations and $h(\nu(x)) = \upsilon(h(x))$ for any $x \in A$.

The following proposition should be compared with Proposition 2.5.

Proposition 2.12. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism.*

(1) *If F is a filter of $\mathfrak{B}_\upsilon$, then $h^{\leftarrow}(F)$ is a filter of $\mathfrak{A}_\nu$;*
(2) *if h is onto and F is a filter of $\mathfrak{A}_\nu$ containing coker(h), then $h(F)$ is a filter of $\mathfrak{B}_\upsilon$.*

Proof.
(1): Let F be a filter of $\mathfrak{B}_\upsilon$. By Proposition 2.5(1) it follows that $h^{\leftarrow}(F)$ is a filter of $\mathfrak{A}$. Let $x \in h^{\leftarrow}(F)$. So $h(\nu(x)) = \upsilon(h(x)) \in \upsilon(F) \subseteq F$. It shows that $\nu(h^{\leftarrow}(F)) \subseteq h^{\leftarrow}(F)$.
(2): Let h be onto and F be a filter of $\mathfrak{A}_\nu$ which contains coker(h). By Proposition 2.5(2) it follows that $h(F)$ is a filter of $\mathfrak{B}$. Let $y \in h(F)$. So there exists $f \in F$ such that $y = h(f)$. It implies that $\upsilon(y) = \upsilon(h(f)) = h(\nu(f)) \in h(F)$, since $\nu(f) \in F$. □

Definition 2.13. Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. We define ${}^f h : \mathcal{F}(\mathfrak{B}_\upsilon) \longrightarrow \{F | coker(h) \subseteq F \in \mathcal{F}(\mathfrak{A}_\nu)\}$, by ${}^f h(H) = h^{\leftarrow}(H)$.

Proposition 2.14. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be an onto state residuated lattice morphism. Then ${}^f h$ is a bijection.*

Proof. By Proposition 2.12(1) it follows that ${}^f h$ is well-defined. Let ${}^f h(H_1) = {}^f h(H_2)$ for some $H_1, H_2 \in \mathcal{F}(\mathfrak{B}_\upsilon)$. Suppose that $y_1 \in H_1$. So there exists $x_1 \in A$ such that $h(x_1) = y_1$. It implies that $x_1 \in h^{\leftarrow}(H_1) = {}^f h(H_1) = {}^f h(H_2)$. So $H_1 \subseteq H_2$. Analogously, the inverse inclusion is true. Thus ${}^f h$ is injective. Now, let F be a filter of $\mathfrak{A}_\nu$ containing coker(h). By Proposition 2.12(2) it follows that $h(F) \in \mathcal{F}(\mathfrak{B}_\upsilon)$. Let $x \in h^{\leftarrow}(h(F))$ then $h(x) \in h(F)$ and so there exists $f \in F$ such that $h(f) = h(x)$. Hence, $f \to x \in coker(h) \subseteq F$ and it states that $x \in F$. This implies that ${}^f h(h(F))$ is a subset of F. The other inclusion is evident. Hence, ${}^f h(h(F)) = F$ and it means that ${}^f h$ is surjective. □

Corollary 2.15. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be an onto state residuated lattice morphism. Then we have*

$$\mathcal{F}(\mathfrak{B}_\upsilon) = \{h(F) | coker(h) \subseteq F \in \mathcal{F}(\mathfrak{A}_\nu)\}.$$

Proof. It is an immediate consequence of Proposition 2.14. □

Proposition 2.16. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. Then ${}^f h$ is a surjection.*

Proof. By Proposition 2.12(1) it follows that ${}^f h$ is well-defined. Let F be a filter of $\mathfrak{A}_\nu$ containing coker(h). Let $x \in h^{\leftarrow}(\mathcal{F}^{\mathfrak{B}_\upsilon}(h(F)))$ then $h(x) \in \mathcal{F}^{\mathfrak{B}_\upsilon}(h(F))$ and so by Proposition 2.10(1) we can obtain that there exists $f \in F$ such that $h(f) \leq h(x)$. Hence, $f \to x \in coker(h) \subseteq F$ and it states that $x \in F$. The other inclusion is evident. Hence, ${}^f h(\mathcal{F}^{\mathfrak{B}_\upsilon}(h(F))) = F$. □

3. Prime filters in state residuated lattices

In this section we study prime filters which are essential notions in the class of state residuated lattices. Let $\mathfrak{A}_\nu$ be a state residuated lattice. Recalling that a proper filter M of $\mathfrak{A}_\nu$ is called *maximal*, if it is not strictly contained in any filter of $\mathfrak{A}_\nu$. We use $Max(\mathfrak{A}_\nu)$ to denote the set of all maximal filters of $\mathfrak{A}_\nu$. By [18, Proposition 4.12] it follows that any proper filter of a state residuated lattice can be extended to a maximal filter.

Definition 3.1. [15] A proper filter P of $\mathfrak{A}_\nu$ is called prime, if for any $x, y \in A$, $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$ implies $x \in P$ or $y \in P$. The set of all prime filters of $\mathfrak{A}_\nu$ is denoted by Spec$(\mathfrak{A}_\nu)$.

Since $\mathcal{F}(\mathfrak{A}_\nu)$ is a distributive lattice, so a proper filter P of $\mathfrak{A}_\nu$ is prime if and only if P is intersection irreducible, i. e., $F_1 \cap F_2 = P$ implies $F_1 = P$ or $F_2 = P$, for any $F_1, F_2 \in \mathcal{F}(\mathfrak{A}_\nu)$ ([18, Proposition 4.18] or [4, Proposition 2.1]). This shows that $Max(\mathfrak{A}_\nu) \subseteq Spec(\mathfrak{A}_\nu)$ and so any proper filter of $\mathfrak{A}_\nu$ is contained in a prime filter.

Definition 3.2. A non-empty subset $\mathcal{C}$ of $\mathfrak{A}_\nu$ is called $\vee$-closed if $x, y \in \mathcal{C}$ implies $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in \mathcal{C}$. The set of $\vee$-closed subset of $\mathfrak{A}_\nu$ shall be denoted by $\mathcal{C}(\mathfrak{A}_\nu)$. Clearly, $\{1\}, A \in \mathcal{C}(\mathfrak{A}_\nu)$.

Remark 3.3. It is obvious that a filter P of $\mathfrak{A}_\nu$ is prime if and only if $P^c := A \setminus P$ is $\vee$-closed. Also, if Γ be a family of prime filters, then $(\cup\Gamma)^c$ is $\vee$-closed.

Let $\mathfrak{A}_\nu$ be a state residuated lattice. It is obvious that $(A; \mathcal{C}(\mathfrak{A}_\nu))$ is a closed set system. The closure operator associated with this system shall be denoted by $\mathcal{C}^{\mathfrak{A}_\nu} : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. Thus for any subset X of A, $\mathcal{C}^{\mathfrak{A}_\nu}(X) = \cap\{\mathcal{C} \in \mathcal{C}(\mathfrak{A}_\nu) | X \subseteq \mathcal{C}\}$ is the smallest $\vee$-closed subset of $\mathfrak{A}_\nu$ containing X. $\mathcal{C}^{\mathfrak{A}_\nu}(X)$ is called the $\vee$-closed subset of $\mathfrak{A}_\nu$ generated by X. When there is no ambiguity $\mathcal{C}^{\mathfrak{A}_\nu}(X)$ denoted by $\mathcal{C}^\nu(X)$.

Lemma 3.4. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, F be a filter of $\mathfrak{A}_\nu$ and $x \in A$. If $\mathcal{C}^\nu(x) \cap F \neq \emptyset$, then $x \in F$.*

Proof. Let $C^1(x) = \{x\}$ and $C^n(x) = C^{n-1}(x) \cup \{(u \odot \nu(u)) \vee (v \odot \nu(v)) \mid u, v \in C^{n-1}(x)\}$, for all positive integers n. It is easy to see that $\bigcup_{n\in\mathbb{N}} C^n(x)$ is the smallest $\vee$-closed subset of $\mathfrak{A}_\nu$ containing x and so $\mathcal{C}^\nu(x) = \bigcup_{n\in\mathbb{N}} C^n(x)$. Now by induction on n we will show that $y \in \mathcal{C}^\nu(x)$ implies $y \leq x$ and this proves the result. It is obvious that for the base case the result is true. For the inductive step, assume that the result is true for $n-1$ and consider $y \in C^n(x) \setminus C^{n-1}(x)$. So $y = (u \odot \nu(u)) \vee (v \odot \nu(v))$ for some $u, v \in C^{n-1}(x)$. This implies that $u, v \leq x$ and this concludes that $y \leq x$. This complete the induction. □

The following result is an easy consequence of Zorn's lemma.

Lemma 3.5. *If $\mathcal{C}$ is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which does not meet the filter F, then $\mathcal{C}$ is contained in a $\vee$-closed subset C which is maximal with respect to the property of not meeting F.*

Proof. Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$ and $\mathcal{C}$ be a $\vee$-closed subset of $\mathfrak{A}_\nu$ which does not meet the filter F. Let $\Sigma = \{C \in \mathcal{C}(\mathfrak{A}_\nu) \mid F \cap C = \emptyset\}$. Since $\mathcal{C} \in \Sigma$ so $\Sigma \neq \emptyset$. Now, let $\mathcal{C}$ be an arbitrary chain in Σ. By an easy verification it follows that $\bigcup \mathcal{C} \in \Sigma$ and this shows that Σ satisfies the conditions of Zorn's lemma. So Σ has a maximal element and this holds the result. □

The following proposition gives a criteria for the prime filters of a state residuated lattice, inspired by the one obtained for a state pseudo BL-algebra by [**?**, Theorem 3.8] and for a state residuated lattice by [14, Lemma 1] in which a $\vee$-closed subset is defined as a subset such that $x, y \in \mathcal{C}$ implies $x \vee y \in \mathcal{C}$.

Theorem 3.6. *(Prime filters theorem) If $\mathcal{C}$ is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which does not meet the filter F, then F is contained in a filter P which is maximal with respect to the property of not meeting $\mathcal{C}$; furthermore P is prime.*

Proof. Let $\Sigma = \{G \in \mathcal{F}(\mathfrak{A}_\nu) | F \subseteq G,\ G \cap \mathcal{C} = \emptyset\}$. Easily, find that Σ satisfies conditions of *Zorn's lemma*. Let P be a maximal element of Σ. Let $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$ and neither $x \notin P$ nor $y \notin P$. By maximality of P we have $\mathcal{F}^\nu(P, x) \cap \mathcal{C} \neq \emptyset$ and $\mathcal{F}^\nu(P, y) \cap \mathcal{C} \neq \emptyset$. Suppose $a_x \in \mathcal{F}^\nu(P, x) \cap \mathcal{C}$ and $a_y \in \mathcal{F}^\nu(P, y) \cap \mathcal{C}$. By Proposition 2.10(3) and Definition 3.2 follows that $(a_x \odot \nu(a_x)) \vee (a_y \odot \nu(a_y)) \in \mathcal{F}^\nu(P, x \vee y) \cap \mathcal{C} = P \cap \mathcal{C}$; a contradiction. □

Corollary 3.7. *Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$ and X be a subset of A. The following assertions hold:*

(1) *If $X \nsubseteq F$, there exists a prime filter P such that $F \subseteq P$ and $X \nsubseteq P$;*
(2) $\mathcal{F}^\nu(X) = \bigcap\{P \in Spec(\mathfrak{A}_\nu) | X \subseteq P\}$.

Proof.
(1): Let $x \in X - F$. By taking $\mathcal{C}^\nu(x)$ and using Lemma 3.4 it follows that $\mathcal{C}^\nu(x) \cap F = \emptyset$. So the result holds by Theorem 3.6.
(2): Set $\sigma_X = \{P \in Spec(\mathfrak{A}_\nu) | X \subseteq P\}$. Obviously, we have $\mathcal{F}^\nu(X) \subseteq \bigcap \sigma_X$. Now let $a \notin \mathcal{F}^\nu(X)$. By (1) it follows that there exits a prime filter P containing $\mathcal{F}^\nu(X)$ such that $a \notin P$. It shows that $a \notin \bigcap \sigma_X$. □

4. The going up and lying over theorems

In this section we investigate the general version of going up and lying over theorems in the context of state residuated lattices. We find that these two theorems holds in all state residuated lattices.

In the following we study the treatment of prime state filters of a state residuated lattice under state residuated lattice morphisms. It should be compared with Proposition 2.5.

Proposition 4.1. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. The following assertions hold:*

(1) *If $P \in Spec(\mathfrak{B}_\upsilon)$, then $h^{\leftarrow}(P) \in Spec(\mathfrak{A}_\nu)$;*
(2) *If h is onto, then for any prime filter P of $\mathfrak{A}_\nu$ containing $coker(h)$ we have $h(P) \in Spec(\mathfrak{B}_\upsilon)$.*

Proof.
(1): Let P be a prime filter of $\mathfrak{B}_\upsilon$. By Proposition 2.12(1), $h^{\leftarrow}(P)$ is a filter of $\mathfrak{A}_\nu$. Let $(x_1 \odot \nu(x_1)) \vee (x_2 \odot \nu(x_2)) \in h^{\leftarrow}(P)$ for some $x_1, x_2 \in A$. So we have $(h(x_1) \odot \upsilon(h(x_1))) \vee (h(x_2) \odot \upsilon(h(x_2))) \in P$ and it implies that $h(x_1) \in P$ or $h(x_2) \in P$. Consequently, $x_1 \in h^{\leftarrow}(P)$ or $x_2 \in h^{\leftarrow}(P)$ and the result holds.

(2): Let h be an onto state residuated lattice morphism and P be a prime filter of $\mathfrak{A}_\nu$ containing coker(h). By Proposition 2.12(2), $h(P)$ is a filter of $\mathfrak{B}_\upsilon$. Let $(y_1 \odot \upsilon(y_1)) \vee (y_2 \odot \upsilon(y_2)) \in h(P)$ for some $y_1, y_2 \in B$. So there exist $x_1, x_2 \in A$ and $p \in P$ such that $h(x_1) = y_1$, $h(x_2) = y_2$ and $h((x_1 \odot \nu(x_1)) \vee (x_2 \odot \nu(x_2))) = h(p)$. It follows that $h(p \to ((x_1 \odot \nu(x_1)) \vee (x_2 \odot \nu(x_2)))) = 1$ and it states that $p \to ((x_1 \odot \nu(x_1)) \vee (x_2 \odot \nu(x_2))) \in coker(h) \subseteq P$. Thus $(x_1 \odot \nu(x_1)) \vee (x_2 \odot \nu(x_2)) \in P$ and it implies that $x_1 \in P$ or $x_2 \in P$. Hence $y_1 \in h(P)$ or $y_2 \in h(P)$. □

Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. We define ${}^s h : Spec(\mathfrak{B}_\upsilon) \longrightarrow \{P | coker(h) \subseteq P \in Spec(\mathfrak{A}_\nu)\}$, by ${}^s h(Q) = {}^f h(Q)$ (see Definition 2.13).

Proposition 4.2. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be an onto state residuated lattice morphism. Then ${}^s h$ is a bijection.*

Proof. By Proposition 4.1(1) it follows that ${}^s h$ is well-defined. Injectivity of ${}^s h$ inherit by injectivity of ${}^f h$. Also, if P is a prime filter of $\mathfrak{A}_\nu$ containing coker(h), by Proposition 4.1(2) it follows that $h(P) \in Spec(\mathfrak{B}_\upsilon)$ and we have ${}^s h(h(P)) = {}^f h(h(P)) = P$. So ${}^s h$ is surjective. □

Corollary 4.3. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be an onto state residuated lattice morphism. Then we have*

$$Spec(\mathfrak{B}_\upsilon) = \{h(P) | coker(h) \subseteq P \in Spec(\mathfrak{A}_\nu)\}.$$

Proof. It is an immediate consequence of Proposition 4.2. □

Proposition 4.4. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. If $\mathfrak{C}$ is a $\vee$-closed subset of $\mathfrak{A}_\nu(\mathfrak{B}_\upsilon)$, then $h(\mathfrak{C})(h^{\leftarrow}(\mathfrak{C}))$ is a $\vee$-closed subset of $\mathfrak{B}_\upsilon(\mathfrak{A}_\nu)$.*

Proof. It is straightforward. □

Definition 4.5. Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. We say that h fulfills *the going up property* provided, if $P_1, Q_1 \in Spec(\mathfrak{A}_\nu)$ and $P_2 \in Spec(\mathfrak{B}_\upsilon)$ such that $P_1 \subseteq Q_1$ and $P_1 = h^{\leftarrow}(P_2)$, there exists $Q_2 \in Spec(\mathfrak{B}_\upsilon)$ such that $P_2 \subseteq Q_2$ and $Q_1 = h^{\leftarrow}(Q_2)$.

Theorem 4.6. *(Going up theorem) Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. Then h fulfills the going up property.*

Proof. Let $P_1, Q_1 \in Spec(\mathfrak{A}_\nu)$ and $P_2 \in Spec(\mathfrak{B}_\upsilon)$ such that $P_1 \subseteq Q_1$ and $P_1 = h^{\leftarrow}(P_2)$. Since $Q_1^c \subseteq P_1^c$ so $h^{\leftarrow}(P_2) \cap Q_1^c = \emptyset$. In a routine way we have $P_2 \cap h(Q_1^c) = \emptyset$. Due to Theorem 3.6 and Proposition 4.4 there exists a prime filter Q_2 of $\mathfrak{B}_\upsilon$ containing P_2 such that $Q_2 \cap h(Q_1^c) = \emptyset$. It implies that $h^{\leftarrow}(Q_2) \subseteq Q_1$. Now assume that $x \in Q_1 \setminus h^{\leftarrow}(Q_2)$. So

$\mathcal{F}^{\upsilon}(Q_2, h(x)) \cap h(Q_1^c) \neq \emptyset$. Let $y \in \mathcal{F}^{\upsilon}(Q_2, h(x)) \cap h(Q_1^c)$. By Remark 2.4 there exist $q \in Q_2$, a positive integer n and $u \in Q_1^c$ such that $q \odot h((x \odot \nu(x))^n) \leq y$ and $y = h(u)$. It follows that $q \leq h((x \odot \nu(x))^n \to u)$ and it means that $(x \odot \nu(x))^n \to u \in Q_1$. Hence, $u \in Q_1$. It is a contradiction. □

Definition 4.7. Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. We say that h fulfills *the lying over property* provided, if $P \in Spec(\mathfrak{A}_\nu)$ such that $\mathrm{coker}(h) \subseteq P$, there exists a $Q \in Spec(\mathfrak{B}_\upsilon)$ such that $P = h^{\leftarrow}(Q)$ i. e. ${}^sh$ is a surjection.

Theorem 4.8. *(Lying over theorem) Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. Then h fulfills the lying over property.*

Proof. Let P be a prime filter of $\mathfrak{A}_\nu$ such that $\mathrm{coker}(h) \subseteq P$. In a routine way we can show that $\mathcal{F}^{\upsilon}(h(P)) \cap h(P^c) = \emptyset$. By Theorem 3.6 we find a prime filter P' of $\mathfrak{B}_\upsilon$ containing $\mathcal{F}^{\upsilon}(h(P))$ such that $P' \cap h(P^c) = \emptyset$. So $h^{\leftarrow}(P'), P \in Spec(\mathfrak{A}_\nu)$ and $P' \in Spec(\mathfrak{B}_\upsilon)$ such that $h^{\leftarrow}(P') \subseteq P$. Applying Going up theorem (Theorem 4.6) gives us a prime filter Q such that $h^{\leftarrow}(Q) = P$. □

Corollary 4.9. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. The following assertions are equivalent:*

(1) *h is a one-to-one state residuated lattice morphism;*
(2) *$Im({}^sh) = Spec(\mathfrak{A}_\nu)$.*

Proof.
(2)⇒(1): Let $P \in Spec(\mathfrak{A}_\nu)$. So there exits $Q \in Spec(\mathfrak{B}_\upsilon)$ such that $P = h^{\leftarrow}(Q)$. So by Proposition 2.5(1) it follows that $\mathrm{coker}(h) \subseteq P$. Therefore, $\mathrm{coker}(h) \subseteq \bigcap Spec(\mathfrak{A}_\nu)$ and it implies that $\mathrm{coker}(h) = \{1\}$ due to Proposition 3.7(2).
(1)⇒(2): Let $P \in Spec(\mathfrak{A}_\nu)$. Since h is one-to-one, so $\mathrm{coker}(h) = \{1\} \subseteq P$. This implies that $P \in Im({}^sh)$ due to Lying over theorem (Theorem 4.8). □

In the next corollary we characterize prime filters of a subalgebra of a state residuated lattices. Recalling that a state residuated lattice $\mathfrak{B}_\mu$ is a subalgebra of $\mathfrak{A}_\nu$ if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and μ is the restriction of ν.

Corollary 4.10. *Let $\mathfrak{B}_\mu$ be a subalgebra of a state residuated lattice $\mathfrak{A}_\nu$. We have*

$$Spec(\mathfrak{B}_\mu) = \{P \cap B | P \in Spec(\mathfrak{A}_\nu)\}.$$

Proof. It is a direct consequence of Lying over theorem by taking the inclusion state residuated lattice morphism. □

5. The prime spectrum of a state residuated lattice

Let $\mathfrak{A}_\nu$ be a state residuated lattice. For each subset X of A, we define

$$V(X) = \{P \in Spec(\mathfrak{A}_\nu) | X \subseteq P\}.$$

Proposition 5.1. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions hold:*

(1) $X \subseteq Y \subseteq A$ *implies* $V(Y) \subseteq V(X) \subseteq Spec(\mathfrak{A}_\nu)$*;*
(2) $V(X) = \emptyset$ *if and only if* $\mathcal{F}^\nu(X) = A$*;*
(3) $V(X) = Spec(\mathfrak{A}_\nu)$ *if and only if* $X \subseteq \{1\}$*;*
(4) *if* $\mathcal{X} \subseteq \mathcal{P}(A)$*, then* $V(\cup\mathcal{X}) = \cap_{X\in\mathcal{X}} V(X)$*;*
(5) $V(X) = V(\mathcal{F}^\nu(X))$*;*
(6) $V(X) \cup V(Y) = V(\mathcal{F}^\nu(X) \cap \mathcal{F}^\nu(Y))$*;*
(7) $\mathcal{F}^\nu(X) = \mathcal{F}^\nu(Y)$ *if and only if* $V(X) = V(Y)$*;*

Proof.
(1): It is obvious.
(2): It is evident by Corollary 3.7(1).
(3): It is obvious that $V(\emptyset) = Spec(\mathfrak{A}_\nu)$. Since 1 is an element of any filter of $\mathfrak{A}_\nu$, it follows that 1 is an element of any prime filter of $\mathfrak{A}_\nu$, that is, $V(\{1\}) = Spec(\mathfrak{A}_\nu)$. Conversely, let $X \nsubseteq \{1\}$. By Corollary 3.7(1) there exists a prime filter P of $\mathfrak{A}_\nu$ such that $X \nsubseteq P$. It means that $P \notin V(X)$. So $V(X) \neq Spec(\mathfrak{A}_\nu)$.
(4): For any $X \in \mathcal{X}$ we have $X \subseteq \cup\mathcal{X}$ and so by (1) it follows that $V(\cup\mathcal{A}) \subseteq V(X)$. Hence we obtain that $V(\cup\mathcal{A}) \subseteq \cap_{X\in\mathcal{X}} V(X)$. Conversely, $P \in \cap_{X\in\mathcal{X}} V(X)$ implies that $X \subseteq P$ for any $X \in \mathcal{X}$. So $\cup\mathcal{X} \subseteq P$ and it shows that $P \in V(\cup\mathcal{X})$.
(5): It is obvious.
(6): Let $P \in V(\mathcal{F}^\nu(X) \cap \mathcal{F}^\nu(Y))$. Then $\mathcal{F}^\nu(X) \cap \mathcal{F}^\nu(Y) \subseteq P$ and it implies that $\mathcal{F}^\nu(X) \subseteq P$ or $\mathcal{F}^\nu(Y) \subseteq P$. Hence, $P \in V(X) \cup V(Y)$. The converse inclusion follows by (1).
(7): It follows by (5) and Corollary 3.7(2). □

Let $\mathfrak{A}_\nu$ be a state residuated lattice. By parts (2), (3), (4) and (6) of Proposition 5.1 the family $\{V(X)\}_{X\subseteq A}$ of subsets of Spec($\mathfrak{A}$) satisfies the axioms for closed sets in a topological space for state residuated lattices. The resulting topology is called the *Stone topology* and the topological space Spec($\mathfrak{A}_\nu$) is called the *prime spectrum* of $\mathfrak{A}_\nu$.

In the following for any $X \subseteq A$, we denote the complement of $V(X)$ by $D(X)$. So $D(X) = \{P \in Spec(\mathfrak{A}_\nu) | X \nsubseteq P\}$.

Proposition 5.2. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions hold:*

(1) $X \subseteq Y \subseteq A$ *implies* $D(X) \subseteq D(Y) \subseteq Spec(\mathfrak{A}_\nu)$*;*
(2) $D(X) = \emptyset$ *if and only if* $X \subseteq \{1\}$*;*
(3) $D(X) = Spec(\mathfrak{A}_\nu)$ *if and only if* $\mathcal{F}^\nu(X) = A$*;*
(4) *if* $\mathcal{X} \subseteq \mathcal{P}(A)$*, then* $D(\cup\mathcal{X}) = \cup_{X\in\mathcal{X}} D(X)$*;*
(5) $D(X) = D(\mathcal{F}^\nu(X))$*;*
(6) $D(X) \cap D(Y) = D(\mathcal{F}^\nu(X) \cup \mathcal{F}^\nu(Y))$*;*
(7) $\mathcal{F}^\nu(X) = \mathcal{F}^\nu(Y)$ *if and only if* $D(X) = D(Y)$*;*

Proof. By duality it follows by Proposition 5.1. □

In the following for any $a \in A$ we denote $V(\{a\})$ by $V(a)$ and $D(\{a\})$ by $D(a)$. So

$$V(a) = \{P \in Spec(\mathfrak{A}_\nu) | a \in P\} \text{ and } D(a) = \{P \in Spec(\mathfrak{A}_\nu) | a \notin P\}.$$

Proposition 5.3. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions hold:*

(1) $D(a) = \emptyset$ *if and only if* $a = 1$*;*
(2) $D(a) = Spec(\mathfrak{A}_\nu)$ *if and only if* $\mathcal{F}^\nu(a) = A$*;*
(3) $\mathcal{F}^\nu(a) = \mathcal{F}^\nu(b)$ *if and only if* $D(a) = D(b)$*;*
(4) $V(a) \subseteq D(\neg a)$*;*
(5) *if* $a \leq b$*, then* $D(b) \subseteq D(a)$*.*

Proof. (1), (2) and (3) follows by Proposition 5.2 (2), (3) and (7), respectively.
(4): Let $P \in V(a)$. So $a \in P$. If $\neg a \in P$, then $0 = a \odot \neg a \in P$ and it is a contradiction. Hence, $\neg a \notin P$ and so $P \in D(\neg a)$.
(5): Let $P \in D(b)$. So $b \notin P$. If $P \notin D(a)$, then $a \in P$ and from $a \leq b$ it implies that $b \in P$, that is a contradiction. □

Proposition 5.4. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The family $\{D(a)\}_{a\in A}$ is a basis for the topological space $Spec(\mathfrak{A}_\nu)$.*

Proof. Let $X \subseteq A$ and $D(X)$ be an open subset of Spec($\mathfrak{A}_\nu$). By Proposition 5.2(4) it follows that $D(X) = D(\cup_{a\in X}\{a\}) = \cup_{a\in X} D(a)$. Hence, any open subset of Spec($\mathfrak{A}_\nu$) is a union of subsets of the family $\{D(a)\}_{a\in A}$. □

The sets $D(a)$ will be called *basic open sets* of $\mathrm{Spec}(\mathfrak{A}_\nu)$. Let $\mathfrak{A}_\nu$ be a state residuated lattice. With any filter F of $\mathfrak{A}_\nu$ we associate a binary relation $\equiv_F$ on A as follows; $x \equiv_F y$ if and only if $x \to y, y \to x \in F$, for any $x, y \in A$. It is well-known that for a filter F of $\mathfrak{A}_\nu$, the quotient set $A/\equiv_F$ with the natural operations becomes to residuated lattice which is denoted by $\mathfrak{A}/F$. By [18, Proposition 4.3] it follows that for each filter F of $\mathfrak{A}_\nu$ the mapping $\nu/F : A/F \longrightarrow A/F$, defined by $\nu(a/F) = \nu(a)/F$, is a state operator on $\mathfrak{A}/F$. By [18, Proposition 4.10] it follows that there exists a lattice isomorphism between the set of filters and the set of congruences of $\mathfrak{A}_\nu$, i. e., the class of state residuated lattices is an ideal determined variety.

Proposition 5.5. *Let $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\upsilon$ be a state residuated lattice morphism. Then ${}^s h$ is a closed continuous map.*

Proof. In a routine way we can show that ${}^s h^{\leftarrow}(D(a)) = D(h(a))$ and this shows that ${}^s h$ is a continuous map. Now we show that ${}^s h$ is a closed map. Let F be a filter of $\mathfrak{B}_\upsilon$. It is easy to see that ${}^s h(V(F)) \subseteq V(h^{\leftarrow}(F))$. Conversely, let $P \in V(h^{\leftarrow}(F))$. Define $\lambda : \mathfrak{A}_\nu/h^{\leftarrow}(F) \longrightarrow \mathfrak{B}_\upsilon/F$ by $\lambda(a/h^{\leftarrow}(F)) = h(a)/F$. Let $\lambda(a_1/h^{\leftarrow}(F)) = \lambda(a_2/h^{\leftarrow}(F))$ for some $a_1, a_2 \in A$. This implies that $h(a_1)/F = h(a_2)/F$ and so we have $h(a_1 \to a_2) = h(a_1) \to h(a_2) \in F$ and $h(a_2 \to a_1) = h(a_2) \to h(a_1) \in F$. This states that $a_1 \to a_2, a_2 \to a_1 \in h^{\leftarrow}(F)$. So we get that $a_1/h^{\leftarrow}(F) = a_2/h^{\leftarrow}(F)$ and this implies that λ is a one-to-one state residuated lattice morphism. By Lying over theorem it follows that ${}^s\lambda : Spec(\mathfrak{B}_\upsilon/F) \longrightarrow Spec(\mathfrak{A}_\nu/h^{\leftarrow}(F))$ is a surjection. By Proposition 4.2, ${}^s\pi^{\mathfrak{A}_\nu}_{h^{\leftarrow}(F)}$ and ${}^s\pi^{\mathfrak{B}_\upsilon}_{F}$ are bijections. So there exists a prime filter Q of $\mathfrak{B}_\upsilon$ containing F such that ${}^s\pi^{\mathfrak{B}_\upsilon}_{F}{}^s\lambda({}^s\pi^{\mathfrak{A}_\nu}_{h^{\leftarrow}(F)})^{-1}(Q) = P$. On the other hand in a routine way we can show that ${}^s\pi^{\mathfrak{B}_\upsilon}_{F}{}^s\lambda({}^s\pi^{\mathfrak{A}_\nu}_{h^{\leftarrow}(F)})^{-1}(Q) = h^{\leftarrow}(Q)$. So ${}^s h(Q) = P$ and it shows that $P \in {}^s h(V(F))$. □

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Mohammad Taheri

Department of mathematics, Shahrekord Branch

Islamic Azad university, Shahrekord

Iran.

taheri.mohamad96@yahoo.com

Farhad Khaksar Haghani

Department of mathematics, Shahrekord Branch

Islamic Azad university, Shahrekord

Iran.

haghani1351@yahoo.com

Saeed Rasouli

Department of mathematics, Shahrekord Branch

Islamic Azad university, Shahrekord
Iran & Department of Mathematics,
Persian Gulf University, Bushehr,
Iran.
srasouli@pgu.ac.ir