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Research Paper

MODULES WHOSE NONZERO FINITELY GENERATED SUBMODULES ARE DENSE

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. First, we study multiplication R-modules M where R is a one dimensional Noetherian ring or M is a finitely generated R-module. In fact, it is proved that if M is a multiplication R-module over a one dimensional Noetherian ring R, then $M \cong I$ for some invertible ideal I of R or M is cyclic. Also, a multiplication R-module M is finitely generated if and only if M contains a finitely generated submodule N such that $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$. A submodule N of M is called dense in M, if $M = \sum_{\varphi} \varphi(N)$ where φ runs over all the R-homomorphisms from N into M and R-module M is called a weak π-module if every non-zero finitely generated submodule is dense in M. It is shown that a faithful multiplication module over an integral domain R is a weak π-module if and only if it is a Prüfer prime module.

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1. Introduction

All rings in this paper are commutative with identity and modules are unital. An R-module M is called a multiplication module if, for every submodule N of M, there exists an ideal I of R such that $N = IM$. Obviously every cyclic module is a multiplication module. But, in general any multiplication module is not a cyclic module, see [3, Example 2.1].

Let $S = Z(R)$ be the set of nonzero divisors of R and $K = S^{-1}R$ the total quotient ring of R. For a nonzero ideal I of R, let $I^{-1} = [R :_K I]$. Then I is called an invertible ideal of R if $II^{-1} = R$. Let M be an R-module and $T = \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\}$. Then T is a multiplicatively closed subset of S. Let N be a nonzero submodule of M, $N^{-1} = [M :_{T^{-1}R} N]$. Then N^{-1} is an R-submodule of $T^{-1}R$. N is said to be invertible if $NN^{-1} = M$. An R-module M is called a Dedekind Module if each non-zero submodule of M is invertible, (see [10]). Also, M is called a Prüfer module if every non-zero finitely generated submodule of M is invertible. Note that if R is an integral domain, then a nonzero ideal of R is invertible if and only if it is projective as R-module.

In Section 2, we give some characterizations of multiplication modules over one dimensional Noetherian rings and also finitely generated multiplication modules. More precisely, it is shown that if R is a Noetherian domain with $\dim(R) \leq 1$ then M is a multiplication R-module if and only if either $M \cong I$ for some invertible ideal I of R or M is a cyclic R-module.

Also, if M is a multiplication R-module, then M is a finitely generated R-module if and only if M contains a finitely generated submodule N such that $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$.

Following [9] a submodule N of M is called a dense submodule, if $M = \sum_{\varphi} \varphi(N)$ where the sum is taken over all R-homomorphisms φ from N into M. An R-module M is called π-module if every non-zero submodule is dense in M.

Note that there is another different notion of dense submodule due to Findlay-Lambek that plays an important role in the context of commutative (or noncommutative) algebra, (see [7]). In fact a submodule N of M is called *dense submodule* and is written $N \leq_d M$ if for any $x, y \in M$ with $x \neq 0$ there exists $r \in R$ such that $rx \neq 0$ and $ry \in N$. An R-module M is said to be *monoform* if any nonzero submodule of M is a dense submodule, see for example [8].

We recall that a nonzero R-module M is said to be a prime module if $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$ for each non-zero submodule N of M. Let M be a π-module. Then by [9, Proposition 1.3], M is a prime module. In general, the converse is not true. By [9, Corollary 1.6] a domain R is a π-module if and only if it is a Dedekind domain.

In [5], it is shown that R-module M is prime if and only if every nonzero cyclic submodule of M is dense in M. Moreover, it is shown that, for modules with nonzero socles and for co-semisimple modules, the two concepts π and prime are equivalent.

We say that an R-module M is a weak π-module if every nonzero finitely generated submodule of M is dense in M. In Section 3 of this paper we deal with the modules whose finitely generated submodules are dense. In particular, we show that a domain R is a weak π-module if and only if R is a Prüfer domain.

Naoum and Al-Alwani in [9] have investigated dense submodules of multiplication module. It is proved that a faithful multiplication R-module is a π-module if and only if R is a Dedekind domain, (see [9, Theorem 3.1]).

In Theorem 3.7 we prove that if M is a faithful multiplication R-module over an integral domain R, then the following are equivalent.

(i) R is a Prüfer domain.

(ii) R is a weak π-module.

(iii) The sum of any two dense ideals in R is a dense ideal in R. (iv) M is a weak π-module.

(v) M is a Prüfer prime module. (vi) The sum of each two dense submodules is a dense submodule.

(vii) Every faithful multiplication submodule of M is a weak π-module.

($viii$) Every faithful multiplication submodule of M is a Prüfer module.

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2. Characterization of multiplication modules

In this section we give some characterizations of multiplication modules over one dimensional Noetherian rings. In particular, we show that if R is a Noetherian domain with $\dim(R) \leq 1$, then an R-module M is a multiplication R-module if and only if either $M \cong I$ for some invertible ideal I of R or M is a cyclic R-module.

Next, we prove that if M is a multiplication R-module, then M is a finitely generated R-module if and only if M contains a finitely generated submodule N such that $Ann_R(N) = Ann_R(M)$.

Let M be an R-module. The subset $T(M)$ is defined by $T(M) = \{m \in M\colon \text{there is } 0 \neq r \in R$ such that $rm = 0\}$. Obviously, if R is an integral domain, then $T(M)$ is a submodule of M. If $T(M) = 0$, M is called torsion free.

First, we recall a lemma that we need in this part.

Lemma 2.1. (i) *Let R be a semi-local ring. Then an R-module M is a multiplication R-module if and only if it is cyclic.*
(ii) *Let R be a domain and M a non-zero torsion free multiplication R-module. Then there exists an invertible ideal I of R such that $M \cong I$.*

Proof. See [4, Proposition 4] and [6, Lemma 3.6]. □

Theorem 2.2. *Let R be a Noetherian domain with $\dim R \leq 1$. Then M is a multiplication R-module if and only if M is a cyclic R-module or $M \cong I$ for some invertible ideal I of R.*

Proof. If $\dim R = 0$ the result follows by Lemma 2.1(i). Assume that $\dim R = 1$.

First, suppose that $\operatorname{Ann}_R M = 0$. If $T(M) \neq (0)$ then $\operatorname{Ann}_R(m) \neq 0$ for some nonzero element $m \in M$. Since $Rm = [Rm :_R M]M$ then $[Rm :_R M] \neq 0$. So there is a nonzero element $r \in R$ such that $rM \subseteq Rm$. Let $s \in \operatorname{Ann}_R Rm$. Then $srM = 0$. So, $s = 0$, a contradiction. So, M is a nonzero torsion free R-module. Now, by part (ii) of Lemma 2.1 $M \cong I$ for some invertible ideal I of R.

Now let $I = \operatorname{Ann}_R(M) \neq o$. Since R/I is a Noetherian ring with $dim(R/I) = 0$ then R/I is an Artinian ring. Also, M is an $\frac{R}{I}$-multiplication module. Thus by Lemma 2.1 (ii), M is a cyclic $\frac{R}{I}$-module and hence it is a cyclic R-module. □

Lemma 2.3. *Let M be a multiplication R-module. Then*

(i) M is a finitely generated R-module if and only if for each $\mathfrak{p} \in \operatorname{Max}(R)$ if $\mathfrak{p} \supseteq \operatorname{Ann}_R(M)$ then $\mathfrak{p}M \neq M$.

(ii) If $\mathfrak{p} \in \operatorname{Supp}_R M$, then $\mathfrak{p}M \neq M$.

(iii) If $\mathfrak{p} \in \operatorname{Max} R$ and $\mathfrak{p}M \neq M$, then $\mathfrak{p} \in \operatorname{Supp}_R M$.

Proof. (i) It follows by [6, Theorem 3.1].

(ii) Suppose that $\mathfrak{p} \in \operatorname{Supp}_R M$ and $\mathfrak{p}M = M$. Thus $(\mathfrak{p}M)_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}M_{\mathfrak{p}} = M_{\mathfrak{p}}$. So by [4, Theorem 3.1] $M_{\mathfrak{p}} = 0$, a contradiction.

(iii) Assume that $\mathfrak{p} \in \operatorname{Max} R$ and $\mathfrak{p}M \neq M$. Then $M/\mathfrak{p}M \neq 0$ and so there exists $\mathfrak{q} \in \operatorname{Supp}_R(M/\mathfrak{p}M)$. Now, $\mathfrak{q} \supseteq \operatorname{Ann}_R(M/\mathfrak{p}M) \supseteq \mathfrak{p}$ results that $\mathfrak{p} \in \operatorname{Supp}_R M$. □

Corollary 2.4. *Let M be a multiplication R-module. Then*

(i) M is a finitely generated R-module if and only if
$\{\mathfrak{p} \in \operatorname{Max}(R) : \mathfrak{p} \supseteq \operatorname{Ann}_R M\} \subseteq \operatorname{Supp}_R M$.

(ii) If for every maximal ideal $\mathfrak{p}$ of R, $\mathfrak{p} \in \operatorname{Supp}_R M$ then M is finitely generated. In particular, if $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m} \in \operatorname{Supp}_R M$ then M is finitely generated.

(iii) M is a finitely generated R-module if and only if M contains a finitely generated submodule N such that $\operatorname{Ann}_R(N) = \operatorname{Ann}_R(M)$.

(iv) If M is a faithful R-module then M is finitely generated or $M = T(M)$.

Proof. (i) $\Rightarrow)$ It is obvious. $\Leftarrow)$ Suppose that $\mathfrak{p} \in \operatorname{Max}(R)$ and $\mathfrak{p} \supseteq \operatorname{Ann}_R(M)$. By assumption, $\mathfrak{p} \in \operatorname{Supp}_R M$ and by Lemma 2.3, (ii) $\mathfrak{p}M \neq M$. So, M is a finitely generated R-module from Lemma 2.3, (i).

(ii) It is clear by Lemma 2.3,(i) and (ii).

(iii) $\Leftarrow$) Assume $\mathfrak{p} \in \mathrm{Max}(R)$ such that $\mathfrak{p} \supseteq Ann_R(M)$. Then $\mathfrak{p} \supseteq \mathrm{Ann}_R(N)$. So, $\mathfrak{p} \in \mathrm{Supp}_R N \subseteq \mathrm{Supp}_R M$. Now, (i) completes the result. ($\Rightarrow$) It is clear.

(iv) If $M \neq T(M)$ then for some $x \in M, \mathrm{Ann}_R(x) = 0$. Since $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(M)$, by (iii) M is finitely generated. □

Remark 2.5. We know that if M is a finitely generated R-module then for every $\mathfrak{p} \in \mathrm{Spec}\, R$ with $\mathfrak{p} \supseteq \mathrm{Ann}_R(M)$, we have $\mathfrak{p} \in \mathrm{Supp}_R M$ and the converse is not true in general. But, from Corollary 2.4, we see that if M is also multiplication R-module the converse is true.

3. Weak π-modules

Recall that a submodule N of M is called dense in M if $M = \sum \varphi(N); \varphi \in \mathrm{Hom}_R(N, M)$ and an R-module M is called π-module if any nonzero submodule of M is dense in M.

Definition 3.1. We say that an R-module M is a weak π-module if every nonzero finitely generated submodule of M is dense in M. Let M be a Noetherian module. Then M is a π-module if and only if M is a weak π-module. Also, if R is a Noetherian ring, then for any finitely generated R-module M, M is a π-module if and only if M is a weak π-module.

Obviously, every π-module is weak π-module and by [5, Theorem 1.1] any weak π-module is a prime module; but the converse is not true in general.

Example 3.2. Let R be a Noetherian domain with $\dim R > 1$. Then R is prime but it is not a weak π-module, (see [9, Corollary 1.6]).

A valuation domain is an integral domain R such that for every element x of its field of fractions F, at least one of x or x^{-1} belongs to R. The ideals of a valuation domain are totally ordered by inclusion and every finitely generated ideal is principal. In fact a valuation domain R is Noetherian if and only if R is a principal ideal domain (PID).

Example 3.3. Let R be a valuation domain which is not a PID. Then R is not a Dedekind domain. Also, every finitely generated ideal of R is principal and so dense in R. Hence R is a weak π-module but it is not π-module.

The following results from [5, Propositions 1.6 and 1.8, Corollary 1.7 and Theorem 1.9] obviously.

(i) Let M be an R-module with $Soc(M) \neq 0$. Then M is prime if and only if M is a weak π-module. In particular, if M is Artinian or semisimple, then M is prime if and only if M is a weak π-module.

(ii) Let M be a cosemisimple R-module. Then M is prime if and only if M is a weak π-module.

(iii) If $\dim R = 0$, then two concepts weak π and prime are equivalent for all R-modules.

An R-module M is called principally quasi-injective if each R-homomorphism from a cyclic submodule of M to M can be extended to an endomorphism of M, (see [11]).

In the following proposition it is shown that the three concepts weak π, π and prime are equivalent if the module is principally quasi-injective.

Proposition 3.4. *Let M be a principally quasi-injective R-module. The following are equivalent.*

(i) *M is prime.*

$ii)$ M is π-module.

(iii) *M is weak π-module.*

Proof. It is enough to prove $(i) \Rightarrow (ii)$. Let N be a nonzero submodule of M, $y \in M$ and $0 \neq x \in N$. By [5, Theorem 1.1] Rx is dense in M. So, there exist R-homomorphisms $\theta_1, ..., \theta_k : Rx \to M$ and $r_1, ..., r_k$ in R such that $y = \theta_1(r_1x) + ... + \theta_k(r_kx)$. Since M is principally quasi-injective, there exist $\varphi_1, ..., \varphi_k : M \to M$ such that φ_i extend θ_i to M. Let $\psi_i = \varphi_i\delta$ where δ is inclusion homomorphism of N to M. Now, $y = \psi_1(r_1x) + ... + \psi_k(r_kx)$. Thus, N is dense in M. $\square$

Lemma 3.5. (i) *Let I be a non-zero ideal of R. Then I is dense in R as an R-submodule if and only if I is a finitely generated faithful projective ideal.*

(ii) *Let F be a free R-module and I be a non-zero ideal of R. Then I is dense in R if and only if IF is a dense submodule in F.*

(iii) *Let M be a faithful multiplication R-module and I be a non-zero ideal of R. Then I is a dense (resp. invertible) ideal of R if and only if IM is a dense (resp. invertible) submodule of M.*

(iv) *Let I, J be two ideal of R and $J \subseteq I$. If J is dense in R then J is dense in I.*

Proof. For $(i), (ii)$ and (iii) see [2, Lemmas 2 and 8, Propositions 9 and 12].

(iv) Let $x \in I$ and the homomorphism $\varphi : R \to I$ is defined by $\varphi(r) = rx$. There exists $\varphi_1, ..., \varphi_k \in \mathrm{Hom}_R(J, R)$ and $j_1, ..., j_k \in J$ such that $1_R = \varphi_1(j_1) + ... + \varphi_k(j_k)$. Now, $x = \varphi(1_R) = \varphi\varphi_1(j_1) + ... + \varphi\varphi_k(j_k)$. So, J is dense in I. $\square$

Proposition 3.6. *For an integral domain R the following are equivalent.*

(i) *R is a Prüfer domain.*

(ii) R is a weak π-module.

(iii) There exists a finitely dimensional free weak π-module .

(iv) The sum of every two dense ideals in R is a dense ideal in R.

(v) Every ideal of R is a weak π-module.

(vi) Every finitely generated ideal of R is a weak π-module.

(vii) Every principal ideal of R is a weak π-module.

Proof. $(i) \Rightarrow (ii)$ Assume that I is a nonzero finitely generated ideal of R. Then I is a projective and $Ann_R I = 0$. It follows that I is dense in R by Lemma 3.5 (i).

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Let F be a finitely dimension free weak π-module. Let I be a nonzero finitely generated ideal of R. Then IF is a nonzero finitely generated submodule of F and so is dense in F. Hence I is dense in R by part (ii) of Lemma 3.5 and so it is projective.

$(i) \Rightarrow (iv)$ Let I_1, I_2 be two dense ideals in R. Then I_1, I_2 are finitely generated and projective. So, $I_1 + I_2$ is finitely generated and so it is projective. Also, $\mathrm{Ann}_R(I_1 + I_2) = 0$ and so by Lemma 3.5(i) $I_1 + I_2$ is dense in R.

$(iv) \Rightarrow (i)$ Since every cyclic ideal is dense we conclude that each finitely generated ideal is dense, and so it is projective. It follows that R is a Prüfer domain.

$(i) \Rightarrow (v)$. Let I be an ideal of R and J be a nonzero finitely generated ideal of I. Then J is dense in R. By part (iv) of Lemma 3.5, J is dense in I. So I is weak π-module.

$(v) \Rightarrow (vi), (vi) \Rightarrow (vii)$ and $(vii) \Rightarrow (ii)$ are obvious. □

Theorem 3.7. *Let M be a faithful multiplication R-module over an integral domain R. Then the following are equivalent.*

(i)] R is a Prüfer domain. (ii) R is a weak π-module.

(iii) The sum of any two dense ideals in R is a dense ideal in R.

(iv) M is a weak π-module.

(v) M is a Prüfer module.

(vi) The sum of any two dense submodules is a dense submodule.

(vii) Any faithful multiplication submodule of M is a weak π-module.

(viii) Any faithful multiplication submodule of M is a Prüfer module.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from Proposition 3.6. $(ii) \Rightarrow (iv)$ Let N be a nonzero finitely generated submodule of M. Then from [1, Proposition 2.2] there exists a finitely generated ideal of R such that $N = IM$. Since I is dense in R, N is dense in M by Lemma 3.5.

$(iv) \Rightarrow (v)$ Let N be a nonzero finitely generated submodule of M. Then there exists a finitely generated ideal of R such that $N = IM$. Since I is dense in R, I is also an invertible ideal in R and so $N = IM$ is an invertible submodule of M by part (iii) of Lemma 3.5.

$(v) \Rightarrow (i)$ Let I be a nonzero finitely generated ideal of R. Then $IM \neq 0$ is an invertible submodule and so I is an invertible ideal of R by Lemma 3.5 (iii).

$(v) \Rightarrow (vi)$ Let $N_1 = I_1M, N_2 = I_2M$ be two nonzero dense submodules of M. From part (iii) of Lemma 3.5, I_1, I_2 are dense in R. From $(v) \Rightarrow (iii)$, $I_1 + I_2$ is a dense ideal and so $N_1 + N_2$ is dense in M.

$(vi) \Rightarrow (iii)$ Let I_1, I_2 be two nonzero dense ideal of R. Then I_1M, I_2M are dense in M. So, $(I_1 + I_2)M = I_1M + I_2M$ is dense in M. It follows that $I_1 + I_2$ is dense in R.

$(iv) \Rightarrow (vii)$ Let N be a faithful multiplication submodule of M and K a nonzero finitely generated submodule of N. There exists a finitely generated ideal I of R and an ideal J of R such that $K = IM, N = JM$. By [12, Corollary to Theorem 9], $I \subseteq J$. Since IM is dense in M, I is dense in R. It follows that by Lemma 3.5 that I is dense in J. So, IM is dense in JM by part (iii) of Lemma 3.5.

$(vii) \Rightarrow (viii)$ Let N be a faithful multiplication submodule of M. Since N is a weak π-module from $(iv) \Rightarrow (v)$, N is a Prüfer module.

$(viii) \Rightarrow (v)$ It is obvious. □

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References

[1] M.M. Ali and D.J. Smith, *Some remarks on multiplication and projective modules*, Commun. Algebra, **32** No. 10 (2004) 3897-3909.

[2] M. Alkan and Y. Tras, *On invertible and dense submodules*, Commun. Algebra, **32** No. 10 (2004) 3911-3919.

[3] A. Azizi, *Principal ideal multiplication modules*, Algebra Colloq., **15** No. 04 (2008) 637-648.

[4] A. Barnard, *Multiplication modules*, J. Algebra, **71** (1981) 174-178.

[5] M. Behboodi, *On prime modules and dense submodules*, J. Commut. Algebra, **4** No. 4 (2012) 479-488.

[6] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Commun. Algebra, **16** (1988) 755-799.

[7] G. D. Findlay and J. Lambek, *A generalized ring of quotients, I, II*, Canad. Math. Bull., **1** (1958) 77-85, 155-167.

[8] A. Hajikarimi and A. R. Naghipour, *A note on monoform modules*, Bull. Korean Math. Soc., **56** No. 2 (2019) 505-514.

[9] A. G. Naoum and F. G. Al-Alwan, *Dense submodules of multiplication modules*, Commun. Algebra, **24** No. 2 (1996) 413-424.

[10] A. G. Naoum and F. G. Al-Alwan, *Dedekind modules*, Commun. Algebra, **24** No. 2 (1996) 397-412.

[11] W. K. Nicholson, J. K. Park and M. F. Yousif, *Principally quasi-injective modules*, Commun. Algebra, **27** No. 4 (1999) 1683-1693.

[12] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50** (1988) 223-235.

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