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Research Paper

NORMAL IDEALS IN PSEUDO-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduced the concepts of normlet and normal ideal in a pseudo-complemented almost distributive lattice and studied its properties. We have characterized normal ideals and established equivalent conditions for every ideal to become a normal ideal. Also, derived equivalent conditions for the set of all prime normal ideals of a pseudo-complemented ADL to become a Hausdorff space.

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [7] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided

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a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji[8] introduced the concept of pseudo-complementation on an ADL and they studied the properties of pseudo-complemented ADLs. In [4], G.C. Rao and S. Ravi Kumar proved that some important results on minimal prime ideal of an ADL. In [6], Sambasiva Rao introduced normal ideals in Pseudo-complemented distributive lattices and proved their properties. In this paper, we introduced the concepts of normlets and normal ideals in a pseudo-complemented ADL, analogous to that in a pseudo-complemented distributive lattice. We characterized normal ideals in terms of normlets. Derived a set of equivalent conditions for an ideal to become a normal ideal which leads to a characterization of disjunctive ADL. Established a set of equivalent conditions for every prime normal ideal to become a minimal prime ideal. Finally, proved a set of equivalent conditions for the space $Spec_N(L)$ to become a Hausdorff space.

2. **Preliminaries**

In this section we give some important definitions and results that are frequently used for ready reference.

Definition 2.1. [7] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(3) $(x \vee y) \wedge y = y$
(4) $(x \vee y) \wedge x = x$
(5) $x \vee (x \wedge y) = x$
(6) $0 \wedge x = 0$
(7) $x \vee 0 = x$, for all $x, y, z \in L$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.2. [7] *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* L

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) *If* $a \leq c,\ b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice.

Theorem 2.3. [7] *Let* $(L, \vee, \wedge, 0)$ *be an ADL with 0. Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*

(2) $a \vee b = b \vee a$*, for all* $a, b \in L$

(3) $a \wedge b = b \wedge a$*, for all* $a, b \in L$

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.4. [7] *Let* L *be an ADL and* $m \in L$*. Then the following are equivalent:*

(1) m *is maximal with respect to* $\leq$

(2) $m \vee a = m$*, for all* $a \in L$

(3) $m \wedge a = a$*, for all* $a \in L$

(4) $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [[1], [2]], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M

of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of$L$ is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee \bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.5. [7] *For any x, y in L the following are equivalent:*

(1) $(x] \subseteq (y]$
(2) $y \wedge x = x$
(3) $y \vee x = y$
(4) $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Definition 2.6. [4] A prime ideal of L is called a minimal prime ideal if it is a minimal element in the set of all prime ideals of L ordered by set inclusion.

Theorem 2.7. [4] *Let L be an ADL. Then a prime ideal P is minimal if and only if for any $x \in P$, there exist an element $y \notin P$ such that $x \wedge y = 0$.*

Definition 2.8 ([3])**.** An equivalence relation θ on an ADL L is called a congruence relation on L if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a, b), (c, d) \in \theta$

Theorem 2.9 ([3])**.** *An equivalence relation θ on an ADL L is a congruence relation if and only if for any $(a, b) \in \theta,\ x \in L, (a \vee x, b \vee x), (x \vee a, x \vee b), (a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ are all in θ*

Definition 2.10. [8] Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
(2) $a \wedge a^* = 0$
(3) $(a \vee b)^* = a^* \wedge b^*$

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Theorem 2.11. [8] *Let L be an ADL and $*$ a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

(1) 0^* *is a maximal element*

(2) *If a is a maximal element then* $a^* = 0$

(3) $0^{**} = 0$

(4) $0^* \wedge a = a$

(5) $a^{**} \wedge a = a$

(6) $a^{***} = a^*$

(7) $a \leq b \Rightarrow b^* \leq a^*$

(8) $a^* \wedge b^* = b^* \wedge a^*$

(9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(10) $a^* \wedge b = (a \wedge b)^* \wedge b^*$.

For any pseudo-complemented ADL L, let us denote the set of all elements of the form $x^* = 0$ by $D(L)$. It is easy to prove that $D(L)$ is a filter of an ADL L.

3. Normal ideals in Pseudo-complemented ADLs

In this section, we introduced the concept of normal ideals in a pseudo-complemented ADL, analogous to that in a pseudo-complemented distributive lattice and studied their properties. Finally, observed some topological properties of the space $Spec_N(L)$ of all prime normal ideals of an pseudo-complemented ADL L. Derived a set of equivalent conditions for the space $Spec_N(L)$ to become a Hausdorff space. Now we have the following definition.

Definition 3.1. Let L be a pseudo-complemented ADL. For any ideal I of L, define the set as $I^\circ = \{x \in L/x \wedge a^* = 0, \text{ for some } a \in I\}$.

Lemma 3.2. *Let L be a pseudo-complemented ADL and I, an ideal of L. Then I° is an ideal of L.*

Proof. Clearly, we have that $0 \in I^\circ$. Let $x, y \in I^\circ$. Then there exist elements $a, b \in I$ such that $x \wedge a^* = 0 = y \wedge b^*$. Since $a, b \in I$ and I is an ideal, we have that $a \vee b \in I$. Now, $(x \vee y) \wedge (a \vee b)^* = (x \vee y) \wedge (a^* \wedge b^*) = (x \wedge a^* \wedge b^*) \vee (y \wedge a^* \wedge b^*) = 0 \vee 0 = 0$. That implies $x \vee y \in I^\circ$. Let $x \in I^\circ$ and $r \in L$. Then there exists an element $a \in I$ such that $x \wedge a^* = 0$. Now $(x \wedge r) \wedge a^* = 0$ and hence $x \wedge r \in I^\circ$. Therefore I° is an Ideal of L. Thus I° is an ideal of L. □

Lemma 3.3. *Let L be a pseudo-complemented ADL. For any ideals I, J of L we have the following:*

(1) $I \subseteq I^\circ$

(2) $I \subseteq J \Rightarrow I^\circ \subseteq J^\circ$

(3) $I^\circ \cap J^\circ = (I \cap J)^\circ$

(4) $I^{\circ\circ} = I^{\circ}$.

Proof. (1). Let $x \in I$. Clearly, we have that $x \wedge x^* = 0$ and $x \in I$ and hence $x \in I^{\circ}$. Therefore $I \subseteq I^{\circ}$.

(2). Suppose $I \subseteq J$. Let $x \in I^{\circ}$. Then there exists an element $a \in I$ such that $x \wedge a^* = 0$. That implies $x \wedge a^* = 0$ and $a \in J$ and hence $x \in J^{\circ}$. Therefore $I^{\circ} \subseteq J^{\circ}$

(3). Clearly, we have $I \cap J \subseteq I$ and $I \cap J \subseteq J$. So that $(I \cap J)^{\circ} \subseteq I^{\circ} \cap J^{\circ}$. Let $x \in I^{\circ} \cap J^{\circ}$. Then $x \in I^{\circ}$ and $x \in J^{\circ}$. Then there exist elements $a \in I$ and $b \in J$ such that $x \wedge a^* = 0 = x \wedge b^*$. Since $a \in I$ and $b \in J$ we get that $a \wedge b \in I \cap J$. Since $x \wedge a^* = 0$ and $x \wedge b^* = 0$ we get that $a^{**} \wedge x = x$ and $b^{**} \wedge x = x$. Now $x = a^{**} \wedge x = a^{**} \wedge b^{**} \wedge x = (a \wedge b)^{**} \wedge x$. That implies $x \wedge (a \wedge b)^* = 0$ and hence $x \in (I \cap J)^{\circ}$. Therefore $(I \cap J)^{\circ} = I^{\circ} \cap J^{\circ}$

(4). By (1) and (2) we have that $I \subseteq I^{\circ}$ and $I^{\circ} \subseteq I^{\circ\circ}$. Let $x \in I^{\circ\circ}$. Then $x \wedge a^* = 0$ for some $a \in I^{\circ}$. That implies $a^{**} \wedge x = x$. Since $a \in I^{\circ}$, there exists an element $b \in I$ such that $a \wedge b^* = 0$. That implies $b^{**} \wedge a = a$. Now $x = a^{**} \wedge x = (b^{**} \wedge a)^{**} \wedge x = b^{**} \wedge a^{**} \wedge x = b^{**} \wedge x$. That implies $x = b^{**} \wedge x$ and hence $x \wedge b^* = 0$. That implies $x \in I^{\circ}$. Therefore $I^{\circ\circ} \subseteq I^{\circ}$. Thus $I^{\circ} = I^{\circ\circ}$ □

Definition 3.4. Let L be a pseudo-complemented ADL. An ideal of the form $((a])^{\circ} = \{x \in L / x \wedge a^* = 0\}$ is called normlet. Clearly, $((a])^{\circ}$ is an ideal of L. We write a° instead of $((a])^{\circ}$.

We derive some important properties of normlet which will help to develop the theory further.

Lemma 3.5. *Let L be a pseudo-complemented ADL with maximal elements. For any $a, b \in L$, we have the following:*

(1) $a \in a^{\circ}$

(2) $a \leq b \Rightarrow a^{\circ} \subseteq b^{\circ}$

(3) $a^{\circ} \cap b^{\circ} = (a \cap b)^{\circ}$

(4) $a^{\circ\circ} = a^{\circ}$

(5) $a \in b^{\circ} \Rightarrow a^{\circ} \subseteq b^{\circ}$

(6) $a \in D(L)$ *iff* $a^{\circ} = L$

(7) $a \vee b$ *is maximal* $\Rightarrow a^{\circ} \vee b^{\circ} = L$

(8) $a^* = b^* \Rightarrow a^{\circ} = b^{\circ}$.

Proof. (1),(2),(3),(4) are clear.

(5). Assume that $a \in b^{\circ}$. Then $a \wedge b^* = 0$. That implies $b^* = a^* \wedge b^*$. Let $x \in a^{\circ}$. Then $x \wedge a^* = 0$. Now $x \wedge b^* = x \wedge a^* \wedge b^* = 0$ and hence $x \in b^{\circ}$. Therefore $a^{\circ} \subseteq b^{\circ}$.

(6). Assume $a \in D(L)$. Then $a^* = 0$. Clearly, we have $x \wedge a^* = 0$, for all $x \in L$. Therefore

$x \in a^\circ$ and hence $L \subseteq a^\circ$. Thus $L = a^\circ$. Conversely, assume that $L = a^\circ$. Choose a maximal element m in L such that $m \in a^\circ$. That implies $m \wedge a^* = 0$ and hence $a^* = 0$. Therefore $a \in D(L)$.

(7). Assume $a \vee b$ is maximal element of L. Suppose $a^\circ \vee b^\circ \neq L$. Then there exists a maximal ideal M of L such that $a^\circ \vee b^\circ \subseteq M$. That implies $a^\circ \subseteq M$ and $b^\circ \subseteq M$. That implies $a \in M$ and $b \in M$. That implies $a \vee b \in M$, which is a contradiction. Hence $a^\circ \vee b^\circ = L$.

(8). Assume that $a^* = b^*$. Let $x \in a^\circ$. Then $x \wedge a^* = 0$. That implies $x \wedge b^* = 0$ and hence $x \in b^\circ$. Therefore $a^\circ \subseteq b^\circ$. Similarly, we have that $b^\circ \subseteq a^\circ$. Therefore $a^\circ = b^\circ$. Conversely, assume that $a^\circ = b^\circ$. Clearly, we have $a \wedge a^* = 0$ and $b \wedge b^* = 0$. That implies $a \in a^\circ = b^\circ$ and $b \in b^\circ = a^\circ$. That implies $a \in b^\circ$ and $b \in a^\circ$ and hence $a \wedge b^* = 0$ and $b \wedge a^* = 0$. Therefore $b^* = a^* \wedge b^*$ and $a^* = b^* \wedge a^*$. Since $a^* \wedge b^* = b^* \wedge a^*$, we have that $a^* = b^*$. □

Theorem 3.6. *Let L be a pseudo-complemented ADL with maximal element m. Then set $\mathcal{A}^\circ(L)$ of all normlets forms a Boolean algebra.*

Proof. Let $a, b \in L$. We define $a^\circ \cap b^\circ = (a \wedge b)^\circ$ and $a^\circ \cup b^\circ = (a \vee b)^\circ$. Clearly, we have that $(\mathcal{A}^\circ(L), \cap, \cup, 0^\circ, m^\circ)$ is a bounded distributive lattice. Let $a^\circ \in \mathcal{A}^\circ(L)$. Now, $a^\circ \cap (a^*)^\circ = (a \wedge a^*)^\circ = 0^\circ = \{0\}$ and now, $x \wedge (a \vee a^*)^* = x \wedge a^* \wedge a^{**} = x \wedge 0 = 0$. That implies $x \wedge (a \vee a^*)^* = 0$ for all $x \in L$. That implies $x \in (a \vee a^*)^\circ = a^\circ \cup (a^*)^\circ$ for all $x \in L$. Therefore $a^\circ \cup (a^*)^\circ = L$ and hence $(\mathcal{A}^\circ(L), \cap, \cup, \{0\}, m^*)$ is a Boolean algebra. □

We introduce the concept of a normal ideal in a pseudo-complemented ADL analogous to that in [6]. Now we have the following definition.

Definition 3.7. Let L be a pseudo-complemented ADL. An ideal I of L is said to be a normal ideal of L if $I = I^\circ$.

We following the example of normal ideal in a pseudo-complemented ADL.

Example 3.8. Consider a discrete ADL $A = \{0, a\}$ and a distributive lattice $B = \{0', a', b', c', 1\}$ whose Hasse diagram is given in the following Figure-1.

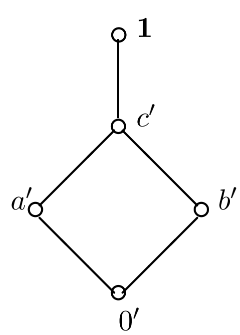

Figure 1

Take
$L = A \times B = \{(0,0'), (0,a'), (0,b'), (0,c'), (0,1), (a,0'), (a,a'), (a,b'), (a,c'), (a,1)\}$. Then $(L, \vee, \wedge, \bar{0})$ is an ADL with zero $\bar{0} = (0,0')$ under point-wise operations. Clearly, $I = \{(0,0'), (0,a')\}$ is an ideal of L. Now $I^{\circ} = \{x \in L \mid x \wedge y^* = \bar{0}, \text{ for some } y \in I\} = \{(0,0'), (0,a')\} = I$. Therefore I is a normal ideal of L.

It is observed that every ideal of pseudo-complemented ADL L need not to be a normal ideal. For that we have the following example.

Example 3.9. Let $L = \{0, a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on L as follows

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Now define $x^* = 0$, for all $x \neq 0$ and $0^* = a$. Then $(L, \vee, \wedge, 0)$ is an ADL and $*$ is a pseudo-complementation on L. But which is not lattice. Take an ideal $J = \{0, c\}$. Clearly which is not a normal ideal of L, because $J^{\circ} = L$.

Proposition 3.10. *Let L be a pseudo-complemented ADL. Every minimal prime ideal of L is a normal ideal.*

Proof. Let P be a minimal prime ideal of pseudo-complemented ADL L. Now, We prove that $P^{\circ} = P$. Clearly, we have $P \subseteq P^{\circ}$. Suppose $P^{\circ} \nsubseteq P$. Choose an element $x \in P^{\circ}$ such that $x \notin P$. Since $x \in P^{\circ}$, there exists an element $a \in P$ such that $x \wedge a^* = 0$. Since $a \in P$ and P is a minimal prime ideal of L, then there exists an element $b \notin P$ such that $a \wedge b = 0$. That implies that $b = a^* \wedge b$. Since $b \wedge b^* = 0$ and P is an ideal, we have that $b \wedge b^* \in P$. Since P is prime, we have that $b \in P$ or $b^* \in P$. Since $b \notin P$, we get that $b^* \in P$. Now, $x \wedge b = x \wedge a^* \wedge b = 0$ and hence $x = b^* \wedge x$. Since $b^* \in P$, we get that $b^* \wedge x \in P$. That implies $x \in P$, which is a contradiction to $x \in P$. Therefore $P^{\circ} \subseteq P$ and hence $P = P^{\circ}$. Thus P is a normal ideal of L.

□

In the following, we characterized the normal ideal of pseudo-complemented ADL L in terms of normlets.

Theorem 3.11. *Let L be a pseudo-complemented ADL. For any ideal I of L, the following are equivalent.*

(1) *I is normal*

(2) *For any $a \in L, a \in I \Rightarrow a^{\circ} \subseteq I$*

(3) *For any $a, b \in L, a^* = b^*$ and $a \in I \Rightarrow b \in I$*

(4) *For any $a, b \in L, a^{\circ} = b^{\circ}$ and $a \in I \Rightarrow b \in I$*

(5) $I = \bigcup\limits_{a \in I} a^{\circ}$.

Proof. (1) $\Rightarrow$ (2): Clear.
(2) $\Rightarrow$ (3): Assume (2). Let $a, b \in L$ with $a^* = b^*$ and $a \in I$. Then $a^{\circ} \subseteq I$. Since $b \wedge b^* = 0$, we get that $b \wedge a^* = 0$. That implies $b \in a^{\circ} \subseteq I$. Therefore $b \in I$.
(3) $\Rightarrow$ (4): Clear.
(4) $\Rightarrow$ (5): Assume (4). Let $a \in I$. Now, we prove that $a^{\circ} \subseteq I$. Let $x \in a^{\circ}$. Then $x \wedge a^* = 0$. That implies $a^* = x^* \wedge a^*$. That implies $a^* = (x \vee a)^*$. Since $a \in I$ and by our assumption, we get that $x \vee a \in I$. That implies $x \in I$ and hence $a^{\circ} \subseteq I$, for all $a \in I$. Therefore $\bigcup\limits_{a \in I} (a)^{\circ} \subseteq I$. Let $x \in I$. We prove that $I \subseteq \bigcup\limits_{x \in I} (x)^{\circ}$. Since $x \wedge x^* = 0$, we get that $x \in x^{\circ}$ and hence $I \subseteq \bigcup\limits_{x \in I} (x)^{\circ}$. Therefore $I = \bigcup\limits_{x \in I} (x)^{\circ}$
(5) $\Rightarrow$ (1): Assume that $I = \bigcup\limits_{a \in I} (a)^{\circ}$. Let $x \in I^{\circ}$. Then there exists an element $b \in I$ such that $x \wedge b^* = 0$. That implies $x \in b^{\circ}$ and $b \in I$. That implies $x \in \bigcup\limits_{a \in I} (a)^{\circ} = I$. That implies $x \in I$ and hence $I^{\circ} \subseteq I$. Therefore $I = I^{\circ}$. Thus I is a normal ideal of L. $\square$

For any ideal I of L, define a relation $\theta(I) = \{(x, y) \in L \times L \mid a^* \wedge x = a^* \wedge y, \text{ for some } a \in I\}$.

Lemma 3.12. *Let L be a pseudo-complemented ADL with maximal elements and I is an ideal of L. Then $\theta(I)$ is a congruence relation on L.*

Proof. Since I is non-empty subset L, we have $\theta(I)$ is non-empty set. Clearly, we have that $0^* \wedge x = x$. Since $0 \in I$, we get that $(x, x) \in \theta(I)$ and hence $\theta(I)$ is reflexive. Clearly, $\theta(I)$ is symmetric. Let $(x,\ y)$, $(y,\ z) \in \theta(I)$. Then there exist elements $a, b \in I$ such that $a^* \wedge x = a^* \wedge y$ and $b^* \wedge y = b^* \wedge z$. Since $a, b \in I$, we have that $a \vee b \in I$. Now $(a \vee b)^* \wedge x = a^* \wedge b^* \wedge x = b^* \wedge a^* \wedge x = b^* \wedge a^* \wedge y = a^* \wedge b^* \wedge y = a^* \wedge b^* \wedge z = a \vee b)^* \wedge z$. That implies $(x,\ z) \in \theta(I)$ and hence $\theta(I)$ is transitive. Therefore $\theta(I)$ is an equivalence relation on L. Let $(x,\ y) \in \theta(I)$ and $z \in L$. Then there exists an element $a \in I$ such that $a^* \wedge x = a^* \wedge y$. Now, $a^* \wedge (x \wedge z) = (a^* \wedge x) \wedge z = (a^* \wedge y) \wedge z = a^* \wedge (y \wedge z)$. That implies $(x \wedge z,\ y \wedge z) \in \theta(I)$ and also we have that $(z \wedge x,\ z \wedge y) \in \theta(I)$. Now $a^* \wedge (x \vee z) = (a^* \wedge x) \vee (a^* \wedge z) = (a^* \wedge y) \vee (a^* \wedge z) = a^* \wedge (y \vee z)$. That implies $(x \vee z,\ y \vee z) \in \theta(I)$ and also we have that $(z \vee x,\ z \vee y) \in \theta(I)$. Hence $\theta(I)$ is a congruence relation on L. $\square$

Theorem 3.13. *Let J be an ideal of a pseudo-complemented ADL L. Then J is a normal ideal if and only if $J = Ker\ \theta(J)$.*

Proof. Assume that J is a normal ideal of L. We prove that $J = Ker\ \theta(J)$. Let $x \in J$. Since $x \wedge x^* = 0$, we get that $x \in Ker\ \theta(J)$ and hence $J \subseteq Ker\ \theta(J)$. Let $x \in Ker\ \theta(J)$. Then there exists an element $a \in J$ such that $x \wedge a^* = 0$. Since $J = J^\circ$, we get that $a \in J^\circ$. Then there exists an element $b \in J$ such that $a \wedge b^* = 0$. Now, $x \wedge (a \vee b)^* = x \wedge a^* \wedge b^* = 0$. That implies $x \in J^\circ = J$ and hence $x \in J$. Therefore $Ker\ \theta(J) = J$. Conversely, assume that $Ker\ \theta(J) = J$. Clearly, we have $J \subseteq J^\circ$. Let $x \in J^\circ$. Then there exists an element $a \in J$ such that $x \wedge a^* = 0$. By our assumption, we get that $a \in Ker\ \theta(J)$. Then there exists an element $b \in J$ such that $a \wedge b^* = 0$. Since $x \wedge a^* = 0$ and $a \wedge b^* = 0$, we have $x = a^{**} \wedge x$ and $a = b^{**} \wedge a$. Now $x = a^{**} \wedge x = (b^{**} \wedge a)^{**} \wedge x = (b^{**} \wedge a^{**}) \wedge x = b^{**} \wedge x$. That implies $x \wedge b^* = 0$. Since $x \wedge b^* = 0$ and $b \in J$, we get that $x \in Ker\ \theta(J) = J$ and hence $J^\circ \subseteq J$. Therefore $J^\circ = J$. Thus J is normal. □

Definition 3.14. Let L_1, L_2 be two pseudo-complemented ADLs with $*$ as their pseudo-complementation. Then $L_1 \times L_2$ is also a pseudo-complemented ADL with respect to the point wise operation in which the pseudo-complementation is given by $(a, b)^* = (a^*, b^*)$.

It can be easily verified that the set of all normlets of $L_1 \times L_2$ forms a distributive lattice with respect to the operations $\cap$ and $\cup$ of known result, when consider point-wise. We have always that $(a, b)^\circ = a^\circ \times b^\circ\ \forall\ a \in L_1, b \in L_2$.

Theorem 3.15. *If I_1 and I_2 are normal ideals of L_1 and L_2 respectively. Then $I_1 \times I_2$ is a normal ideal of $L_1 \times L_2$. Conversely every normal ideal of $L_1 \times L_2$ can be expressed as $I = I_1 \times I_2$ where I_1 and I_2 are normal ideals of L_1 and L_2 respectively.*

Proof. Let I_1 and I_2 be normal ideals of L_1 and L_2 respectively. We prove that $I_1 \times I_2$ is a normal ideal of $L_1 \times L_2$. It is enough to show that $I_1 \times I_2 = \bigcup_{(a,b) \in I_1 \times I_2} (a, b)^\circ$. Clearly, we have that $I_1 \times I_2$ is an ideal of $L_1 \times L_2$. Let $(a, b) \in I_1 \times I_2$. Then $a \in I_1$ and $b \in I_2$. Since I_1 and I_2 are normal ideals, we get that $a^\circ \subseteq I_1$ and $b^\circ \subseteq I_2$. That implies $a^\circ \times b^\circ \subseteq I_1 \times I_2$ and hence $(a, b)^\circ = a^\circ \times b^\circ \subseteq I_1 \times I_2$. Therefore $I_1 \times I_2 = \bigcup_{(a,b) \in I_1 \times I_2} (a, b)^\circ$. Thus $I_1 \times I_2$ is a normal ideal of $L_1 \times L_2$. Conversely, let I be a normal ideal of $L_1 \times L_2$. Consider $I_1 = \{a \in L_1/(a, b) \in I,\ \text{for some}\ b \in L_2\}$. Clearly, $I_1 \neq \varphi$ and I_1 is an ideal of L_1. Let $x \in I_1$. Then $(x, y) \in I$ for some $y \in L_2$. Since I is a normal ideal of $L_1 \times L_2$ we get that $I = \bigcup_{(x,y) \in I} (x, y)^\circ$. That implies $(x, y)^\circ \subseteq I$ and hence $x^\circ \times y^\circ \subseteq I$. We prove that $x^\circ \subseteq I_1$. Let $t \in x^\circ$. Then $(t, y) \in x^\circ \times y^\circ \subseteq I$. That implies $(t, y) \in I$. That implies $t \in I$, since $y \in L_2$). Therefore $x^\circ \subseteq I_1$ and hence I_1 is a normal ideal of L_1. Similarly, $I_2 = \{b \in L_2/(a, b) \in I,\ \text{for some}\ a \in L_1\}$

is a normal ideal of L_2. Now we prove that $I = I_1 \times I_2$. Clearly, we have that $I \subseteq I_1 \times I_2$. Let $(x, y) \in I_1 \times I_2$. Then $x \in I_1$ and $y \in I_2$. Then there exist elements $a \in L_1$ and $b \in L_2$ such that $(x, a) \in I, (b, y) \in I$. Since I is a normal ideal of $L_1 \times L_2$ we have $(x, a)^\circ \subseteq I$ and $(b, y)^\circ \subseteq I$. That implies $(x, a)^\circ \cup (b, y)^\circ \subseteq I$. That implies $((x, a) \vee (b, y))^\circ \subseteq I$. That implies $(x \vee b, a \vee y)^\circ \subseteq I$. That implies $(x, y)^\circ \subseteq I$ and hence $(x, y) \in I$. Therefore $I_1 \times I_2 \subseteq I$. Thus $I = I_1 \times I_2$. □

The following definition is taken from [5].

Definition 3.16. An ADL L is said to be a disjunctive ADL if for any $x, y \in L$, $x^* = y^*$ implies $x = y$.

In the following result, established a set of equivalent conditions for a ADL to become a disjunctive ADL.

Theorem 3.17. *The following conditions are equivalent in a pseudo-complemented ADL:*

(1) *L is a disjunctive ADL*
(2) *Every ideal is a normal ideal*
(3) *Every principal ideal is a normal ideal*
(4) *Every proper ideal contains no dense element*
(5) *Every prime is a normal ideal.*

Proof. (1) ⇒(2): Assume that L is a disjunctive ADL. Let I be any ideal of L. Now we prove that I is a normal ideal of L. Let $a, b \in L$, with $a^* = b^*$ and $a \in I$. Then $a = b$ and $a \in I$ and hence $b \in I$. Therefore I is a normal ideal of L.
(2) ⇒ (3): Clear.
(3) ⇒ (4): Assume that every principal ideal is a normal ideal. Let I be a proper ideal of L. We prove that $D(L) \cap I = \varphi$. Suppose $D(L) \cap I \neq \varphi$. Then choose an element $x \in D(L) \cap I$. That implies $x \in D(L)$ and $x \in I$. That implies $x^* = 0$ and $(x] \subseteq I$. By our assumption, we get that $(x] = (x]^\circ$. Since $x^{**} \wedge x^* = 0$, we get $x^{**} \in (x] = x^\circ$. Therefore $0^* \in (x] \subseteq I$ and hence $I \cap D(L) = \varphi$.
(4) ⇒ (5): Assume that every proper ideal contains no dense element. Let P be any prime ideal of L. By our assumption we have that $P \cap D(L) = \varphi$. Let $x \in P$. We prove that $x^\circ \subseteq P$. Let $a \in x^\circ$. Then $a \wedge x^* = 0$. Since $x^* \wedge x^{**} = 0$, we get that $(x \vee x^*)^* = 0$. That implies $x \vee x^* \in D(L)$. If $x^* \in P$. Then $x \vee x^* \in P \cap D(L)$ and hence $x^* \notin P$. Since $a \wedge x^* = 0 \in P$, we get that $a \in P$. Therefore $x^\circ \subseteq P$.
(5) ⇒ (1): Assume every prime is a normal ideal. Let $x \in L$. Then $x \wedge x^* = 0$ and $x \vee x^* \in D(L)$. Suppose $x \vee x^*$ is not a maximal element of L. Then there exists a prime filter P of L such

that $x \vee x^* \notin P$. That implies $x \notin P$ and $x^* \notin P$. That implies $x \in L \setminus P$ and $x^* \in L \setminus P$ and $L \setminus P$ is a proper ideal of L. That implies $x \vee x^* \in (L \setminus P) \cap D(L)$ and hence $x \vee x^*$ is a maximal element of L. Therefore L is a Boolean algebra. Hence L is a disjunctive ADL. $\square$

Theorem 3.18. *Let L be a pseudo-complemented ADL. For any normal ideal I and a filter F of L such that $I \cap F = \varphi$, there exists a prime normal ideal P such that $I \subseteq P$ and $P \cap F = \varphi$*

Proof. Let I be a normal ideal and F, a filter of a pseudo-complemented ADL L such that $I \cap F = \varphi$. Consider $\mathcal{F} = \{J/J$ is a normal ideal of $L, I \subseteq J$ and $I \cap F = \varphi\}$. Clearly, we have that $I \in \mathcal{F}$. Therefore $\mathcal{F} \neq \varphi$. Clearly, we have that $(\mathcal{F}, \subseteq)$ is a Poset. Let $\{J_\alpha\}_{\alpha \in \Delta}$ be a simply ordered family of normal ideals. Then clearly. $\bigcup\limits_{\alpha \in \Delta} J_\alpha$ is a normal ideal of L and $J_\alpha \subset \bigcup\limits_{\alpha \in \Delta} J_\alpha$. Therefore $\bigcup\limits_{\alpha \in \Delta} J_\alpha$ is an upper bound of $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element say M. Then M is a normal ideal of L, $I \subseteq M$ and $M \cap F = \varphi$. We prove that M is a prime ideal of L. Let $x, y \in M$ with $x \wedge y \in M$. Suppose $x \notin M$ and $y \notin M$. Then $M \subsetneqq M \cup (x]$ and $M \subsetneqq M \cup (y]$. That implies $M \subsetneqq (M \cup (x])^\circ$ and $M \subsetneqq (M \cup (y])^\circ$. That implies $(M \cup (x])^\circ \cap F \neq \varphi$ and $F \cap (M \cup (y])^\circ \neq \varphi$. Choose an element $a \in (M \cup (x])^\circ \cap F$ and $b \in F \cap (M \cup (y])^\circ$. That implies $a \wedge b \in (M \cup (x])^\circ \cap F \cap (M \cup (y])^\circ$ and $a \wedge b \in F$. Since $x \wedge y \in M$, we get that $a \wedge b \in M \cap F$. Therefore $x \in M$ or $y \in M$ and hence M is a prime normal of L. $\square$

Corollary 3.19. *Let I be a normal ideal of a pseudo-complemented ADL L and $x \notin I$. Then there exists a prime normal ideal P of L such that $I \subseteq P$ and $x \notin P$.*

Corollary 3.20. *For any normal ideal I of a pseudo-complemented ADL L, we have $I = \cap\{P/P$ is a normal ideal of L and $I \subseteq P\}$*

Corollary 3.21. *The intersection of all prime normal ideals of a pseudo-complemented ADL L is $\{0\}$.*

We discuss some topological properties of prime normal ideals. Fir this, we first need the following.

Let L be a pseudo-complemented ADL and $Spec_N(L)$, denotes the set of all prime normal ideals of L. For any $A \subseteq L$, let $K(A) = \{P \in Spec_N(L)/A \subsetneqq P\}$ and for any $x \in L, K(x) = K(\{x\})$. Then we have the following result.

Lemma 3.22. *Let L be a pseudo-complemented ADL with maximal elements. For any $x, y \in L$, the following holds:*

(1) $K(x) \cap K(y) = K(x \wedge y)$

(2) $K(x) \cup K(y) = K(x \vee y)$

(3) $K(x) = \varphi$ *iff* $x = 0$

(4) *If* m *is a maximal element of* L *then* $K(M) = Spec_N(L)$.

From the above lemma, it can be easily observed that the collections $\{K(x) \mid x \in L\}$ forms a base for topology on $Spec_N(L)$. which is called a hull kernel topology.

Theorem 3.23. *Let* L *be a pseudo-complemented ADL. Then we have the following:*

(1) *For any* $x \in L$, $K(x)$ *is compact in* $Spec_N(L)$

(2) *Let* C *be a compact open subset of* $Spec_N(L)$. *Then* $C = K(x)$ *for some* $x \in L$

(3) $Spec_N(L)$ *is a* T_0- *space.*

Proof. (1). Let $x \in L$ and $A \subseteq L$ with $K(x) \subseteq \bigcup\limits_{y \in A} K(y)$. Consider $I = (A]$ when I is a normal ideal of L. We prove that $x \in I^\circ$. Suppose $x \notin I^\circ$. Then there exists a prime normal ideal P of L such that $x \notin P, I^\circ \subseteq P$. That implies $p \in K(x) \subseteq \bigcup\limits_{y \in A} K(y)$. Then there exists an element $y \in A$ such that $p \in K(y)$. That implies $y \notin P$. That implies $y \notin I^\circ$. Since $I \subseteq I^\circ$, we get that $y \notin I$. That implies $x \in I^\circ$. Then there exists an element $a \in I$ such that $x \wedge a^* = 0$. That implies $x \in a^\circ$ and $a \in I$. Since $a \in I = (A]$, we have that $a = (\bigvee\limits_{i=1}^{n} a_i) \wedge a$, where $a_i \in A$ for $1 \leq i \leq n$. Now, $a^{**} = \big((\bigvee\limits_{i=1}^{n} a_i) \wedge a\big)^{**} = \big(a \wedge \bigvee\limits_{i=1}^{n} a_i\big)^{**} = a^{**} \wedge \big(\bigvee\limits_{i=1}^{n} a_i\big)^{**}$. That implies $a^{**} \leq \big(\bigvee\limits_{i=1}^{n} a_i\big)^{**}$ and hence $\big(\bigvee\limits_{i=1}^{n} a_i\big)^{*} \leq a^*$. That implies $x \wedge \big(\bigvee\limits_{i=1}^{n} a_i\big)^{*} \leq x \wedge a^* = 0$ and hence $x \wedge \big(\bigvee\limits_{i=1}^{n} a_i\big)^{*} = 0$. That implies $x \in \big(\bigvee\limits_{i=1}^{n} a_i\big)^{\circ}$. That implies $a^\circ \subseteq \big(\bigvee\limits_{i=1}^{n} a_i\big)^{\circ}$. We prove that $K(x) \subseteq K\big(\bigvee\limits_{i=1}^{n} a_i\big)$. Let $p \notin K\big(\bigvee\limits_{i=1}^{n} a_i\big)$. Then $\big(\bigvee\limits_{i=1}^{n} a_i\big) \in P$. That implies $\big(\bigvee\limits_{i=1}^{n} a_i\big) \wedge a \in P$. That implies $a \in P$. That implies $(a)^\circ \subseteq P^\circ = P$. Since $x \in a^\circ$, we get that $x \in P$. That implies $P \notin K(x)$. Therefore $K(x) \subseteq K\big(\bigvee\limits_{i=1}^{n} a_i\big) = \bigcup\limits_{i=1}^{n} K(a_i)$, which is a finite open cover. Therefore $K(x)$ is a compact open subset of $Spec_N(L)$.

(2). Let C be a compact open subset of $Spec_N(L)$. Then $C = K(A)$ for some $A \subseteq L$. That implies $C = \bigcup\limits_{a \in A} K(a)$. Since C is compact, there exist elements $a_1, a_2, \cdots, a_n \in A$ such that $C = \bigcup\limits_{i=1}^{n} K(a_i) = K\big(\bigvee\limits_{i=1}^{n} a_i\big)$. Therefore $C = K(x)$ for some $x \in L$.

(3). Let P and Q be two distinct prime normal ideals of L. Without loss of generality, we can assume that $P \nsubseteq Q$. Choose an element $x \in P$ such that $x \notin Q$. That implies $P \notin K(x)$ and $Q \in K(x)$. Therefore $Spec_N(L)$ is a T_0-space. □

Theorem 3.24. *Let* L *be a pseudo-complemented ADL. Then the following are equivalent:*

(1) *Every prime normal ideal is a minimal prime ideal*

(2) $Spec_N(L) = Minp(L)$, *where* $Minp(L)$ *is the set of all minimal prime ideals of* L

(3) *Each* $K(x)$ *is closed in* $Spec_N(L)$

(4) $Spec_N(L)$ *is Hausdroff*

(5) *For any* $x, y \in L$, *there exists* $z \in L$ *such that* $x \wedge z = 0$ *and* $K(y) \cap \big(Spec_N(L) \setminus K(x)\big) = K(x^* \wedge z)$.

Proof. (1) $\Rightarrow$ (2): Clear.
(2) $\Rightarrow$ (3): Assume (2). We prove that $Spec_N(L) \setminus K(x)$ is open. Let $P \in Spec_N(L) \setminus K(x)$. Then $x \in P$. By our assumption P is minimal, there exists an element $y \notin P$ such that $x \wedge y = 0$. That implies $P \in K(y)$ and hence $Spec_N(L) \setminus K(x) \subseteq K(y)$. Therefore $Spec_N(L) \setminus K(x))$ is open. Thus $K(x)$ is closed.
(3) $\Rightarrow$ (4): Let $P,\ Q \in Spec_N L$ with $P \neq Q$. Then there exists an element $x \in P$ such that $x \notin Q$. That implies $P \notin K(x)$ and $Q \in K(x)$. That implies $P \in Spec_N(L) \setminus K(x)$ and $Q \in K(x)$ and $(Spec_N \setminus K(x)) \cap K(x) = \emptyset$. By our assumption we have that $Spec_N(L) \setminus K(x)$ is open and hence $Spec_N(L)$ is Hausdorff.
(4) $\Rightarrow$ (5): Assume that $Spec_N(L)$ is a Haudorff space. We have that $K(a)$ is compact subset of $Spec_N(L)$, for all $a \in L$. Then $K(a)$ is clopen subset of $Spec_N(L)$, for all $a \in L$. Let $x, y \in L$ with $x < y$. Then $K(y) \cap (Spec_N(L) \setminus K(x))$ is a clopen subset of the compact space $K(y)$. Since $K(y)$ is open on $Spec_M(L)$, we have that $K(y) \cap (Spec_N(L) \setminus K(x))$ is a compact open subset of $Spec_N(L)$. Then by the theorem-3.23(2), there exists an element $z \in L$ such that $K(z) = K(y) \cap (Spec_N(L) \setminus K(x))$. That implies $K(x) \cap K(z) = \emptyset$ and hence $K(x \wedge z) = \emptyset$. That implies $x \wedge z = 0$. That implies $x^* \wedge z = z$. Hence $K(y) \cap (Spec_N(L) \setminus K(x)) = K(z) = K(x^* \wedge z)$.
(5) $\Rightarrow$ (1): Let $P \in Spec_N(L)$. Now we prove that P is minimal prime ideal of L. Let $x \in P$. Since P is a proper ideal of L, there exists an element $y \in L$ such that $y \notin P$. By our assumption there exits an element $z \in L$ such that $x \wedge z = 0$ and $K(y) \cap \big(Spec_N(L) \setminus K(x)\big) = K(x^* \wedge z)$. Clearly, we have that $P \in K(y) \cap \big(Spec_N(L) \setminus K(x)\big) = K(x^* \wedge z)$. We prove that $z \notin P$. Suppose that $z \in P$. Then $x^* \wedge z \in P$ and hence $P \notin K(x^* \wedge z)$, which is a contradiction to $P \in K(x^* \wedge z)$. Therefore $z \notin P$. Hence we have that for any $x \in P$, there exists an element $z \notin P$ such that $x \wedge z = 0$. Thus P is a minimal prime ideal of L. □

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