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COMMUTING CONJUGACY CLASS GRAPHS OF FINITE GROUPS

MOHAMMAD ALI SALAHSHOUR AND ALI REZA ASHRAFI*

Abstract. Suppose that G is a finite non-abelian group. Define the graph $\Gamma(G)$ with the non-central conjugacy classes of G as vertex set and two distinct vertices A and B are adjacent if and only if there are $x \in A$ and $y \in B$ such that $xy = yx$. The graph $\Gamma(G)$ is called the commuting conjugacy class graph of G and introduced by Mohammadian et al. in [A. Mohammadian, A. Erfanian, M. Farrokhi D. G. and B. Wilkens, Triangle-free commuting conjugacy class graphs, *J. Group Theory* **19** (3) (2016) 1049–1061]. In this paper, the graph structure of the commuting conjugacy class graph of nilpotent groups of order n are obtained in which n is not divisible by p^5, for every prime factor p of n.

1. Introduction

Throughout this paper all groups and graphs are assumed to be finite. Our graph theory notations are taken from [6] and for group theory notions and notations we refer to [11]. Suppose that Γ is a graph with vertex set V and edge set E and ρ is an equivalence relation on V. The partition of V constructed from ρ is denoted by Π. The **quotient graph** $\frac{\Gamma}{\rho}$ is a

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*Corresponding author

graph with vertex set Π such that two parts A and B are adjacent if and only if there exists a vertex $x \in A$ and another vertex $y \in B$ such that $xy \in E$.

Suppose that Γ_1 and Γ_2 are two graphs. The **strong product** $\Gamma_1 \boxtimes \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$. Two distinct vertices (u_1, u_2) and (v_1, v_2) of this graph are adjacent if and only if ($u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2)$) or ($u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1)$) or (u_1 is adjacent to v_1 and u_2 is adjacent to v_2). This product was first introduced by Sabidussi in 1960 [12]. If the vertex sets of Γ_1 and Γ_2 are disjoint then the join $\Gamma_1 + \Gamma_2$ is a graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and two vertices $x, y \in V(\Gamma_1 + \Gamma_2)$ are adjacent if and only if ($xy \in E(\Gamma_1) \cup E(\Gamma_2)$) or ($x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$) or ($y \in V(\Gamma_1)$ and $x \in V(\Gamma_2)$).

There are several ways to associate a graph to an algebraic structure like groups. The **commuting graph** $\Delta(G)$ of a group G is one of the most important graphs defined on a group [9] with non-trivial elements of G as vertex set and two no-trivial elements x and y in G are adjacent if and only if $xy = yx$. Rapinchuk et al. [10] applied this graph to prove that finite quotients of the multiplicative group of a finite dimensional division algebra are solvable.

Mohammadian et al. [8] introduced the **commuting conjugacy class graph** $\Gamma(G)$ of a non-abelian group G as a simple graph with non-central conjugacy classes of G as its vertex set and two distinct vertices A and B are adjacent if and only if there are $x \in A$ and $y \in B$ such that $xy = yx$ [8]. They obtained some interesting properties of this graph among them a classification of finite groups such that $\Gamma(G)$ are triangle-free is given. It is merit to note that the commuting conjugacy class graph was initially introduced by Herzog et al. [7] in which they considered all non-trivial conjugacy classes of G as its vertex set. It is easy to see that the commuting conjugacy class graph $\Gamma(G)$ is a quotient of the induced graph $\Delta(G)$ on non-central elements of G over the set of all non-central conjugacy classes of the group G.

2. Preliminaries

The aim of this section is to present some results which are crucial in proving the main results of this paper. We start by stating the classification theorem of $p-$group of order p^4 given by Burnside [4].

Theorem 2.1. *If p is an odd prime, then there are 15 groups of order p^4 up to isomorphisms. Five of those are abelian and the non-abelian groups can be found in the list below.*

(1) $\langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1,\ [a, b] = [a, c] = [a, d] = [b, c] = [b, d] = 1, dcd^{-1} = ac\rangle$,

(2) $\langle a, b \mid a^{p^2} = b^{p^2} = 1\ ,\ bab^{-1} = a^{p+1}\rangle$,

(3) $\langle a, b \mid a^{p^3} = b^p = 1\ ,\ bab^{-1} = a^{p^2+1}\rangle$,

(4) $\langle a, b, c \mid a^{p^2} = b^p = c^p = 1\ ,\ ab = ba,\ ac = ca,\ cbc^{-1} = a^pb\rangle$,

(5) $\langle a, b, c \mid a^{p^2} = b^p = c^p = 1\ ,\ ab = ba,\ bc = cb,\ cac^{-1} = a^{p+1}\rangle$,

(6) $\langle a, b, c \mid a^{p^2} = b^p = c^p = 1\ ,\ ab = ba,\ bc = cb,\ cac^{-1} = ab\rangle$,

(7) $\langle a, b, c \mid a^{p^2} = b^p = c^p = 1\ ,\ bab^{-1} = a^{p+1},\ cac^{-1} = ab,\ bc = cb\rangle$,

$$(8)\quad \begin{array}{ll} \langle a,b,c \mid a^{p^2}=b^p=1\ ,\ [b,c]=1, c^p=a^p, b^{-1}ab=a^{p+1}, c^{-1}ac=ab^{-1}\rangle & p=3 \\ \langle a,b,c \mid a^{p^2}=b^p=c^p=1\ ,\ bab^{-1}=a^{p+1},\ cac^{-1}=a^{p+1}b,\ cbc^{-1}=a^pb\rangle & p>3 \end{array},$$

$$(9)\quad \begin{array}{ll} \langle a,b,c \mid a^{p^2}=b^p=1\ ,\ [b,c]=1,\ c^p=a^{-p},\ b^{-1}ab=a^{p+1},\ c^{-1}ac=ab^{-1}\ \rangle & p=3 \\ \langle a,b,c \mid a^{p^2}=b^p=c^p=1\ ,\ bab^{-1}=a^{p+1},\ cac^{-1}=a^{dp+1}b,\ cbc^{-1}=a^{dp}b,\ d\overset{p}{\not\equiv}0,1\ \rangle & p>3 \end{array},$$

$$(10)\quad \begin{array}{ll} \langle a,b,c \mid a^{p^2}=b^p=c^p=1\ ,\ ab=ba,\ c^{-1}ac=ab,\ c^{-1}bc=a^{-p}b\rangle & p=3 \\ \langle a,b,c,d \mid a^p=b^p=c^p=d^p=[a,b]=[a,c]=[a,d]=[b,c]=[d,b]a^{-1}=[d,c]b^{-1}=1\rangle & p>3 \end{array}.$$

Define $Cent(G) = \{C_G(x) \mid x \in G\}$. Each element of $Cent(G)$ is said to be an element centralizer of G. A group G is called a $CA-$group if all proper element centralizers of G are abelian. The following theorem is crucial throughout this paper:

Theorem 2.2. *Let G be a finite group. Then the following are hold:*

1. *Let $[G : Z(G)] = pqr$, where p, q and r are primes not necessarily distinct. Then G is a $CA-$group. (See Baishya* [3, Lemma 2.1]*)*
2. *The group G is a $CA-$group if and only if whenever $x, y \in G \setminus Z(G)$ satisfy $xy = yx$, then $C_G(x) = C_G(y)$. (See Dolfi et al.* [5, Proposition 3.2(a)]*)*
3. *Let G be a $CA-$group. Then, for all $a, b \in G \setminus Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$. (See Abdollahi et al.* [1, Remark 2.1(4)]*)*

In [3, Proposition 2.2], Baishya proved that if p is the smallest prime divisor of $|G|$ and $|G : Z(G)| = p^3$, then $|Cent(G)| = p^2 + p + 2$ or $p^2 + 2$. In the following lemma we prove that the condition of "p is the smallest prime divisor of $|G|$" can be substituted by "p is a prime divisor of $|G|$". It is merit to mention here that our proof is similar to the proof of the mentioned result of Baishya.

Lemma 2.3. *Suppose that G is a finite group, $Z = Z(G)$ and $[G : Z] = p^3$, where p is prime. $|Cent(G)| = p^2 + 2$ or $p^2 + p + 2$.*

Proof. Since $[G : Z] = p^3$, by putting $p = q = r$ in Theorem 2.2(1), G is a $CA-$group. Suppose $x \in G \setminus Z$. Since $Z \leq C_G(x)$, $[G : C_G(x)] \mid [G : Z] = p^3$. Therefore, $[G : C_G(x)] = 1, p, p^2$ or p^3. Since $x \notin Z$ and $C_G(x) \neq Z$, the cases that $[G : C_G(x)] = 1$, p^3 are impossible. This shows that $[G : C_G(x)] = p$ or p^2. Our main proof will consider two cases as follows:

1. *For any $x \in G \setminus Z$, $[G : C_G(x)] = p^2$.* Fix $y \in G \setminus Z$. Suppose $\{C_G(t) \mid t \in G \setminus C_G(y)\} = \{C_G(x_1), \ldots, C_G(x_k)\}$. For simplicity of our argument, we assume that $A_i = C_G(x_i) \setminus Z, 1 \leq i \leq k$. It is clear $|A_1| = |A_2| = \cdots = |A_k| = (p-1)|Z|$ and by Theorem 2.2(3), A_i's are distinct and $C_G(y) \cap A_i = \emptyset$, $1 \leq i \leq k$. We now prove that $G \setminus C_G(y) = \bigcup_{i=1}^{k} A_i$. It is easy to see that $A_i \subseteq G \setminus C_G(y)$, $1 \leq i \leq k$, then $\bigcup_{i=1}^{k} A_i \subseteq G \setminus C_G(y)$. Next we assume that $t \in G \setminus C_G(y)$ is arbitrary. Hence, there exists an integer s such that $1 \leq s \leq k$ and $t \in C_G(x_s) \setminus Z$ and so $t \in \bigcup_{i=1}^{k} A_i$.

Therefore, $G \setminus C_G(y) = \bigcup_{i=1}^{k} A_i$ which implies that $|G \setminus C_G(y)| = \sum_{i=1}^{k} |A_i| = k|A_i|$ $= k|C_G(x_i) \setminus Z|$. Since for every $x \in G \setminus Z$ we have $[G : C_G(x)] = p^2$, $[C_G(x) : Z] = p$. Thus $k = \frac{|G \setminus C_G(y)|}{|C_G(x) \setminus Z|} = \frac{|C_G(y)|(p^2-1)}{|Z|(p-1)} = p^2 + p$. This implies that $|Cent(G)| = p^2 + p + 2$, as desired.

(2) *There exists $x \in G \setminus Z$ such that $[G : C_G(x)] = p$.* We first prove that $C_G(x)$ is unique. Suppose that $x \neq y \in G \setminus Z$ such that $[G : C_G(y)] = p$ and $C_G(x) \neq C_G(y)$. Since $Z \subsetneqq C_G(y) \subsetneqq G$ and $[G : Z] = p^3$, $[C_G(y) : Z] = p^2$. On the other hand, $C_G(x)C_G(y) \subseteq G$ and so $|C_G(x)C_G(y)| \leq |G|$. Thus, $\frac{|C_G(x)||C_G(y)|}{|C_G(x) \cap C_G(y)|} \leq |G|$. By Theorem 2.2(3), $C_G(x) \cap C_G(y) = Z$. Hence, $\frac{|C_G(x)||C_G(y)|}{|Z|} \leq |G|$ which implies that $p^2 \leq p$, that is impossible. Therefore, G has exactly one element centralizer $C_G(x)$ of index p and other proper element centralizers are of index p^2. It is now easy to check that $G = [G \setminus C_G(x)] \cup C_G(x)$ and hence for every $z \in G \setminus C_G(x)$, $[G : C_G(z)] = p^2$. Apply the same argument as Part (1) to prove that $|Cent(G)| = \frac{|G \setminus C_G(x)|}{|C_G(z) \setminus Z|} + 2 = \frac{|C_G(x)|(p-1)}{|Z|(p-1)}$ $+ 2 = p^2 + 2$.

This completes the proof. □

3. Main Results

Suppose that $\Gamma_1, \ldots, \Gamma_s$ are graphs with mutually disjoint vertex sets. The **graph union** $\Gamma_1 \cup \cdots \cup \Gamma_s$ is a graph with vertex set $V(\Gamma_1) \cup \cdots \cup V(\Gamma_s)$ and edge set $E(\Gamma_1) \cup \cdots \cup E(\Gamma_s)$. If all graphs Γ_i, $1 \leq i \leq s$, are isomorphic then we will write $s\Gamma_1$ as $\Gamma_1 \cup \cdots \cup \Gamma_s$. The aim of this section is to obtain the structure of the commuting conjugacy class graph of a nilpotent group.

Theorem 3.1. *Suppose that G is a non-abelian finite group with center $Z = Z(G)$ and $\frac{G}{Z} \cong Z_p \times Z_p$, where p is prime. Then $p \mid |Z|$ and $\Gamma(G) = (p+1)K_n$, where $n = \frac{(p-1)|Z|}{p}$.*

Proof. Suppose $C_G(x)$ is a proper centralizer of G. Since $|\frac{G}{Z(G)}| = p^2$, $C_G(x) = \langle Z, x \rangle$ and so it is abelian. Therefore, the group G is a $CA-$group. We now apply Theorem 2.2(3) to prove that for any $x, y \in G \setminus Z$,

$$C_G(x) = C_G(y) \quad \text{and} \quad C_G(x) \cap C_G(y) = Z. \tag{1}$$

Since $\frac{G}{Z}$ is abelian, $C_G(x)$ is normal in G and so for any $y \in C_G(x)$, $y^G \subseteq C_G(x)$. Furthermore, G is a $CA-$group and so by Theorem 2.2(2) if $y \in C_G(x) \setminus Z$ and $y \notin x^G$ then $C_G(y) = C_G(x)$ which implies that $|y^G| = |x^G|$. Obviously, $|G| = p^2|Z|$ and hence for any $x \in G \setminus Z$, $Z \subsetneqq C_G(x) \subsetneqq G$. Thus $|C_G(x)| = p|Z|$ and $|x^G| = p$. This proves that the number of non-central conjugacy classes of G contained in $C_G(x)$ is $n = \frac{(p-1)|Z|}{p}$. Note that if

$C_G(x) \neq C_G(y)$ then by the Equation 1, there are no conjugacy classes a^G and b^G contained in $C_G(x)$ and $C_G(y)$, respectively, such that $a^G b^G \in E(\Gamma(G))$.

Next we assume that G has exactly s distinct centralizers of order $p|Z|$. Then by Equation 1, $s = p+1$. Therefore, $\Gamma(G) = (p+1)K_n$, where $n = \frac{(p-1)|Z|}{p}$. □

Corollary 3.2. *If G is a non-abelian $p-$group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then $\Gamma(G) = (p+1)K_{p^{n-3}(p-1)}$.*

Theorem 3.3. *Suppose that G is a non-abelian group with center Z such that $|\frac{G}{Z}| = p^3$, for a prime p. Then one of the following is satisfied:*

(1) *If $\frac{G}{Z}$ is abelian then $\Gamma(G) = K_m \cup p^2 K_n$ or $(p^2+p+1)K_n$, where $m = \frac{(p^2-1)|Z|}{p}$ and $n = \frac{(p-1)|Z|}{p^2}$.*

(2) *If $\frac{G}{Z}$ is non-abelian then $\Gamma(G) = K_m \cup kpK_{n_1} \cup (p-k)K_{n_2}$, $(kp+1)K_{n_1} \cup (p+1-k)K_{n_2}$, $K_m \cup pK_{n_2}$, $(p^2+p+1)K_{n_1}$ or $K_{n_1} \cup (p+1)K_{n_2}$, where $m = \frac{(p^2-1)|Z|}{p}$, $n_1 = \frac{(p-1)|Z|}{p^2}$, $n_2 = \frac{(p-1)|Z|}{p}$, $1 \leq k \leq p$.*

Proof. Since $|\frac{G}{Z(G)}| = p^3$, by Theorem 2.2(1) the group G is a $CA-$group and by Theorem 2.2(3), for any $x, y \in G \setminus Z$,

$$C_G(x) = C_G(y) \quad \text{and} \quad C_G(x) \cap C_G(y) = Z. \tag{2}$$

Now, we have the following two different cases:

(1) Suppose that $\frac{G}{Z}$ is abelian. Then for every $x \in G \setminus Z$, $C_G(x)$ is normal in G and so for any $y \in C_G(x)$, $y^G \subseteq C_G(x)$. Moreover, G is a $CA-$group and so by Theorem 2.2(2) if $y \in C_G(x) \setminus Z$ and $y \notin x^G$ then $C_G(y) = C_G(x)$ and $|y^G| = |x^G|$. It is clear that $|G| = p^3|Z|$ and so for every $x \in G \setminus Z$, $Z \subsetneqq C_G(x) \subsetneqq G$. Then $|C_G(x)| = p|Z|$ or $p^2|Z|$. If $|C_G(x)| = p^2|Z|$, then $|x^G| = p$. So, the number of non-central conjugacy classes of G contained in $C_G(x)$ is $m = \frac{(p^2-1)|Z|}{p}$. If $|C_G(x)| = p|Z|$, then $|x^G| = p^2$. Hence the number of non-central conjugacy classes of G contained in $C_G(x)$ is $n = \frac{(p-1)|Z|}{p^2}$. Note that if $C_G(x) \neq C_G(y)$ then by Equation 2, there are no conjugacy classes a^G and b^G contained in $C_G(x)$ and $C_G(y)$, respectively, such that $a^G b^G \in E(\Gamma(G))$.

Next we assume that G has exactly s_1 distinct element centralizers of order $p^2|Z|$ and s_2 distinct element centralizers of order $p|Z|$. Therefore, by Equations 2 we have $(p^2+p+1)(p-1)|Z| = (s_1(p+1)+s_2)(p-1)|Z|$. Hence, $s_1(p+1)+s_2 = p^2+p+1$. On the other hand, by the proof of Lemma 2.3, $s_1 = 0$ or 1. So $s_2 = p^2+p+1$ or p^2 and the proof of this case is completed.

(2) Suppose that $\frac{G}{Z}$ is non-abelian. Since $|G| = p^3|Z|$, it is easy to see that for all $x \in G \setminus Z$, $Z \subsetneqq C_G(x) \subsetneqq G$. This shows that $|C_G(x)| = p|Z|$ or $p^2|Z|$.

(a) $|C_G(x)| = p^2|Z|$. It is clear that $Z \lhd G$ and so $\frac{C_G(x)}{Z} \leq \frac{G}{Z}$. Since $\left|\frac{G}{Z}\right| = p^3$ and $\left|\frac{C_G(x)}{Z}\right| = p^2$, $C_G(x) \lhd G$. Hence $y^G \subseteq C_G(x)$, for every element $y \in C_G(x) \setminus Z$. Also G is a $CA-$group, so by Theorem 2.2(2) if $y \in C_G(x) \setminus Z$ and $y \notin x^G$ then $C_G(y) = C_G(x)$ and $|y^G| = |x^G| = p$. Thus the number of distinct non-central conjugacy classes of G contained in $C_G(x)$ is $m = \frac{(p^2-1)|Z|}{p}$.

(b) $|C_G(x)| = p|Z|$. If $C_G(x)$ is a normal subgroup of G, then $y^G \subseteq C_G(x)$, for every element $y \in C_G(x) \setminus Z$. Since G is a $CA-$group, by Theorem 2.2(2) if $y \in C_G(x) \setminus Z$ and $y \notin x^G$ then $C_G(y) = C_G(x)$ and $|y^G| = |x^G| = p^2$. Thus, the number of distinct non-central conjugacy classes of G contained in $C_G(x)$ is $n_1 = \frac{(p-1)|Z|}{p^2}$. Next, we assume that $C_G(x)$ is not normal in G. Since $\frac{G}{Z}$ is a $p-$group of order p^3, $\frac{C_G(x)}{Z} \lneqq N_{\frac{G}{Z}}(\frac{C_G(x)}{Z}) \lneqq \frac{G}{Z}$ and $|N_{\frac{G}{Z}}(\frac{C_G(x)}{Z})| = p^2$. On the other hand, $Z \lneqq C_G(x) \lneqq G$. So, $N_{\frac{G}{Z}}(\frac{C_G(x)}{Z}) = \frac{N_G(C_G(x))}{Z}$ and $[G : N_G(C_G(x))]$ $= p$. Thus, the number of conjugates of $C_G(x)$ in G is p. But G is a $CA-$group and by Theorem 2.2(2) if $y \in C_G(x) \setminus Z$ and $y \notin x^G$ then $C_G(y) = C_G(x)$, $|y^G|$ $= |x^G| = p^2$ and $|C_G(x) \cap y^G| = |C_G(x) \cap x^G| = p$. Thus, the number of distinct non-central conjugacy classes of G is $n_2 = \frac{(p-1)|Z|}{p}$.

Note that if $C_G(x) \neq C_G(y)$ then by Equation 2 there are no conjugacy classes a^G and b^G contained in $C_G(x)$ and $C_G(y)$, respectively, such that $a^G b^G \in E(\Gamma(G))$. Assume that s_1 denotes the number of distinct element centralizers of order $p^2|Z|$, s_2 is the number of distinct normal element centralizers of order $p|Z|$ and s_3 is the number of distinct element centralizers of order $p|Z|$ containing p conjugates. By Equations 2, $(p^2 + p + 1)(p - 1)|Z| = (s_1(p + 1) + s_2 + s_3 p)(p - 1)|Z|$ and hence $s_1(p+1) + s_2 + s_3 p = p^2 + p + 1$. On the other hand, the proof of Lemma 2.3 shows that $s_1 = 0$ or 1. Suppose $s_1 = 1$. Then, $s_2 + s_3 p = p^2$ and so $p \mid s_2$. Therefore, $s_2 = kp$ and $s_3 = p - k$ in which $0 \leq k \leq p$. If $k = 0$, then $\Gamma(G) = K_m \cup pK_{n_2}$. But $\Gamma(G) = K_m \cup kpK_{n_1} \cup (p - k)K_{n_2}$, $1 \leq k \leq p$. Next we assume that $s_1 = 0$. Then $s_2+s_3 p = p^2+p+1$ and so $p \mid s_2-1$. Therefore, $s_2 = k'p+1$ and $s_3 = p+1-k'$ in which $0 \leq k' \leq p+1$. If $k' = 0$ or $p+1$, then $\Gamma(G) = K_{n_1} \cup (p+1)K_{n_2}$ or $(p^2+p+1)K_{n_1}$. On the other hand, $\Gamma(G) = (k'p+1)K_{n_1} \cup (p+1-k')K_{n_2}$, where $1 \leq k' \leq p$, $m = \frac{(p^2-1)|Z|}{p}$, $n_1 = \frac{(p-1)|Z|}{p^2}$ and $n_2 = \frac{(p-1)|Z|}{p}$.

Hence the result. □

Corollary 3.4. *Let G be a non abelian $p-$group of order p^n and $|Z(G)| = p^{n-3}$, where p is prime and $n \geq 4$. Then one of the following are satisfied:*

(1) *If $\frac{G}{Z}$ is abelian then $\Gamma(G) = K_{p^{n-4}(p^2-1)} \cup p^2 K_{p^{n-5}(p-1)}$ or $(p^2 + p + 1)K_{p^{n-5}(p-1)}$.*

(2) *If* $\frac{G}{Z}$ *is non-abelian then* $\Gamma(G) = K_{p^{n-4}(p^2-1)} \cup pK_{p^{n-4}(p-1)}$, $(p^2+p+1)K_{p^{n-5}(p-1)}$, $K_{p^{n-5}(p-1)} \cup (p+1)K_{p^{n-4}(p-1)}$, $K_{p^{n-4}(p^2-1)} \cup kpK_{p^{n-5}(p-1)} \cup (p-k)K_{p^{n-4}(p-1)}$ *or* $(kp+1)K_{p^{n-5}(p-1)} \cup (p+1-k)K_{p^{n-4}(p-1)}$*, where* $1 \leq k \leq p$.

Proof. Apply Corollary 3.3. □

Corollary 3.5. *Let* G *be a non-abelian* $p-$*group of order* p^4*. Then the commuting conjugacy class graph of* G *has the form* $(p+1)K_{p(p-1)}$ *or* $K_{(p^2-1)} \cup pK_{(p-1)}$.

Proof. Since $|G| = p^4$, p is prime, $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$, then by Corollary 3.2 $\Gamma(G)$ $= (p+1)K_{p(p-1)}$. In other case, $|Z(G)| = p$ and by Corollary 3.4, $\Gamma(G) = K_{(p^2-1)} \cup pK_{(p-1)}$, as desired. □

If G is isomorphic to one of the first six groups in Theorem 2.1, then the commuting conjugacy class graph of G will be isomorphic to $(p+1)K_{p(p-1)}$. In other cases, the commuting conjugacy class graph of G is isomorphic to $K_{(p^2-1)} \cup pK_{(p-1)}$.

Suppose that G and H are two non-abelian finite groups. Define two graphs $\Delta_1(G,H)$ and $\Delta_2(G,H)$ as follows:

$$\begin{aligned}
V(\Delta_1(G,H)) &= \left\{(x,y)^{G\times H} \ \mid \ x^G \in V(\Gamma(G)) \text{ and } y \in Z(H)\right\},\\
E(\Delta_1(G,H)) &= \left\{(x,y)^{G\times H}(a,b)^{G\times H} \ \mid \ x = a \text{ or } x^G a^G \in E(\Gamma(G))\right\},\\
V(\Delta_2(G,H)) &= \left\{(x,y)^{G\times H} \ \mid \ y^H \in V(\Gamma(H)) \text{ and } x \in Z(G)\right\},\\
E(\Delta_2(G,H)) &= \left\{(x,y)^{G\times H}(a,b)^{G\times H} \ \mid \ y = b \text{ or } y^H b^H \in E(\Gamma(H))\right\}.
\end{aligned}$$

Set $\Delta_1 = \Delta_1(G,H)$, $\Delta_2 = \Delta_2(G,H)$ and $\Delta_3 = \Delta_3(G,H) = \Gamma(G) \boxtimes \Gamma(H)$. We also define two sets A and B as follows:

$$\begin{aligned}
A &= \left\{(x,y)^{G\times H}(a,b)^{G\times H} \mid (x,y)^{G\times H} \in V(\Delta_3), (a,b)^{G\times H} \in V(\Delta_1), \left(x = a \ \vee \ x^G a^G \in E(\Gamma(G))\right)\right\}\\
B &= \left\{(x,y)^{G\times H}(a,b)^{G\times H} \mid (x,y)^{G\times H} \in V(\Delta_3), (a,b)^{G\times H} \in V(\Delta_2), \left(y = b \ \vee \ y^H b^H \in E(\Gamma(H))\right)\right\}.
\end{aligned}$$

Define our third graph operation $\uplus$ as follows:

$$\begin{aligned}
V(\Delta_3 \uplus (\Delta_1 + \Delta_2)) &= V(\Delta_3) \cup V(\Delta_1 + \Delta_2),\\
E(\Delta_3 \uplus (\Delta_1 + \Delta_2)) &= E(\Delta_3) \cup E(\Delta_1 + \Delta_2) \cup A \cup B.
\end{aligned}$$

We now obtain the graph structure of the Cartesian product of two groups.

Theorem 3.6. *Let* G *and* H *be two non-abelian group. Then the commuting conjugacy class graph of* $G \times H$ *can be computed as* $\Gamma(G \times H) = \Delta_3(G,H) \uplus [\Delta_1(G,H) + \Delta_2(G,H)]$.

Proof. It is easy to see that an element (a, b) is central in $G \times H$ if and only if a is central in G and b is central in H. We partition the non-central conjugacy classes of $G \times H$ as follows:

(1) *The conjugacy classes $(x, y)^{G\times H}$ such that $y \in Z(H)$ and $x \notin Z(G)$.* Choose two non-central conjugacy classes $(x, y)^{G\times H}$ and $(a, b)^{G\times H}$ of $G \times H$. These vertices are adjacent in $\Gamma(G \times H)$ if and only if $x = a$ or $x^G a^G \in E(\Gamma(G))$. This means that $(x, y)^{G\times H}(a, b)^{G\times H} \in E(\Delta_1(G, H))$.

(2) *The conjugacy classes $(x, y)^{G\times H}$ such that $x \in Z(G)$ and $y \notin Z(H)$.* A similar argument as part (1) shows that $(x, y)^{G\times H}(a, b)^{G\times H} \in E(\Delta_2(G, H))$.

(3) *The conjugacy classes $(x, y)^{G\times H}$ such that $x \notin Z(G)$ and $y \notin Z(H)$.* Suppose that $(x, y)^{G\times H}$ and $(a, b)^{G\times H}$ are two adjacent vertices of $\Gamma(G \times H)$. Then ($x = a$ and $y^H b^H \in E(\Gamma(H))$) or ($y = b$ and $x^G a^G \in E(\Gamma(G))$) or ($x^G a^G \in E(\Gamma(G))$ and $y^H b^H \in E(\Gamma(H))$). Therefore, $(x, y)^{G\times H}(a, b)^{G\times H} \in \Delta_3(G, H)$.

Suppose $(x, y)^{G\times H} \in V(\Delta_1)$ and $(a, b)^{G\times H} \in V(\Delta_2)$. Then by our definition, $y \in Z(H)$ and $a \in Z(G)$. Hence each vertex of Δ_1 is adjacent with each vertex of Δ_2. We now assume that $(x, y)^{G\times H} \in V(\Delta_3)$ and $(a, b)^{G\times H} \in V(\Delta_1 + \Delta_2)$. Without loss of generality, we can assume that $(a, b)^{G\times H} \in V(\Delta_1)$. It is clear that $b \in Z(H)$ and so $by = yb$. Since $a, x \notin Z(G)$, $a = x$ or $x^G a^G \in E(\Gamma(G))$. This proves that $(x, y)^{G\times H}(a, b)^{G\times H} \in A$ which completes the proof. □

Lemma 3.7. *Let H be arbitrary and let G be a non-abelian group. Then $\Gamma(G)$ and $\Delta_1(G, H)$ have the same number of connected components.*

Proof. Choose two vertices x^G and a^G from different components of $\Gamma(G)$ and $b, y \in Z(H)$. Then $(x, y)^{G\times H}$ and $(a, b)^{G\times H}$ are different vertices of $\Delta_1(G, H)$. By definition of Δ_1, $(x, y)^{G\times H}(a, b)^{G\times H} \in E(\Delta_1(G, H))$ if and only if $x^G a^G \in E(\Gamma(G))$, proving the lemma. □

Lemma 3.8. *If H is an arbitrary finite group and G is a finite group such that $\Gamma(G) = mK_n$. Then $\Delta_1(G, H) = mK_{n|Z(H)|}$.*

Proof. Since $\Gamma(G) = mK_n$, G has exactly mn non-central conjugacy classes, and since $Z(H)$ is abelian, $G \times Z(H)$ has exactly $mn|Z(H)|$ conjugacy classes. Note that by definition $|V(\Delta_1)| = mn|Z(H)|$. Choose $(x, y)^{G\times H}$ and $(a, b)^{G\times H}$ from a connected component of $\Delta_1(G, H)$. If $x = a$ then $(x, y)^{G\times H}(a, b)^{G\times H} \in E(\Delta_1(G, H))$. Suppose $x \neq a$. Since x^G and a^G are in a connected component of $\Gamma(G)$ and connected components of $\Gamma(G)$ are isomorphic to K_n, $x^G a^G \in E(\Gamma(G))$. Therefore, $(x, y)^{G\times H}(a, b)^{G\times H} \in E(\Delta_1(G, H))$. By Lemma 3.7, the number of connected components of $\Gamma(G)$ and $\Delta_1(G, H)$ are m and since $|V(\Delta_1(G, H))| = mn|Z(H)|$, $\Delta_1(G, H) = mK_{n|Z(H)|}$. □

Corollary 3.9. *If G and H are two groups such that $\Gamma(G) = \bigcup_{i\in I} m_iK_{n_i}$ and $\Gamma(H) = \bigcup_{j\in J} m'_jK_{n'_j}$, then $\Delta_1(G,H) = \bigcup_{i\in I} m_iK_{n_i|Z(H)|}$ and $\Delta_2(G,H) = \bigcup_{j\in J} m'_jK_{n'_j|Z(G)|}$.*

Lemma 3.10. *If G and H are two groups such that $\Gamma(G) = m_1K_{n_1}$ and $\Gamma(H) = m_2K_{n_2}$, then $\Delta_3(G,H) = m_1m_2K_{n_1n_2}$.*

Proof. We know that $\Delta_3(G,H) = \Gamma(G)\boxtimes\Gamma(H)$. Define A to be a subgraph of Δ_3 containing all vertices of the form $(x^G, y^H) \in V(\Delta_3(G,H))$ such that x^G's will induce a connected component in $\Gamma(G)$ and y^H's will induce another component in $\Gamma(H)$. It is easy to see that $|V(A)| = n_1n_2$. We prove that A is a connected subgraph of $\Delta_3(G,H)$. To do this, we assume that (x^G, y^H) and (a^G, b^H) are two arbitrary vertices in A. Then we have one of the following cases:

(1) $x^G = a^G$ *and* $y^H \neq b^H$.
(2) $x^G \neq a^G$ *and* $y^H = b^H$.
(3) $x^G \neq a^G$ *and* $y^H \neq b^H$.

Since connected components of $\Gamma(G)$ and $\Gamma(H)$ are K_{n_1} and K_{n_2}, respectively, by definition of Δ_3, $(x^G, y^H)(a^G, b^H) \in E(\Delta_3(G,H))$. We now show that the subgraph A is a connected component of Δ_3. To see this, we assume that (x^G, y^H) is an arbitrary vertex of A and (c^G, d^H) is a vertex in $\Delta_3\backslash A$. It is clear that $\{x^G, c^G\}$ is not contained in a connected component of $\Gamma(G)$ or $\{y^H, d^H\}$ is not contained in a connected component of $\Gamma(H)$. Therefore, $x^Gc^G \notin E(\Gamma(G))$ or $y^Hd^H \notin E(\Gamma(H))$ and so by definition of Δ_3, $(x^G, y^H)(c^G, d^H) \notin E(\Delta_3)$. This proves that A is a connected component of Δ_3. Thus, $A = K_{n_1n_2}$. Since $|V(\Delta_3)| = m_1m_2n_1n_2$, $\Delta_3(G,H) = m_1m_2K_{n_1n_2}$. This completes the proof. □

Corollary 3.11. *If G and H are two groups such that $\Gamma(G) = \bigcup_{i\in I} m_iK_{n_i}$ and $\Gamma(H) = \bigcup_{j\in J} m'_jK_{n'_j}$, then $\Delta_3(G,H) = \bigcup_{\substack{i\in I\\ j\in J}} m_im'_jK_{n_in'_j}$.*

Corollary 3.12. *If G and H are two $p-$groups such that $\Gamma(G) = \bigcup_{i\in I} m_iK_{n_i}$ and $\Gamma(H) = \bigcup_{j\in J} m'_jK_{n'_j}$, then $\Gamma(G\times H) = \left(\bigcup_{\substack{i\in I\\ j\in J}} m_im'_jK_{n_in'_j}\right) \uplus \left(\bigcup_{i\in I} m_iK_{n_i|Z(H)|} + \bigcup_{j\in J} m'_jK_{n'_j|Z(G)|}\right)$.*

Corollary 3.13. *If H is abelian and G is a non-abelian group, then $\Gamma(G\times H) \cong \Delta_1(G,H) \cong \Delta_2(H,G)$.*

Proof. By Theorem 3.6, the non-central conjugacy classes of $G \times H$ have the form $(x,y)^{G\times H}$ such that $y \in Z(H) = H$ and $x \notin Z(G)$. Obviously, two vertices $(x,y)^{G\times H}$ and $(a,b)^{G\times H}$ are adjacent if and only if $x = a$ or $x^Ga^G \in E(\Gamma(G))$, i.e. $\Gamma(G\times H) = \Delta_1(G,H)$. □

Corollary 3.14. *If H is a abelian group and G is a $p-$group such that $\Gamma(G) = \bigcup_{i\in I} m_iK_{n_i}$ then $\Gamma(G\times H) = \bigcup_{i\in I} m_iK_{n_i|H|}$.*

As a simple example based on these results, we can see that $\Gamma(D_8) = 3K_1$, $\Gamma(D_8 \times Z_3) = 3K_3$ and $\Gamma(D_8 \times D_8) = 9K_1 \uplus (3K_2 + 3K_2)$.

4. Concluding Remarks

In this paper, the commuting conjugacy class graph of a group G in which $|\frac{G}{Z(G)}| = p^3$ is completely characterized. As a consequence of our result it is proved that the commuting conjugacy class graph of a group of order p^4 has one of the form $(p+1)K_{p(p-1)}$ or $K_{(p^2-1)} \cup pK_{(p-1)}$. As a consequence of Theorem 3.6, we have:

Theorem 4.1. *Suppose that G is a finite nilpotent group of order n in which n is not divisible by p^5, for every prime factor p of n. Then the commuting conjugacy class graph $\Gamma(G)$ can be obtained from the strong products, joins and the graph operation $\uplus$ on some complete graphs.*

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Mohammad Ali Salahshour

Department of Pure Mathematics, Faculty of Mathematical Sciences,

University of Kashan, Kashan 87317-53153, I. R. Iran

salahshour@iausk.ac.ir

Ali Reza Ashrafi

Department of Pure Mathematics, Faculty of Mathematical Sciences,

University of Kashan, Kashan 87317-53153, I. R. Iran

ashrafi@kashanu.ac.ir