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# COMMUTING CONJUGACY CLASS GRAPHS OF FINITE GROUPS 

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#### Abstract

Suppose that $G$ is a finite non-abelian group. Define the graph $\Gamma(G)$ with the noncentral conjugacy classes of $G$ as vertex set and two distinct vertices $A$ and $B$ are adjacent if and only if there are $x \in A$ and $y \in B$ such that $x y=y x$. The graph $\Gamma(G)$ is called the commuting conjugacy class graph of $G$ and introduced by Mohammadian et al. in [A. Mohammadian, A. Erfanian, M. Farrokhi D. G. and B. Wilkens, Triangle-free commuting conjugacy class graphs, J. Group Theory 19 (3) (2016) 1049-1061]. In this paper, the graph structure of the commuting conjugacy class graph of nilpotent groups of order $n$ are obtained in which $n$ is not divisible by $p^{5}$, for every prime factor $p$ of $n$.


## 1. Introduction

Throughout this paper all groups and graphs are assumed to be finite. Our graph theory notations are taken from [6] and for group theory notions and notations we refer to [ITI]. Suppose that $\Gamma$ is a graph with vertex set $V$ and edge set $E$ and $\rho$ is an equivalence relation

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on $V$. The partition of $V$ constructed from $\rho$ is denoted by $\Pi$. The quotient graph $\frac{\Gamma}{\rho}$ is a graph with vertex set $\Pi$ such that two parts $A$ and $B$ are adjacent if and only if there exists a vertex $x \in A$ and another vertex $y \in B$ such that $x y \in E$.

Suppose that $\Gamma_{1}$ and $\Gamma_{2}$ are two graphs. The strong product $\Gamma_{1} \boxtimes \Gamma_{2}$ of graphs $\Gamma_{1}$ and $\Gamma_{2}$ is a graph with vertex set $V\left(\Gamma_{1}\right) \times V\left(\Gamma_{2}\right)$. Two distinct vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ of this graph are adjacent if and only if $\left(u_{1}=v_{1}\right.$ and $\left.u_{2} v_{2} \in E\left(\Gamma_{2}\right)\right)$ or ( $u_{2}=v_{2}$ and $\left.u_{1} v_{1} \in E\left(\Gamma_{1}\right)\right)$ or ( $u_{1}$ is adjacent to $v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ ). This product was first introduced by Sabidussi in 1960 [ [L2]. If the vertex sets of $\Gamma_{1}$ and $\Gamma_{2}$ are disjoint then the join $\Gamma_{1}+\Gamma_{2}$ is a graph with vertex set $V\left(\Gamma_{1}\right) \cup V\left(\Gamma_{2}\right)$ and two vertices $x, y \in V\left(\Gamma_{1}+\Gamma_{2}\right)$ are adjacent if and only if $\left(x y \in E\left(\Gamma_{1}\right) \cup E\left(\Gamma_{2}\right)\right)$ or $\left(x \in V\left(\Gamma_{1}\right)\right.$ and $\left.y \in V\left(\Gamma_{2}\right)\right)$ or $\left(y \in V\left(\Gamma_{1}\right)\right.$ and $\left.x \in V\left(\Gamma_{2}\right)\right)$.

There are several ways to associate a graph to an algebraic structure like groups. The commuting graph $\Delta(G)$ of a group $G$ is one of the most important graphs defined on a group [9] with non-trivial elements of $G$ as vertex set and two no-trivial elements $x$ and $y$ in $G$ are adjacent if and only if $x y=y x$. Rapinchuk et al. [IT] applied this graph to prove that finite quotients of the multiplicative group of a finite dimensional division algebra are solvable.

Mohammadian et al. [ $[8]$ introduced the commuting conjugacy class graph $\Gamma(G)$ of a non-abelian group $G$ as a simple graph with non-central conjugacy classes of $G$ as its vertex set and two distinct vertices $A$ and $B$ are adjacent if and only if there are $x \in A$ and $y \in B$ such that $x y=y x[z]$. They obtained some interesting properties of this graph among them a classification of finite groups such that $\Gamma(G)$ are triangle-free is given. It is merit to note that the commuting conjugacy class graph was initially introduced by Herzog et al. [7] in which they considered all non-trivial conjugacy classes of $G$ as its vertex set. It is easy to see that the commuting conjugacy class graph $\Gamma(G)$ is a quotient of the induced graph $\Delta(G)$ on non-central elements of $G$ over the set of all non-central conjugacy classes of the group $G$.

## 2. Preliminaries

The aim of this section is to present some results which are crucial in proving the main results of this paper. We start by stating the classification theorem of $p$-group of order $p^{4}$ given by Burnside [4].

Theorem 2.1. If $p$ is an odd prime, then there are 15 groups of order $p^{4}$ up to isomorphisms. Five of those are abelian and the non-abelian groups can be found in the list below.
(1) $\left\langle a, b, c, d \mid a^{p}=b^{p}=c^{p}=d^{p}=1,[a, b]=[a, c]=[a, d]=[b, c]=[b, d]=1, d c d^{-1}=a c\right\rangle$,
(2) $\left\langle a, b \mid a^{p^{2}}=b^{p^{2}}=1, b a b^{-1}=a^{p+1}\right\rangle$,
(3) $\left\langle a, b \mid a^{p^{3}}=b^{p}=1, b a b^{-1}=a^{p^{2}+1}\right\rangle$,
(4) $\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, a b=b a, a c=c a, c b c^{-1}=a^{p} b\right\rangle$,
(5) $\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, a b=b a, b c=c b, c a c^{-1}=a^{p+1}\right\rangle$,
(6) $\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, a b=b a, b c=c b, c a c^{-1}=a b\right\rangle$,
(7) $\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, b a b^{-1}=a^{p+1}, c a c^{-1}=a b, b c=c b\right\rangle$,

$$
\begin{array}{ll}
\left\langle a, b, c \mid a^{p^{2}}=b^{p}=1,[b, c]=1, c^{p}=a^{p}, b^{-1} a b=a^{p+1}, c^{-1} a c=a b^{-1}\right\rangle & p=3 \\
\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, b a b^{-1}=a^{p+1}, c a c^{-1}=a^{p+1} b, c b c^{-1}=a^{p} b\right\rangle & p>3 \tag{8}
\end{array}
$$

$$
\left\langle a, b, c \mid a^{p^{2}}=b^{p}=1,[b, c]=1, c^{p}=a^{-p}, b^{-1} a b=a^{p+1}, c^{-1} a c=a b^{-1}\right\rangle \quad p=3
$$

$$
\begin{equation*}
\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, b a b^{-1}=a^{p+1}, c a c^{-1}=a^{d p+1} b, c b c^{-1}=a^{d p} b, d \not \equiv 0,1\right\rangle \quad p>3, \tag{9}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle a, b, c \mid a^{p^{2}}=b^{p}=c^{p}=1, a b=b a, c^{-1} a c=a b, c^{-1} b c=a^{-p} b\right\rangle \quad p=3 \tag{10}
\end{equation*}
$$

$$
\left\langle a, b, c, d \mid a^{p}=b^{p}=c^{p}=d^{p}=[a, b]=[a, c]=[a, d]=[b, c]=[d, b] a^{-1}=[d, c] b^{-1}=1\right\rangle \quad p>3
$$

Define $\operatorname{Cent}(G)=\left\{C_{G}(x) \mid x \in G\right\}$. Each element of $\operatorname{Cent}(G)$ is said to be an element centralizer of $G$. A group $G$ is called a $C A$-group if all proper element centralizers of $G$ are abelian. The following theorem is crucial throughout this paper:

Theorem 2.2. Let $G$ be a finite group. Then the following are hold:
(1) Let $[G: Z(G)]=p q r$, where $p, q$ and $r$ are primes not necessarily distinct. Then $G$ is a $C A$-group. (See Baishya [3, Lemma 2.1])
(2) The group $G$ is a $C A$-group if and only if whenever $x, y \in G \backslash Z(G)$ satisfy $x y=y x$, then $C_{G}(x)=C_{G}(y)$. (See Dolfi et al. [5, Proposition 3.2(a)])
(3) Let $G$ be a $C A$-group. Then, for all $a, b \in G \backslash Z(G)$ either $C_{G}(a)=C_{G}(b)$ or


In [3, Proposition 2.2], Baishya proved that if $p$ is the smallest prime divisor of $|G|$ and $|G: Z(G)|=p^{3}$, then $|\operatorname{Cent}(G)|=p^{2}+p+2$ or $p^{2}+2$. In the following lemma we prove that the condition of " $p$ is the smallest prime divisor of $|G|$ " can be substituted by " $p$ is a prime divisor of $|G| "$. It is merit to mention here that our proof is similar to the proof of the mentioned result of Baishya.

Lemma 2.3. Suppose that $G$ is a finite group, $Z=Z(G)$ and $[G: Z]=p^{3}$, where $p$ is prime. $|\operatorname{Cent}(G)|=p^{2}+2$ or $p^{2}+p+2$.

Proof. Since $[G: Z]=p^{3}$, by putting $p=q=r$ in Theorem $\mathbb{2 . 2 ( 1 ) , ~} G$ is a $C A-$ group. Suppose $x \in G \backslash Z$. Since $Z \leq C_{G}(x),\left[G: C_{G}(x)\right] \mid[G: Z]=p^{3}$. Therefore, $\left[G: C_{G}(x)\right]=1, p, p^{2}$ or $p^{3}$. Since $x \notin Z$ and $C_{G}(x) \neq Z$, the cases that $\left[G: C_{G}(x)\right]=1, p^{3}$ are impossible. This shows that $\left[G: C_{G}(x)\right]=p$ or $p^{2}$. Our main proof will consider two cases as follows:
(1) For any $x \in G \backslash Z,\left[G: C_{G}(x)\right]=p^{2}$. Fix $y \in G \backslash Z$. Suppose $\left\{C_{G}(t) \mid t \in\right.$ $\left.G \backslash C_{G}(y)\right\}=\left\{C_{G}\left(x_{1}\right), \ldots, C_{G}\left(x_{k}\right)\right\}$. For simplicity of our argument, we assume that $A_{i}=C_{G}\left(x_{i}\right) \backslash Z, 1 \leq i \leq k$. It is clear $\left|A_{1}\right|=\left|A_{2}\right|=\cdots=\left|A_{k}\right|=(p-1)|Z|$ and by Theorem $\mathbb{Z . 2}(3), A_{i}$ 's are distinct and $C_{G}(y) \cap A_{i}=\emptyset, 1 \leq i \leq k$. We now prove that $G \backslash C_{G}(y)=\bigcup_{i=1}^{k} A_{i}$. It is easy to see that $A_{i} \subseteq G \backslash C_{G}(y), 1 \leq i \leq k$, then $\bigcup_{i=1}^{k} A_{i} \subseteq G \backslash C_{G}(y)$. Next we assume that $t \in G \backslash C_{G}(y)$ is arbitrary. Hence, there exists an integer $s$ such that $1 \leq s \leq k$ and $t \in C_{G}\left(x_{s}\right) \backslash Z$ and so $t \in \bigcup_{i=1}^{k} A_{i}$.

Therefore, $G \backslash C_{G}(y)=\bigcup_{i=1}^{k} A_{i}$ which implies that $\left|G \backslash C_{G}(y)\right|=\sum_{i=1}^{k}\left|A_{i}\right|=k\left|A_{i}\right|$ $=k\left|C_{G}\left(x_{i}\right) \backslash Z\right|$. Since for every $x \in G \backslash Z$ we have $\left[G: C_{G}(x)\right]=p^{2},\left[C_{G}(x): Z\right]=p$. Thus $k=\frac{\left|G \backslash C_{G}(y)\right|}{\left|C_{G}(x) \backslash Z\right|}=\frac{\left|C_{G}(y)\right|\left(p^{2}-1\right)}{|Z|(p-1)}=p^{2}+p$. This implies that $|\operatorname{Cent}(G)|=p^{2}+p+2$, as desired.
(2) There exists $x \in G \backslash Z$ such that $\left[G: C_{G}(x)\right]=p$. We first prove that $C_{G}(x)$ is unique. Suppose that $x \neq y \in G \backslash Z$ such that $\left[G: C_{G}(y)\right]=p$ and $C_{G}(x) \neq C_{G}(y)$. Since $Z \nsupseteq C_{G}(y) \nsupseteq G$ and $[G: Z]=p^{3},\left[C_{G}(y): Z\right]=p^{2}$. On the other hand, $C_{G}(x) C_{G}(y) \subseteq G$ and so $\left|C_{G}(x) C_{G}(y)\right| \leq|G|$. Thus, $\frac{\left|C_{G}(x)\right|\left|C_{G}(y)\right|}{\left|C_{G}(x) \cap C_{G}(y)\right|} \leq|G|$. By Theorem $\amalg 2(3), C_{G}(x) \cap C_{G}(y)=Z$. Hence, $\frac{\left|C_{G}(x)\right|\left|C_{G}(y)\right|}{|Z|} \leq|G|$ which implies that $p^{2} \leq p$, that is impossible. Therefore, $G$ has exactly one element centralizer $C_{G}(x)$ of index $p$ and other proper element centralizers are of index $p^{2}$. It is now easy to check that $G=\left[G \backslash C_{G}(x)\right] \cup C_{G}(x)$ and hence for every $z \in G \backslash C_{G}(x),\left[G: C_{G}(z)\right]=p^{2}$. Apply the same argument as Part (1) to prove that $|\operatorname{Cent}(G)|=\frac{\left|G \backslash C_{G}(x)\right|}{\left|C_{G}(z) \backslash Z\right|}+2=\frac{\left|C_{G}(x)\right|(p-1)}{|Z|(p-1)}$ $+2=p^{2}+2$.

This completes the proof.

## 3. Main Results

Suppose that $\Gamma_{1}, \ldots, \Gamma_{s}$ are graphs with mutually disjoint vertex sets. The graph union $\Gamma_{1} \cup \cdots \cup \Gamma_{s}$ is a graph with vertex set $V\left(\Gamma_{1}\right) \cup \cdots \cup V\left(\Gamma_{s}\right)$ and edge set $E\left(\Gamma_{1}\right) \cup \cdots \cup E\left(\Gamma_{s}\right)$. If all graphs $\Gamma_{i}, 1 \leq i \leq s$, are isomorphic then we will write $s \Gamma_{1}$ as $\Gamma_{1} \cup \cdots \cup \Gamma_{s}$. The aim of this section is to obtain the structure of the commuting conjugacy class graph of a nilpotent group.

Theorem 3.1. Suppose that $G$ is a non-abelian finite group with center $Z=Z(G)$ and $\frac{G}{Z} \cong Z_{p} \times Z_{p}$, where $p$ is prime. Then $p\left||Z|\right.$ and $\Gamma(G)=(p+1) K_{n}$, where $n=\frac{(p-1)|Z|}{p}$.

Proof. Suppose $C_{G}(x)$ is a proper centralizer of $G$. Since $\left|\frac{G}{Z(G)}\right|=p^{2}, C_{G}(x)=\langle Z, x\rangle$ and so it is abelian. Therefore, the group $G$ is a $C A$-group. We now apply Theorem $\overline{L 2}(3)$ to prove that for any $x, y \in G \backslash Z$,

$$
\begin{equation*}
C_{G}(x)=C_{G}(y) \quad \text { and } \quad C_{G}(x) \cap C_{G}(y)=Z \tag{1}
\end{equation*}
$$

Since $\frac{G}{Z}$ is abelian, $C_{G}(x)$ is normal in $G$ and so for any $y \in C_{G}(x), y^{G} \subseteq C_{G}(x)$. Furthermore, $G$ is a $C A$-group and so by Theorem $\amalg 2(2)$ if $y \in C_{G}(x) \backslash Z$ and $y \notin x^{G}$ then $C_{G}(y)=C_{G}(x)$ which implies that $\left|y^{G}\right|=\left|x^{G}\right|$. Obviously, $|G|=p^{2}|Z|$ and hence for any $x \in G \backslash Z, Z \supsetneqq C_{G}(x) \supsetneqq G$. Thus $\left|C_{G}(x)\right|=p|Z|$ and $\left|x^{G}\right|=p$. This proves that the number of non-central conjugacy classes of $G$ contained in $C_{G}(x)$ is $n=\frac{(p-1)|Z|}{p}$. Note that if
$C_{G}(x) \neq C_{G}(y)$ then by the Equation $\mathbb{l}$, there are no conjugacy classes $a^{G}$ and $b^{G}$ contained in $C_{G}(x)$ and $C_{G}(y)$, respectively, such that $a^{G} b^{G} \in E(\Gamma(G))$.

Next we assume that $G$ has exactly $s$ distinct centralizers of order $p|Z|$. Then by Equation (1) $s=p+1$. Therefore, $\Gamma(G)=(p+1) K_{n}$, where $n=\frac{(p-1)|Z|}{p}$.

Corollary 3.2. If $G$ is a non-abelian $p-$ group of order $p^{n}$ and $|Z(G)|=p^{n-2}$, $p$ is prime and $n \geq 3$, then $\Gamma(G)=(p+1) K_{p^{n-3}(p-1)}$.

Theorem 3.3. Suppose that $G$ is a non-abelian group with center $Z$ such that $\left|\frac{G}{Z}\right|=p^{3}$, for a prime $p$. Then one of the following is satisfied:
(1) If $\frac{G}{Z}$ is abelian then $\Gamma(G)=K_{m} \cup p^{2} K_{n}$ or $\left(p^{2}+p+1\right) K_{n}$, where $m=\frac{\left(p^{2}-1\right)|Z|}{p}$ and $n=\frac{(p-1)|Z|}{p^{2}}$.
(2) If $\frac{G}{Z}$ is non-abelian then $\Gamma(G)=K_{m} \cup k p K_{n_{1}} \cup(p-k) K_{n_{2}},(k p+1) K_{n_{1}} \cup(p+1-k) K_{n_{2}}$, $K_{m} \cup p K_{n_{2}},\left(p^{2}+p+1\right) K_{n_{1}}$ or $K_{n_{1}} \cup(p+1) K_{n_{2}}$, where $m=\frac{\left(p^{2}-1\right)|Z|}{p}, n_{1}=\frac{(p-1)|Z|}{p^{2}}$, $n_{2}=\frac{(p-1)|Z|}{p}, 1 \leq k \leq p$.

Proof. Since $\left|\frac{G}{Z(G)}\right|=p^{3}$, by Theorem $\overline{L .2(1)}$ the group $G$ is a $C A-$ group and by Theorem [2.2(3), for any $x, y \in G \backslash Z$,

$$
\begin{equation*}
C_{G}(x)=C_{G}(y) \quad \text { and } \quad C_{G}(x) \cap C_{G}(y)=Z \tag{2}
\end{equation*}
$$

Now, we have the following two different cases:
(1) Suppose that $\frac{G}{Z}$ is abelian. Then for every $x \in G \backslash Z, C_{G}(x)$ is normal in $G$ and so for any $y \in C_{G}(x), y^{G} \subseteq C_{G}(x)$. Moreover, $G$ is a $C A$-group and so by Theorem $\overline{L 2(2)}$ if $y \in C_{G}(x) \backslash Z$ and $y \notin x^{G}$ then $C_{G}(y)=C_{G}(x)$ and $\left|y^{G}\right|=\left|x^{G}\right|$. It is clear that $|G|=p^{3}|Z|$ and so for every $x \in G \backslash Z, Z \supsetneqq C_{G}(x) \supsetneqq G$. Then $\left|C_{G}(x)\right|=p|Z|$ or $p^{2}|Z|$. If $\left|C_{G}(x)\right|=p^{2}|Z|$, then $\left|x^{G}\right|=p$. So, the number of non-central conjugacy classes of $G$ contained in $C_{G}(x)$ is $m=\frac{\left(p^{2}-1\right)|Z|}{p}$. If $\left|C_{G}(x)\right|=p|Z|$, then $\left|x^{G}\right|=p^{2}$. Hence the number of non-central conjugacy classes of $G$ contained in $C_{G}(x)$ is $n=\frac{(p-1)|Z|}{p^{2}}$. Note that if $C_{G}(x) \neq C_{G}(y)$ then by Equation [ひ, there are no conjugacy classes $a^{G}$ and $b^{G}$ contained in $C_{G}(x)$ and $C_{G}(y)$, respectively, such that $a^{G} b^{G} \in E(\Gamma(G))$.

Next we assume that $G$ has exactly $s_{1}$ distinct element centralizers of order $p^{2}|Z|$ and $s_{2}$ distinct element centralizers of order $p|Z|$. Therefore, by Equations [] we have $\left(p^{2}+p+1\right)(p-1)|Z|=\left(s_{1}(p+1)+s_{2}\right)(p-1)|Z|$. Hence, $s_{1}(p+1)+s_{2}=p^{2}+p+1$. On the other hand, by the proof of Lemma [2.3, $s_{1}=0$ or 1 . So $s_{2}=p^{2}+p+1$ or $p^{2}$ and the proof of this case is completed.
(2) Suppose that $\frac{G}{Z}$ is non-abelian. Since $|G|=p^{3}|Z|$, it is easy to see that for all $x \in G \backslash Z$, $Z \supsetneqq C_{G}(x) \supsetneqq G$. This shows that $\left|C_{G}(x)\right|=p|Z|$ or $p^{2}|Z|$.
(a) $\left|C_{G}(x)\right|=p^{2}|Z|$. It is clear that $Z \triangleleft G$ and so $\frac{C_{G}(x)}{Z} \leq \frac{G}{Z}$. Since $\left|\frac{G}{Z}\right|=p^{3}$ and $\left|\frac{C_{G}(x)}{Z}\right|=p^{2}, C_{G}(x) \triangleleft G$. Hence $y^{G} \subseteq C_{G}(x)$, for every element $y \in C_{G}(x) \backslash Z$. Also $G$ is a $C A$-group, so by Theorem $\amalg .2(2)$ if $y \in C_{G}(x) \backslash Z$ and $y \notin x^{G}$ then $C_{G}(y)=C_{G}(x)$ and $\left|y^{G}\right|=\left|x^{G}\right|=p$. Thus the number of distinct non-central conjugacy classes of $G$ contained in $C_{G}(x)$ is $m=\frac{\left(p^{2}-1\right)|Z|}{p}$.
(b) $\left|C_{G}(x)\right|=p|Z|$. If $C_{G}(x)$ is a normal subgroup of $G$, then $y^{G} \subseteq C_{G}(x)$, for every element $y \in C_{G}(x) \backslash Z$. Since $G$ is a $C A$-group, by Theorem $\mathbb{Z . 2 ( 2 )}$ if $y \in C_{G}(x) \backslash Z$ and $y \notin x^{G}$ then $C_{G}(y)=C_{G}(x)$ and $\left|y^{G}\right|=\left|x^{G}\right|=p^{2}$. Thus, the number of distinct non-central conjugacy classes of $G$ contained in $C_{G}(x)$ is $n_{1}=\frac{(p-1)|Z|}{p^{2}}$. Next, we assume that $C_{G}(x)$ is not normal in $G$. Since $\frac{G}{Z}$ is a $p$-group of order $p^{3}, \frac{C_{G}(x)}{Z} \supsetneqq N_{\frac{G}{Z}}\left(\frac{C_{G}(x)}{Z}\right) \supsetneqq \frac{G}{Z}$ and $\left|N_{\frac{G}{Z}}\left(\frac{C_{G}(x)}{Z}\right)\right|=p^{2}$. On the other hand, $Z \supsetneqq C_{G}(x) \supsetneqq G$. So, $N_{\frac{G}{Z}}\left(\frac{C_{G}(x)}{Z}\right)=\frac{N_{G}\left(C_{G}(x)\right)}{Z}$ and $\left[G: N_{G}\left(C_{G}(x)\right)\right]$ $=p$. Thus, the number of conjugates of $C_{G}(x)$ in $G$ is $p$. But $G$ is a $C A$-group and by Theorem [2.2(2) if $y \in C_{G}(x) \backslash Z$ and $y \notin x^{G}$ then $C_{G}(y)=C_{G}(x),\left|y^{G}\right|$ $=\left|x^{G}\right|=p^{2}$ and $\left|C_{G}(x) \cap y^{G}\right|=\left|C_{G}(x) \cap x^{G}\right|=p$. Thus, the number of distinct non-central conjugacy classes of $G$ is $n_{2}=\frac{(p-1)|Z|}{p}$.
Note that if $C_{G}(x) \neq C_{G}(y)$ then by Equation there are no conjugacy classes $a^{G}$ and $b^{G}$ contained in $C_{G}(x)$ and $C_{G}(y)$, respectively, such that $a^{G} b^{G} \in E(\Gamma(G))$. Assume that $s_{1}$ denotes the number of distinct element centralizers of order $p^{2}|Z|$, $s_{2}$ is the number of distinct normal element centralizers of order $p|Z|$ and $s_{3}$ is the number of distinct element centralizers of order $p|Z|$ containing $p$ conjugates. By Equations [], $\left(p^{2}+p+1\right)(p-1)|Z|=\left(s_{1}(p+1)+s_{2}+s_{3} p\right)(p-1)|Z|$ and hence $s_{1}(p+1)+s_{2}+s_{3} p=p^{2}+p+1$. On the other hand, the proof of Lemma 2.3 shows that $s_{1}=0$ or 1 . Suppose $s_{1}=1$. Then, $s_{2}+s_{3} p=p^{2}$ and so $p \mid s_{2}$. Therefore, $s_{2}=k p$ and $s_{3}=p-k$ in which $0 \leq k \leq p$. If $k=0$, then $\Gamma(G)=K_{m} \cup p K_{n_{2}}$. But $\Gamma(G)=K_{m} \cup k p K_{n_{1}} \cup(p-k) K_{n_{2}}, 1 \leq k \leq p$. Next we assume that $s_{1}=0$. Then $s_{2}+s_{3} p=p^{2}+p+1$ and so $p \mid s_{2}-1$. Therefore, $s_{2}=k^{\prime} p+1$ and $s_{3}=p+1-k^{\prime}$ in which $0 \leq k^{\prime} \leq p+1$. If $k^{\prime}=0$ or $p+1$, then $\Gamma(G)=K_{n_{1}} \cup(p+1) K_{n_{2}}$ or $\left(p^{2}+p+1\right) K_{n_{1}}$. On the other hand, $\Gamma(G)=\left(k^{\prime} p+1\right) K_{n_{1}} \cup\left(p+1-k^{\prime}\right) K_{n_{2}}$, where $1 \leq k^{\prime} \leq p, m=\frac{\left(p^{2}-1\right)|Z|}{p}$, $n_{1}=\frac{(p-1)|Z|}{p^{2}}$ and $n_{2}=\frac{(p-1)|Z|}{p}$.

Hence the result.

Corollary 3.4. Let $G$ be a non abelian $p$-group of order $p^{n}$ and $|Z(G)|=p^{n-3}$, where $p$ is prime and $n \geq 4$. Then one of the following are satisfied:
(1) If $\frac{G}{Z}$ is abelian then $\Gamma(G)=K_{p^{n-4}\left(p^{2}-1\right)} \cup p^{2} K_{p^{n-5}(p-1)}$ or $\left(p^{2}+p+1\right) K_{p^{n-5}(p-1)}$.
(2) If $\frac{G}{Z}$ is non-abelian then $\Gamma(G)=K_{p^{n-4}\left(p^{2}-1\right)} \cup p K_{p^{n-4}(p-1)},\left(p^{2}+p+1\right) K_{p^{n-5}(p-1)}$, $K_{p^{n-5}(p-1)} \cup(p+1) K_{p^{n-4}(p-1)}, K_{p^{n-4}\left(p^{2}-1\right)} \cup k p K_{p^{n-5}(p-1)} \cup(p-k) K_{p^{n-4}(p-1)}$ or $(k p+$ 1) $K_{p^{n-5}(p-1)} \cup(p+1-k) K_{p^{n-4}(p-1)}$, where $1 \leq k \leq p$.

Proof. Apply Corollary [3].

Corollary 3.5. Let $G$ be a non-abelian $p$-group of order $p^{4}$. Then the commuting conjugacy class graph of $G$ has the form $(p+1) K_{p(p-1)}$ or $K_{\left(p^{2}-1\right)} \cup p K_{(p-1)}$.

Proof. Since $|G|=p^{4}, p$ is prime, $|Z(G)|=p$ or $p^{2}$. If $|Z(G)|=p^{2}$, then by Corollary $3.2 \Gamma(G)$ $=(p+1) K_{p(p-1)}$. In other case, $|Z(G)|=p$ and by Corollary B.4, $\Gamma(G)=K_{\left(p^{2}-1\right)} \cup p K_{(p-1)}$, as desired.

If $G$ is isomorphic to one of the first six groups in Theorem [2.I, then the commuting conjugacy class graph of $G$ will be isomorphic to $(p+1) K_{p(p-1)}$. In other cases, the commuting conjugacy class graph of $G$ is isomorphic to $K_{\left(p^{2}-1\right)} \cup p K_{(p-1)}$.

Suppose that $G$ and $H$ are two non-abelian finite groups. Define two graphs $\Delta_{1}(G, H)$ and $\Delta_{2}(G, H)$ as follows:

$$
\begin{aligned}
V\left(\Delta_{1}(G, H)\right) & =\left\{(x, y)^{G \times H} \mid x^{G} \in V(\Gamma(G)) \text { and } y \in Z(H)\right\}, \\
E\left(\Delta_{1}(G, H)\right) & =\left\{(x, y)^{G \times H}(a, b)^{G \times H} \mid x=a \text { or } x^{G} a^{G} \in E(\Gamma(G))\right\}, \\
V\left(\Delta_{2}(G, H)\right) & =\left\{(x, y)^{G \times H} \mid y^{H} \in V(\Gamma(H)) \text { and } x \in Z(G)\right\}, \\
E\left(\Delta_{2}(G, H)\right) & =\left\{(x, y)^{G \times H}(a, b)^{G \times H} \mid y=b \text { or } y^{H} b^{H} \in E(\Gamma(H))\right\} .
\end{aligned}
$$

Set $\Delta_{1}=\Delta_{1}(G, H), \Delta_{2}=\Delta_{2}(G, H)$ and $\Delta_{3}=\Delta_{3}(G, H)=\Gamma(G) \boxtimes \Gamma(H)$. We also define two sets $A$ and $B$ as follows:

$$
\begin{aligned}
& A=\left\{(x, y)^{G \times H}(a, b)^{G \times H} \mid(x, y)^{G \times H} \in V\left(\Delta_{3}\right),(a, b)^{G \times H} \in V\left(\Delta_{1}\right),\left(x=a \vee x^{G} a^{G} \in E(\Gamma(G))\right)\right\} \\
& B=\left\{(x, y)^{G \times H}(a, b)^{G \times H} \mid(x, y)^{G \times H} \in V\left(\Delta_{3}\right),(a, b)^{G \times H} \in V\left(\Delta_{2}\right),\left(y=b \vee y^{H} b^{H} \in E(\Gamma(H))\right)\right\} .
\end{aligned}
$$

Define our third graph operation $\uplus$ as follows:

$$
\begin{aligned}
V\left(\Delta_{3} \uplus\left(\Delta_{1}+\Delta_{2}\right)\right) & =V\left(\Delta_{3}\right) \cup V\left(\Delta_{1}+\Delta_{2}\right), \\
E\left(\Delta_{3} \uplus\left(\Delta_{1}+\Delta_{2}\right)\right) & =E\left(\Delta_{3}\right) \cup E\left(\Delta_{1}+\Delta_{2}\right) \cup A \cup B .
\end{aligned}
$$

We now obtain the graph structure of the Cartesian product of two groups.
Theorem 3.6. Let $G$ and $H$ be two non-abelian group. Then the commuting conjugacy class graph of $G \times H$ can be computed as $\Gamma(G \times H)=\Delta_{3}(G, H) \uplus\left[\Delta_{1}(G, H)+\Delta_{2}(G, H)\right]$.

Proof. It is easy to see that an element $(a, b)$ is central in $G \times H$ if and only if $a$ is central in $G$ and $b$ is central in $H$. We partition the non-central conjugacy classes of $G \times H$ as follows:
(1) The conjugacy classes $(x, y)^{G \times H}$ such that $y \in Z(H)$ and $x \notin Z(G)$. Choose two non-central conjugacy classes $(x, y)^{G \times H}$ and $(a, b)^{G \times H}$ of $G \times H$. These vertices are adjacent in $\Gamma(G \times H)$ if and only if $x=a$ or $x^{G} a^{G} \in E(\Gamma(G))$. This means that $(x, y)^{G \times H}(a, b)^{G \times H} \in E\left(\Delta_{1}(G, H)\right)$.
(2) The conjugacy classes $(x, y)^{G \times H}$ such that $x \in Z(G)$ and $y \notin Z(H)$. A similar argument as part (1) shows that $(x, y)^{G \times H}(a, b)^{G \times H} \in E\left(\Delta_{2}(G, H)\right)$.
(3) The conjugacy classes $(x, y)^{G \times H}$ such that $x \notin Z(G)$ and $y \notin Z(H)$. Suppose that $(x, y)^{G \times H}$ and $(a, b)^{G \times H}$ are two adjacent vertices of $\Gamma(G \times H)$. Then $(x=a$ and $\left.y^{H} b^{H} \in E(\Gamma(H))\right)$ or $\left(y=b\right.$ and $\left.x^{G} a^{G} \in E(\Gamma(G))\right)$ or $\left(x^{G} a^{G} \in E(\Gamma(G))\right.$ and $y^{H} b^{H} \in$ $E(\Gamma(H)))$. Therefore, $(x, y)^{G \times H}(a, b)^{G \times H} \in \Delta_{3}(G, H)$.
Suppose $(x, y)^{G \times H} \in V\left(\Delta_{1}\right)$ and $(a, b)^{G \times H} \in V\left(\Delta_{2}\right)$. Then by our definition, $y \in Z(H)$ and $a \in Z(G)$. Hence each vertex of $\Delta_{1}$ is adjacent with each vertex of $\Delta_{2}$. We now assume that $(x, y)^{G \times H} \in V\left(\Delta_{3}\right)$ and $(a, b)^{G \times H} \in V\left(\Delta_{1}+\Delta_{2}\right)$. Without loss of generality, we can assume that $(a, b)^{G \times H} \in V\left(\Delta_{1}\right)$. It is clear that $b \in Z(H)$ and so $b y=y b$. Since $a, x \notin Z(G), a=x$ or $x^{G} a^{G} \in E(\Gamma(G))$. This proves that $(x, y)^{G \times H}(a, b)^{G \times H} \in A$ which completes the proof.

Lemma 3.7. Let $H$ be arbitrary and let $G$ be a non-abelian group. Then $\Gamma(G)$ and $\Delta_{1}(G, H)$ have the same number of connected components.

Proof. Choose two vertices $x^{G}$ and $a^{G}$ from different components of $\Gamma(G)$ and $b, y \in Z(H)$. Then $(x, y)^{G \times H}$ and $(a, b)^{G \times H}$ are different vertices of $\Delta_{1}(G, H)$. By definition of $\Delta_{1}$, $(x, y)^{G \times H}(a, b)^{G \times H} \in E\left(\Delta_{1}(G, H)\right)$ if and only if $x^{G} a^{G} \in E(\Gamma(G))$, proving the lemma.

Lemma 3.8. If $H$ is an arbitrary finite group and $G$ is a finite group such that $\Gamma(G)=m K_{n}$. Then $\Delta_{1}(G, H)=m K_{n|Z(H)|}$.

Proof. Since $\Gamma(G)=m K_{n}, G$ has exactly $m n$ non-central conjugacy classes, and since $Z(H)$ is abelian, $G \times Z(H)$ has exactly $m n|Z(H)|$ conjugacy classes. Note that by definition $\left|V\left(\Delta_{1}\right)\right|=$ $m n|Z(H)|$. Choose $(x, y)^{G \times H}$ and $(a, b)^{G \times H}$ from a connected component of $\Delta_{1}(G, H)$. If $x=a$ then $(x, y)^{G \times H}(a, b)^{G \times H} \in E\left(\Delta_{1}(G, H)\right)$. Suppose $x \neq a$. Since $x^{G}$ and $a^{G}$ are in a connected component of $\Gamma(G)$ and connected components of $\Gamma(G)$ are isomorphic to $K_{n}$, $x^{G} a^{G} \in E(\Gamma(G))$. Therefore, $(x, y)^{G \times H}(a, b)^{G \times H} \in E\left(\Delta_{1}(G, H)\right)$. By Lemma ${ }^{3} .2$, the number of connected components of $\Gamma(G)$ and $\Delta_{1}(G, H)$ are $m$ and since $\left|V\left(\Delta_{1}(G, H)\right)\right|=m n|Z(H)|$, $\Delta_{1}(G, H)=m K_{n|Z(H)|}$.

Corollary 3.9. If $G$ and $H$ are two groups such that $\Gamma(G)=\bigcup_{i \in I} m_{i} K_{n_{i}}$ and $\Gamma(H)=$ $\bigcup_{j \in J} m_{j}^{\prime} K_{n_{j}^{\prime}}$, then $\Delta_{1}(G, H)=\bigcup_{i \in I} m_{i} K_{n_{i}|Z(H)|}$ and $\Delta_{2}(G, H)=\bigcup_{j \in J} m_{j}^{\prime} K_{n_{j}^{\prime}|Z(G)|}$.

Lemma 3.10. If $G$ and $H$ are two groups such that $\Gamma(G)=m_{1} K_{n_{1}}$ and $\Gamma(H)=m_{2} K_{n_{2}}$, then $\Delta_{3}(G, H)=m_{1} m_{2} K_{n_{1} n_{2}}$.

Proof. We know that $\Delta_{3}(G, H)=\Gamma(G) \boxtimes \Gamma(H)$. Define $A$ to be a subgraph of $\Delta_{3}$ containing all vertices of the form $\left(x^{G}, y^{H}\right) \in V\left(\Delta_{3}(G, H)\right)$ such that $x^{G}$,s will induce a connected component in $\Gamma(G)$ and $y^{H}$ 's will induce another component in $\Gamma(H)$. It is easy to see that $|V(A)|=n_{1} n_{2}$. We prove that $A$ is a connected subgraph of $\Delta_{3}(G, H)$. To do this, we assume that $\left(x^{G}, y^{H}\right)$ and $\left(a^{G}, b^{H}\right)$ are two arbitrary vertices in $A$. Then we have one of the following cases:
(1) $x^{G}=a^{G}$ and $y^{H} \neq b^{H}$.
(2) $x^{G} \neq a^{G}$ and $y^{H}=b^{H}$.
(3) $x^{G} \neq a^{G}$ and $y^{H} \neq b^{H}$.

Since connected components of $\Gamma(G)$ and $\Gamma(H)$ are $K_{n_{1}}$ and $K_{n_{2}}$, respectively, by definition of $\Delta_{3},\left(x^{G}, y^{H}\right)\left(a^{G}, b^{H}\right) \in E\left(\Delta_{3}(G, H)\right)$. We now show that the subgraph $A$ is a connected component of $\Delta_{3}$. To see this, we assume that $\left(x^{G}, y^{H}\right)$ is an arbitrary vertex of $A$ and $\left(c^{G}, d^{H}\right)$ is a vertex in $\Delta_{3} \backslash A$. It is clear that $\left\{x^{G}, c^{G}\right\}$ is not contained in a connected component of $\Gamma(G)$ or $\left\{y^{H}, d^{H}\right\}$ is not contained in a connected component of $\Gamma(H)$. Therefore, $x^{G} c^{G} \notin E(\Gamma(G))$ or $y^{H} d^{H} \notin E(\Gamma(H))$ and so by definition of $\Delta_{3},\left(x^{G}, y^{H}\right)\left(c^{G}, d^{H}\right) \notin E\left(\Delta_{3}\right)$. This proves that $A$ is a connected component of $\Delta_{3}$. Thus, $A=K_{n_{1} n_{2}}$. Since $\left|V\left(\Delta_{3}\right)\right|=m_{1} m_{2} n_{1} n_{2}$, $\Delta_{3}(G, H)=m_{1} m_{2} K_{n_{1} n_{2}}$. This completes the proof.

Corollary 3.11. If $G$ and $H$ are two groups such that $\Gamma(G)=\bigcup_{i \in I} m_{i} K_{n_{i}}$ and $\Gamma(H)=$ $\bigcup_{j \in J} m_{j}^{\prime} K_{n_{j}^{\prime}}$, then $\Delta_{3}(G, H)=\bigcup_{\substack{i \in J \\ j \in J}} m_{i} m_{j}^{\prime} K_{n_{i} n_{j}^{\prime}}$.

Corollary 3.12. If $G$ and $H$ are two $p$-groups such that $\Gamma(G)=\bigcup_{i \in I} m_{i} K_{n_{i}}$ and $\Gamma(H)=$ $\bigcup_{j \in J} m_{j}^{\prime} K_{n_{j}^{\prime}}$, then $\Gamma(G \times H)=\left(\bigcup_{\substack{i \in I \\ j \in J}} m_{i} m_{j}^{\prime} K_{n_{i} n_{j}^{\prime}}\right) \uplus\left(\bigcup_{i \in I} m_{i} K_{n_{i}|Z(H)|}+\bigcup_{j \in J} m_{j}^{\prime} K_{n_{j}^{\prime}|Z(G)|}\right)$.

Corollary 3.13. If $H$ is abelian and $G$ is a non-abelian group, then $\Gamma(G \times H) \cong \Delta_{1}(G, H) \cong$ $\Delta_{2}(H, G)$.

Proof. By Theorem E2.6, the non-central conjugacy classes of $G \times H$ have the form $(x, y)^{G \times H}$ such that $y \in Z(H)=H$ and $x \notin Z(G)$. Obviously, two vertices $(x, y)^{G \times H}$ and $(a, b)^{G \times H}$ are adjacent if and only if $x=a$ or $x^{G} a^{G} \in E(\Gamma(G))$, i.e. $\Gamma(G \times H)=\Delta_{1}(G, H)$.

Corollary 3.14. If $H$ is a abelian group and $G$ is a $p-$ group such that $\Gamma(G)=\bigcup_{i \in I} m_{i} K_{n_{i}}$ then $\Gamma(G \times H)=\bigcup_{i \in I} m_{i} K_{n_{i}|H|}$.

As a simple example based on these results, we can see that $\Gamma\left(D_{8}\right)=3 K_{1}, \Gamma\left(D_{8} \times Z_{3}\right)=3 K_{3}$ and $\Gamma\left(D_{8} \times D_{8}\right)=9 K_{1} \uplus\left(3 K_{2}+3 K_{2}\right)$.

## 4. Concluding Remarks

In this paper, the commuting conjugacy class graph of a group $G$ in which $\left|\frac{G}{Z(G)}\right|=p^{3}$ is completely characterized. As a consequence of our result it is proved that the commuting conjugacy class graph of a group of order $p^{4}$ has one of the form $(p+1) K_{p(p-1)}$ or $K_{\left(p^{2}-1\right)} \cup$ $p K_{(p-1)}$. As a consequence of Theorem [3.6, we have:

Theorem 4.1. Suppose that $G$ is a finite nilpotent group of order $n$ in which $n$ is not divisible by $p^{5}$, for every prime factor $p$ of $n$. Then the commuting conjugacy class graph $\Gamma(G)$ can be obtained from the strong products, joins and the graph operation $\uplus$ on some complete graphs.

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