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r-SUBMODULES AND uz-MODULES

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Abstract. In this article we study and investigate the behavior of r-submodules (a proper submodule N of an R-module M in which $am \in N$ with $\mathrm{Ann}_M(a) = (0)$ implies that $m \in N$ for each $a \in R$ and $m \in M$). We show that every simple submodule, direct summand, divisible submodule, torsion submodule and the socle of a module is an r-submodule and if R is a domain, then the singular submodule is an r-submodule. We also introduce the concepts of uz-module (i.e., an R-module M such that either $\mathrm{Ann}_M(a) \neq (0)$ or $aM = M$, for every $a \in R$) and strongly uz-module (i.e., an R-module M such that $aM \subseteq a^2M$, for every $a \in R$) in the category of modules over commutative rings. We show that every Von Neumann regular module is a strongly uz-module and every Artinian R-module is a uz-module. It is observed that if M is a faithful cyclic R-module, then M is a uz-module if and only if every its cyclic submodule is an r-submodule. In addition, in this case, R is a domain if and only if the only r-submodule of M is zero submodule. Finally, we prove that R is a uz-ring if and only if every faithful cyclic R-module is a uz-module.

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1. Introduction

Throughout this paper R is a commutative ring with $1 \neq 0$ and M is a unitary R-module. For $S \subseteq R$ and $N \subseteq M$ we define $\mathrm{Ann}_R(S) = \{a \in R : aS = (0)\}$, $\mathrm{Ann}_M(S) = \{m \in M : mS = (0)\}$ and $\mathrm{Ann}_R(N) = \{a \in R : aN = (0)\}$. For simplicity of notation, in the case $S = \{a\}$ and $N = \{m\}$, we write $\mathrm{Ann}_R(a)$, $\mathrm{Ann}_M(a)$ and $\mathrm{Ann}_R(m)$ instead of $\mathrm{Ann}_R(\{a\})$, $\mathrm{Ann}_M(\{a\})$ and $\mathrm{Ann}_R(\{m\})$, respectively. An element $a \in R$ is said to be regular if $\mathrm{Ann}_R(a) = (0)$, otherwise, it is called a zerodivisor element, and is said to be regular (resp., zerodivisor) element relative to an R-module M if $\mathrm{Ann}_M(a) = (0)$ (resp., $\mathrm{Ann}_M(a) \neq (0)$). By $\mathrm{r}(R)$, $\mathrm{zd}(R)$ and $\mathrm{u}(R)$ we mean the set of all regular elements, zerodivisor elements and unit elements of R, respectively. We call a ring R a *uz*-ring if for every $a \in R$ either $a \in \mathrm{zd}(R)$ or $a \in \mathrm{u}(R)$. Also we denote the set of all regular elements of R relative to M, by $\mathrm{r}_M(R)$, that is $\mathrm{r}_M(R) = \{a \in R : \mathrm{Ann}_M(a) = (0)\}$. An ideal I of R is called a) a nonregular ideal if $I \subseteq \mathrm{zd}(R)$; b) an r-ideal if $ab \in I$, with $\mathrm{Ann}_R(a) = (0)$, implies that $b \in I$, for each $a, b \in R$. For $m \in M$ (resp., $a \in R$), Rm (resp., Ra) denotes the cyclic submodule (resp., principal ideal) generated by $m \in M$ (resp., $a \in R$). A homomorphism of an R-module M to itself is called an endomorphism. The set of all endomorphisms of M is a ring, which is denoted by $\mathrm{End}_R(M)$. For each R-module M, the Jacobson (resp., socle), by definition, is the intersection (resp., the sum) of all maximal (resp., minimal) submodules of M, which will be denoted by $\mathrm{J}(M)$ (resp., $\mathrm{soc}(M)$). An R-module M is said to be a) a simple module if it is nonzero and it has no nontrivial submodule; b) semisimple if every submodule of M is a direct summand; c) divisible if for each $m \in M$ and $0 \neq a \in R$, there exists $x \in M$ such that $m = ax$; d) faithful if $\mathrm{Ann}_R(M) = (0)$; e) Von Neumann regular module if every its cyclic submodule is a direct summand. Also a nonzero submodule N of an R-module M is said to be essential if for every nonzero submodule K of M we have $N \cap K \neq (0)$. For more information about the aforementioned submodules in the category of R-modules, we refer the reader to [1, 6, 10, 11]. We also refer the reader to [12] and [9] for the necessary information about r-ideals and r-submodules, respectively. Finally, for more details and undefined terms and notations, see [2, 3, 5, 7].

2. r-submodules

Our aim in this section is to study the behavior of r-submodules. The concept of r-ideal was introduced and study in [12]. Recall from [9] the following definition.

Definition 2.1. Let R be a ring and M be an R-module. A proper submodule N of M is called an r-submodule if $am \in N$ with $\mathrm{Ann}_M(a) = (0)$ implies that $m \in N$ for each $a \in R$ and $m \in M$.

Let R be any ring and let us consider R as a module over itself. Since that the submodules of R are ideals in R, one can easily show that I is an r-ideal if and only if I as a submodule is an r-submodule.

Some preliminary properties of r-submodules are as follows:

Remark 2.2. Let M be an R-module.

(i) The zero submodule of M is an r-submodule.
(ii) The intersection of any family of r-submodules of M is an r-submodule.
(iii)) $\mathrm{Ann}_M(I)$ is an r-submodule of M for any ideal I of R.
(iv) If $f \in \mathrm{End}_R(M)$, then $\ker(f) = \{m \in M : f(m) = 0\}$ is an r-submodule of M.

Let M be an R-module. Recall that if N is a submodule of M and $a \in R$, then $(N : a) = \{m \in M : am \in N\}$ is a submodule of M which contains N. Also if S is a multiplicatively closed subset of R, then $S^{-1}R$ (resp., $S^{-1}M$) is a ring (resp., an $S^{-1}R$-module), which is called the ring (resp., module) of fractions of R (resp., M) with respect to S. Clearly, $S = \mathrm{r}_M(R)$ is a multiplicatively closed subset in R. For more information about the above concepts, see [13]. In the following proposition we give several equivalent definitions for r-submodules. For the proof see Proposition 4 in [9].

Proposition 2.3. *Let M be an R-module and N be a submodule of M. Then the following statements are equivalent.*
(i) *N is an r-submodule.*
(ii) *$aM \cap N = aN$, for each $a \in \mathrm{r}_M(R)$.*
(iii) *$N = (N : a)$, for each $a \in \mathrm{r}_M(R)$.*
(iv) *$N = \mathcal{N}^c$, where $\mathcal{N}$ is a submodule in $S^{-1}M$ and $S = \mathrm{r}_M(R)$.*

Proposition 2.4. *Let $N \subseteq K$ be two submodules of an R-module M. If N is an r-submodule of M and $\frac{K}{N}$ is an r-submodule of R-module $\frac{M}{N}$, then K is an r-submodule of M.*

Proof. Let $a \in R$, $m \in M$ and $am \in K$ with $\mathrm{Ann}_M(a) = (0)$. Clearly, $a(m+N) \in \frac{K}{N}$ and also $\mathrm{Ann}_{\frac{M}{N}}(a) = (0)$. To see this, let $m + N \in \mathrm{Ann}_{\frac{M}{N}}(a)$. Hence $a(m+N) = am + N = N$ which implies that $am \in N$. On the other hand, since $\mathrm{Ann}_M(a) = (0)$ and N is an r-submodule, we have $m \in N$ whence $m + N = N$. Therefore by our hypothesis, we have $m + N \in \frac{K}{N}$ and so $m \in K$. This shows that K is an r-submodule. □

If $f : M \to N$ is an R-module isomorphism, then it is clear that $\mathrm{Ann}_M(a) = (0)$ if and only if $\mathrm{Ann}_N(a) = (0)$, for any $a \in R$.

Proposition 2.5. *r-submodules are invariant under isomorphisms.*

Proof. Let M and N be R-modules and $f : M \to N$ be an R-module isomorphism. We are to show that whenever K is an r-submodule of M, then $f(K)$ is an r-submodule of N. To see this, suppose that $a \in R$, $n \in N$ and $an \in f(K)$ with $\mathrm{Ann}_N(a) = (0)$. Take $m \in K$ and $m_1 \in M$ such that $an = f(m)$ and $n = f(m_1)$. Clearly, $an = af(m_1) = f(am_1) = f(m)$, whence $f(am_1 - m) = 0$ and so $am_1 - m \in \ker(f) = (0)$. Therefore $am_1 = m \in K$. Since $\mathrm{Ann}_M(a) = (0)$, we infer that $m_1 \in K$ and hence $n = f(m_1) \in f(K)$. □

In the following two theorems we observe that every simple submodule and the socle of a module are r-submodules.

Theorem 2.6. *Every simple submodule of a module is an r-submodule.*

Proof. Assume that N is a simple submodule of an R- module M. Therefore there exists $0 \neq m \in N$ such that $N = Rm$. Now let $a \in R$, $x \in M$ and $ax \in N$ with $\mathrm{Ann}_M(a) = (0)$. If $ax = 0$, then $x = 0 \in N$. In case $ax \neq 0$, we have $N = Rax$. Since $am \neq 0$, we infer that $N = Ram$. Consequently, $N = Rax = Ram$ and hence $ax \in Ram$. Therefore there exists $s \in R$ such that $ax = sam$, whence $x - sm \in \mathrm{Ann}_M(a) = (0)$. Thus $x = sm \in Rm = N$ which completes the proof. □

The following corollary is now an immediate consequence of Theorem 2.6.

Corollary 2.7. *If M is a very semisimple R-module (i.e., its every cyclic submodule is simple), then every cyclic submodule of M is an r-submodule.*

Theorem 2.8. *Let M be any R-module. Then* $\mathrm{soc}(M)$ *is an r-submodule.*

Proof. Suppose that $\{N_i : i \in A\}$ be the set of all minimal submodules of M. By definition, we have $\mathrm{soc}(M) = \oplus_{i\in A} N_i$. Now let $a \in R$, $m \in M$ and $am \in \mathrm{soc}(M)$ with $\mathrm{Ann}_M(a) = (0)$. Hence $am = \sum_{k=1}^{n} a_{i_k}$, where $a_{i_k} \in N_{i_k}$, for $i_1, \cdots, i_n \in A$. Without loss of generality, we can assume that $aa_{i_k} \neq 0$, for each i_k. Consequently, $Raa_{i_k} = N_{i_k}$ therefore $am = \sum_{k=1}^{n} ar_{i_k} a_{i_k}$, where $r_{i_k} \in R$, for $k = 1, \cdots, n$. This implies that $m - \sum_{k=1}^{n} r_{i_k} a_{i_k} \in \mathrm{Ann}_M(a) = (0)$, hence $m = \sum_{k=1}^{n} r_{i_k} a_{i_k}$. Therefore $m \in \mathrm{soc}(M)$ and we are done. □

Proposition 2.9. (i) *Every divisible submodule of a module is an r-submodule.*
(ii) *Every direct summand of a module is an r-submodule.*

Proof. (i) Assume that N is a divisible submodule of an R-module M. Let $a \in R$, $m \in M$ and $am \in N$ with $\mathrm{Ann}_M(a) = (0)$. Since N is divisible, there exists $n \in N$ such that $am = an$. Hence $m - n \in \mathrm{Ann}_M(a) = (0)$ and therefore $m = n \in N$.
(ii) Suppose that N is a direct summand of an R-module M. By Lemma 5.6 in [1], there exists $e \in \mathrm{End}_R(M)$ such that $N = \ker(1 - e)$ and $e^2 = e$. This means that N is an r-submodule. □

Example 2.10. In view of Proposition 2.9, injective submodules in any R-module, a fortiori $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$ are r-submodules.

Using the previous proposition we have the next corollary.

Corollary 2.11. (i) *If M is a semisimple R-module, then every submodule of M is an r-submodule.*
(ii) *If M is a Von Neumann regular R-module, then every finitely generated submodule of M is a direct summand and therefore it is an r-submodule as well, see Lemma 1 in* [8].

We recall that if M is an R-module, then $t(M) = \{m \in M : \mathrm{Ann}_R(m) \neq (0)\}$ is called torsion submodule of M. If $t(M) = M$ (resp., $t(M) = (0)$), then M is called torsion (resp., torsion free) module. We also recall that $\mathcal{Z}(M) = \{m \in M : \mathrm{Ann}_R(m)$ is an essential ideal in $R\}$ is a submodule of M, which is called singular submodule. If $\mathcal{Z}(M) = M$, (resp., $\mathcal{Z}(M) = (0)$) then M is called singular (resp., nonsingular) module. In the following results we show that $t(M)$ is always an r-submodule of M and if R is a domain, then $\mathcal{Z}(M)$ is also an r-submodule of M.

Proposition 2.12. *Let M be an R-module. Then the following statements hold.*

(i) *The torsion submodule of M is an r-submodule.*

(ii) *If R is a domain then the singular submodule of M is an r-submodule.*

Proof. (i) Suppose that $am \in t(M)$ with $\mathrm{Ann}_M(a) = (0)$, where $a \in R$ and $m \in M$. By definition of the torsion submodule, we have $\mathrm{Ann}_R(am) \neq (0)$ whence there exists $0 \neq s \in R$ such that $s(am) = a(sm) = 0$. Therefore $sm \in \mathrm{Ann}_M(a) = (0)$ and hence $0 \neq s \in \mathrm{Ann}_R(m)$, that is, $\mathrm{Ann}_R(m) \neq (0)$. This means that $m \in t(M)$.
(ii) Assume that $am \in \mathcal{Z}(M)$ with $\mathrm{Ann}_M(a) = (0)$, where $a \in R$ and $m \in M$. The definition of singular submodule implies that $\mathrm{Ann}_R(am) \cap Rx \neq (0)$, for any $0 \neq x \in R$. Hence there exists $0 \neq s \in R$ such that $(sxm)a = 0$, and thus $sxm \in \mathrm{Ann}_M(a) = (0)$, so $0 \neq sx \in \mathrm{Ann}_R(m) \cap Rx$, i.e., $m \in \mathcal{Z}(M)$. □

The direct sum of two r-submodules may not be an r-submodule, see Example 5.14 in [12]. For a nontrivial idempotent e in R, eM is clearly an r-submodule of M, for manifestly eM is a summand of M. Now the following proposition shows certain direct sum of r-submodules in a module M, which are not necessarily simple submodules is in fact an r-submodule.

Proposition 2.13. *Let M be an R-module and $\{e_i : i \in A\}$ be a set of orthogonal idempotents in R and no finite subset of these idempotents generate R, in the sense that $1 \neq \sum_{i\in B} e_i$, where B is a finite subset of A. Then $N = \oplus_{i\in A} e_i M$ is an r-submodule.*

Proof. Let $am \in N$, where $a \in R$, $m \in M$ with $\mathrm{Ann}_M(a) = (0)$. We are to show that $m \in N$. Clearly, $am = \sum_{k=1}^{n} e_{i_k} m_{i_k}$, where $i_k \in A$ and $m_{i_k} \in M$, for $k = 1, \cdots, n$. Let us put $x = \prod_{k=1}^{n}(1 - e_{i_k})$. It is manifest that $amx = 0$ and hence $mx = 0$. It is now evident that $x = 1 - y$, where $y = \sum_{k=1}^{n} e_{i_k}$. Therefore $m(1 - y) = 0$, so $m = my$. This implies that $m \in N$. □

Definition 2.14. Let R be a ring and M be an R-module. Then

(i) $a \in R$ is said to be m-regular relative to M, if $\mathrm{Ann}_R(a) = (0)$ implies that $\mathrm{Ann}_M(a) = (0)$;

(ii) $a \in R$ is said to be R-regular relative to M, if $\mathrm{Ann}_M(a) = (0)$ implies that $\mathrm{Ann}_R(a) = (0)$.

For example if we consider $R[x]$ as a module over R, then every $a \in R$ is an m-regular element relative to $R[x]$ if and only if it is an R-regular element relative to $R[x]$. Also one can easily see that, if M is an R-module and $\mathrm{Ann}_M(a) = (0)$, for every $a \in R$, then $\mathrm{Ann}_R(M)$ is an r-ideal in R. Note that, in this case, there is no any essential r-submodule in M.

Lemma 2.15. *Suppose that M is an R-module. Then the following statements hold.*

(i) If M is a faithful R-module, then every $a \in R$ is an R-regular element relative to M.

(ii) If M is a finitely generated free R-module, then every $a \in R$ is an m-regular element relative to M.

Proof. (i) Assume that $a \in R$ with $\mathrm{Ann}_M(a) = (0)$ and $s \in \mathrm{Ann}_R(a)$. Hence $sa = 0$ and it is evident that $sam = 0$, for any $m \in M$. Thus $sm \in \mathrm{Ann}_M(a) = (0)$ and therefore $sm = 0$. This implies that $s \in \mathrm{Ann}_R(M) = (0)$, i.e., $\mathrm{Ann}_R(a) = (0)$.

(ii) Assume that $X = \{x_1, \cdots, x_n\}$ is a base for M, $a \in R$ with $\mathrm{Ann}_R(a) = (0)$ and $m \in M$. Now suppose that $m \in \mathrm{Ann}_M(a)$, hence $am = 0$. On the other hand, there exist $s_1, \cdots, s_n \in R$ such that $m = s_1x_1 + \cdots + s_nx_n$. Therefore $as_1x_1 + \cdots + as_nx_n = 0$, and consequently $as_i = 0$, for $i = 1, \cdots, n$. This conclude that $s_1, \cdots, s_n \in \mathrm{Ann}_R(a) = (0)$, therefore $s_i = 0$, for $i = 1, \cdots, n$ and hence $m = 0$. This implies that $\mathrm{Ann}_M(a) = (0)$. □

We should emphasize that any cyclic submodule need not be an r-submodule. For example, the principle ideal $I = \mathbb{Z}4$ in $\mathbb{Z}$ is not an r-ideal and so it is not an r-submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module. Whenever M is a finitely generated free R-module and I is an r-ideal in R, we have the following fact.

Proposition 2.16. *Let M be a finitely generated free R-module with a base X and I be an ideal in R. Then I is an r-ideal in R if and only if IX is an r-submodule of M.*

Proof. Suppose that $X = \{x_1, \cdots, x_n\}$ and $am \in IX$ with $\mathrm{Ann}_M(a) = (0)$, where $a \in R$ and $m \in M$. Take $s_1, \cdots, s_n \in R$ and $t_1, \cdots, t_n \in I$ such that $m = s_1x_1 + \cdots + s_nx_n$ and $am = t_1x_1 + \cdots + t_nx_n$. Hence $as_1x_1 + \cdots + as_nx_n = t_1x_1 + \cdots + t_nx_n$. Therefore $as_i = t_i \in I$, for $i = 1, \cdots, n$. Now by part (i) of the above lemma we have $\mathrm{Ann}_R(a) = (0)$ and so by our hypothesis, we conclude that $s_i \in I$, for $i = 1, \cdots, n$. This means that $m \in IX$. Conversely, suppose that $ax \in I$, with $\mathrm{Ann}_R(a) = (0)$, where $a, x \in R$ and $0 \neq m \in M$. Clearly, $axm \in IX$. Now using part (ii) of the above lemma, we have $\mathrm{Ann}_M(a) = (0)$, whence by our hypothesis, we have $xm \in IX$. This yields that $x \in I$. □

We remind the reader that a submodule N of a module M is called prime (resp., primary) if for each $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $aM \subseteq N$ (resp., $a^nM \subseteq N$ for some $n \in \mathbb{N}$). Also N is called semiprime, if $a^2m \in N$ implies that $am \in N$. Clearly, every submodule is prime if and only if it is both primary and semiprime. Furthermore, if N is a prime r-submodule of M, then $am \in N$ implies that $m \in N$, for every $m \in M$ and $a \in \mathrm{r}_M(R)$. For otherwise, we have $aM \subseteq N$ and so by part (ii) of Proposition 2.3 we conclude that $aM = aN$. This immediately implies that $M = N$ which is not true.

Now similarly to the notion of nonregular ideal, we may define a nonregular submodule.

Definition 2.17. A proper submodule N of an R-module M is called nonregular, if $aM \subseteq N$ implies that $\mathrm{Ann}_M(a) \neq (0)$, for each $a \in R$.

If we consider R as an R-module, then our definition agrees with the concept of nonregular ideal.

Remark 2.18. (i) Every r-submodule of a module is nonregular.
(ii) Every prime nonregular submodule of a module is an r-submodule.

The converse of part (i) of the above remark is not true, in general. For example, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then the submodule $N = \mathbb{Z}\frac{1}{2}$ is a nonregular submodule but it is not an r-submodule. To see this, it is clear that $2.\frac{3}{4} = \frac{1}{2}.3 \in N$ and $\mathrm{Ann}_{\mathbb{Q}}(2) = (0)$ but $\frac{3}{4} \notin N$

We conclude this section with the following proposition.

Proposition 2.19. *Every maximal r-submodule is a prime submodule.*

Proof. Assume that N is a maximal r-submodule of an R-module M. We are to show that N is prime. To see this, let $a \in R$, $m \in M$ and $am \in N$. Since N is an r-submodule, $(N : a)$ is an r-submodule and it is evident that $N \subseteq (N : a)$. Now maximality of N implies that $(N : a) = N$ and hence we have $m \in N$, i.e., N is prime. □

3. uz-modules

This section is devoted to the introduction of the uz-modules and strongly uz-modules. We begin with the following definitions.

Definition 3.1. An R-module M is called a

(i) uz-module, if for every $a \in R$ either $\text{Ann}_M(a) \neq (0)$ or $aM = M$;
(ii) strongly uz-module if for every $a \in R$ we have $aM \subseteq a^2M$ (in fact, $aM = a^2M$).

For instance, the modules $\frac{\mathbb{Q}}{\mathbb{Z}}$, $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ over $\mathbb{Z}$ are strongly uz-modules but $\mathbb{Z}$ as $\mathbb{Z}$-module is not a strongly uz-module. Every strongly uz-module is a uz-module, but the converse is not true, in general. For example, $\mathbb{Z}_4$ as a $\mathbb{Z}_4$-module is a uz-module, but is not a strongly uz-module. The ring of $C(X)$, i.e., the ring of all real-valued continuous functions on a completely regular Hausdorff space X is a strongly uz-module as a $C(X)$-module. Recall that it is possible that $\text{Ann}_M(a) = (0)$ or $aM = M$ for every $0 \neq a \in R$. For example, if we consider $\mathbb{Q}$ as a $\mathbb{Z}$-module then both $\text{Ann}_{\mathbb{Q}}(a) = (0)$ and $a\mathbb{Q} = \mathbb{Q}$ for every $0 \neq a \in \mathbb{Z}$

It is clear that a) every simple module is a strongly uz-module; b) a ring R is a uz-ring (resp., Von Neumann regular ring) if and only if as a module over itself is a uz-module (resp., strongly uz-module); c) if M is a strongly uz-module, then every primary submodule of M is prime and $\text{Ann}_R(M)$ is a semiprime ideal.

Remark 3.2. Let M be an R-module. Then the following statements hold.

(i) If $a \in R$, then $\text{Ann}_M(a) = (0)$ if and only if $\text{Ann}_M(a^n) = (0)$, for any $n \in \mathbb{N}$.
(ii) The zero submodule of M is prime if and only if $\text{Ann}_M(a) = (0)$, for any $0 \neq a \in R$.

Remark 3.3. Let M be a faithful R-module. Then the following statements hold.

(i) If M is a strongly uz-module, then R is a reduced ring. In particular, every Von Neumann regular ring is reduced.
(ii) If M is an Artinian module and the zero submodule of M is prime, then $aM = M$, for any $0 \neq a \in R$. In this case, clearly M is a strongly uz-module. As a consequence we have the well known fact that every Artinian domain is a field.

The next result states that every Von Neumann regular (resp., Artinian) module is a strongly uz-module (resp., uz-module).

Theorem 3.4. (i) *Every Von Neumann regular R-module is a strongly uz-module.*
(ii) *Every Artinian R-module is a uz-module.*

Proof. (i) Assume that M is a Von Neumann regular R-module and $a \in R$. We must show that $aM \subseteq a^2M$. Let $m \in M$, it is sufficient to show that $am \in a^2M$. Put $N = Ram$. Clearly, N is a submodule of M and hence it is a direct summand. Thus there exists a submodule K of M such that $M = N \oplus K$. Hence there exist $r \in R$ and $x \in K$ such that $m = ram + x$. Consequently, $ax = (1 - ra)am \in N \cap K = (0)$. Therefore $am = ra^2m \in a^2M$.
(ii) If $\mathrm{Ann}_M(a) \neq (0)$, for any $a \in R$, then we are done. Hence suppose that there exists $a_0 \in R$ such that $\mathrm{Ann}_M(a_0) = (0)$. Since $a_0M \supseteq a_0^2M \supseteq a_0^3M \supseteq \cdots$, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_0^nM = a_0^{n+1}M$, for any $n \geqslant n_0$. Now take an arbitrary $m \in M$. Hence there exists $x \in M$ such that $a_0^{n_0}m = a_0^{n_0+1}x$. Therefore $a_0^{n_0}(m - a_0x) = 0$ and so $m - a_0x \in \mathrm{Ann}_M(a_0^{n_0}) = (0)$. Thus $m = a_0x \in a_0M$, i.e., $M = a_0M$, which completes the proof. □

Part (i) of the previous theorem conclude that every semisimple module is a strongly uz-module. Also the converse of parts (i) and (ii) is not true, in general. For example $\mathbb{Z}(p^\infty)$ as $\mathbb{Z}$-module is a strongly uz-module but is not a Von Neumann regular $\mathbb{Z}$-module and $\mathbb{Q}$ as $\mathbb{Z}$-module is a strongly uz-module but is not a Artinian $\mathbb{Z}$-module

Recall that an R-module M is called multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In view of Proposition 2.16, it is easy to show that if M is a cyclic free multiplication R-module, then R is a uz-ring if and only if every submodule of M is an r-submodule.

Proposition 3.5. *Let M be a strongly uz-module. Then the following statements hold.*
(i) *Every primary submodule of M is prime.*
(ii) *Every semiprime submodule of M is an r-submodule.*
(iii) *If N is a semiprime submodule of M and $am \in N$, where $a \in R$ and $m \in M$, then either $m \in N$ or $\mathrm{Ann}_M(a) \neq (0)$.*

Proof. (i) It is evident.
(ii) Let $am \in N$, where $a \in R$, $m \in M$ with $\mathrm{Ann}_M(a) = (0)$. Since $aM = a^2M$, there exists $x \in M$ such that $am = a^2x \in N$. Consequently, $m - ax \in \mathrm{Ann}_M(a) = (0)$, implies $m = ax$. On the other hand, since N is semiprime, we have $ax \in N$ and consequently, $m \in N$.
(iii) It is evident. □

An infinite R-module M is called Jónsson module if every proper submodule of M has smaller cardinality than M. Clearly, every simple module is a Jónsson module. It is well known that if M is a Jónsson module, then either $aM = M$ or $aM = (0)$, for each $a \in R$, and moreover $\mathrm{Ann}_R(M)$ is a prime ideal of R, see Proposition 2.5 in [4]. One can easily show that every Jónsson module is a strongly uz-module. For more details about Jónsson modules, see [4].

In the following result, we observe that for any faithful cyclic R-module M, every submodule of M is an r-submodule if and only if M is a uz-module.

Proposition 3.6. *Let M be a faithful cyclic R-module. Then the following conditions are equivalent.*

(i) *M is a uz-module.*

(ii) *Every submodule of M is an r-submodule.*

(iii) *Every cyclic submodule of M is an r-submodule.*

Proof. $(i \Rightarrow ii)$ Suppose that $0 \neq m \in M$ and $M = Rm$. Let N be a submodule of M, $a \in R$, $x \in M$ and $ax \in N$ with $\mathrm{Ann}_M(a) = (0)$. By our hypothesis, we have $aM = M$, that is, $Ram = Rm$. Hence there exists $s \in R$ such that $m = asm$. Therefore $(1 - as) \in \mathrm{Ann}_R(m) = (0)$, so $1 = as$. Thus we conclude that $x = s(ax) \in N$, i.e., N is an r-submodule.
$(ii \Rightarrow i)$ If $aM = M$, for any $a \in R$, then we are done. Otherwise, let there exists $a_0 \in R$ such that $a_0M \neq M$. We are to show that $\mathrm{Ann}_M(a_0) \neq (0)$. Since $a_0M \neq M$, there exists $m_0 \in M$ such that $m_0 \notin a_0M$. By taking that $N = a_0M$, we have $a_0m_0 \in N$ and $m_0 \notin N$. Now if $\mathrm{Ann}_M(a_0) = (0)$, then N is not an r-submodule, which is a contradiction.
$(ii \Rightarrow iii)$ It is evident.
$(iii \Rightarrow ii)$ Let N be a submodule of M, $a \in R$, $x \in M$ and $ax \in N$ with $\mathrm{Ann}_M(a) = (0)$. Now by our hypothesis, the submodule $K = Rax$ is an r-submodule and it is obvious that $K \subseteq N$. Clearly, $x \in K$ and so $x \in N$, that is, N is an r-submodule. □

The condition of "M being a cyclic R-module" for implication $(i \Rightarrow ii)$ is essential, i.e., $(i \Rightarrow ii)$ is not true, in general. For example, $\mathbb{Q}$ as $\mathbb{Z}$-module is a uz-module, but the submodule $N = \mathbb{Z}\frac{1}{2}$ of $\mathbb{Q}$ is not an r-submodule. It is worth to point out that the implication $(ii \Rightarrow i)$ is valid for every R-module. Furthermore, if in the above proposition we consider R as an R-module, then Proposition 3.4 in [12] is an immediate consequence of Proposition 3.6.

Remark 3.7. (i) A ring R is a domain if it has a uz-module whose every nonzero submodule is faithful. To see this, let M be an R-module which is a uz-module with the property mentioned above. Hence, in view of Definition 3.1, we must have $aM = M$, for all

$a \in R$ (note, by our assumption we can not have $\mathrm{Ann}_M(a) \neq (0)$). This immediately implies that R is a domain.
(ii) A ring R is a field if and only if it has a uz-cyclic module whose every nonzero submodule is faithful. To see this, if M is cyclic which is a uz-module with the property mentioned above, it is evident that it is isomorphic to R itself, i.e., R becomes a domain which is at the same time a uz-module over itself (note, the property of being a uz-module or a strongly uz-module is preserved under isomorphism). Consequently, $aR = R$ for all $a \in R$ which is the same thing as R being a field, and we are done. The converse is evident, for it is sufficient to consider R as an R-module.

It is well known that a ring R is a domain if and only if the only r-ideal of R is zero ideal, see Proposition 2.8 in [12]. By replacing the r-ideals with the r-submodules, we get the next interesting fact.

Proposition 3.8. *Let M be a faithful cyclic R-module. Then the following statements are equivalent.*

(i) *R is a domain.*
(ii) *The only r-submodule of M is zero submodule.*
(iii) $\mathrm{Ann}_M(ab) = \mathrm{Ann}_M(a) \cup \mathrm{Ann}_M(b)$, *for every $a, b \in R$.*

Proof. $(i \Rightarrow ii)$ Assume that $0 \neq m \in M$ and $M = Rm$. Let $(0) \neq N$ be an r-submodule of M and $0 \neq n \in N$. Hence there exists $0 \neq a \in R$ such that $n = am$. We claim that $\mathrm{Ann}_M(a) = (0)$. To see this, let $0 \neq x \in M$ such that $ax = 0$. Thus there exists $0 \neq b \in R$ such that $x = bm$. Therefore we have $abm = 0$, so $ab \in \mathrm{Ann}_R(m) = (0)$. This implies that $ab = 0$, which is not true, for R is a domain. Now since N is an r-submodule, we infer that $m \in N$, that is, $M = N$, which is a contradiction.
$(ii \Rightarrow iii)$ Since $\mathrm{Ann}_M(a)$ is an r-submodule, for every $a \in R$, the proof is evident.
$(iii \Rightarrow i)$ Suppose that $a, b \in R$ and $ab = 0$. Hence $M = \mathrm{Ann}_M(0) = \mathrm{Ann}_M(ab) = \mathrm{Ann}_M(a) \cup \mathrm{Ann}_M(b)$. Therefore $M = \mathrm{Ann}_M(a)$ or $M = \mathrm{Ann}_M(b)$. That is $aM = (0)$ or $bM = (0)$. Thus $a \in \mathrm{Ann}(M) = (0)$ or $b \in \mathrm{Ann}(M) = (0)$. This means that $a = 0$ or $b = 0$. □

Propositions 2.12 and 3.8 state that every faithful cyclic module over a domain is both nonsingular and torsion free.

In the following theorem we observe two equivalent conditions for uz-rings. For the other equivalent conditions in terms of r-ideals, see Proposition 3.4 in [12].

Theorem 3.9. *Let R be a ring. Then the following statements are equivalent.*

(i) *R is a uz-ring.*

(ii) *Every faithful R-module is a uz-module.*

(iii) *Every faithful cyclic R-module is a uz-module.*

Proof. $(i \Rightarrow ii)$ Let M be a faithful R-module. In view of Proposition 3.6, it is enough to show that every submodule of M is an r-submodule. Suppose that N is a submodule of M and $am \in N$ with $\text{Ann}_M(a) = (0)$, where $a \in R$ and $m \in M$. In case $a \in \text{u}(R)$, we have $m = a^{-1}am \in N$. If $\text{Ann}_R(a) \neq (0)$, then there exists $0 \neq b \in R$ such that $ab = 0$. Now assume that $x \in M$ is an arbitrary element. Clearly, $abx = 0$, whence $bx \in \text{Ann}_M(a) = (0)$. Hence $bx = 0$, i.e., $0 \neq b \in \text{Ann}_R(M) = (0)$, which is a contradiction.

$(ii \Rightarrow iii)$ It is evident.

$(iii \Rightarrow i)$ Let $M = Rm$ be a faithful cyclic R-module and $a \in R$. If $\text{Ann}_M(a) \neq (0)$, then there exists $0 \neq n \in M$ such that $an = 0$. On the other hand, there exists $0 \neq b \in R$ such that $n = bm$, hence $abm = 0$, whence $ab \in \text{Ann}_R(m) = (0)$, that is $ab = 0$. This means that $a \in \text{zd}(R)$. If $aM = M$, then there exists $m_1 \in M$ such that $m = am_1$. Also there is $t \in R$ such that $m_1 = mt$. Therefore $m = amt$, implies $(1-at)m = 0$, hence $1-at \in \text{Ann}_R(m) = (0)$, that is, $at = 1$. This implies that $a \in \text{u}(R)$. □

We conclude the paper by the following corollary which introduces some r-submodules of a uz-module.

Corollary 3.10. *Let R-module M be a uz-module. Then the following statements hold.*

(i) *Every prime submodule of M is nonregular, and hence it is an r-submodule.*

(ii) $\text{J}(M)$ *is an r-submodule of M.*

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