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Research Paper

ON $\mathbb{Z}G$-CLEAN RINGS

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Abstract. Let R be an associative ring with unity. An element $x \in R$ is called $\mathbb{Z}G$-clean if $x = e + r$, where e is an idempotent and r is a $\mathbb{Z}G$-regular element in R. A ring R is called $\mathbb{Z}G$-clean if every element of R is $\mathbb{Z}G$-clean. In this paper, we show that in an abelian $\mathbb{Z}G$-regular ring R, the $Nil(R)$ is a two-sided ideal of R and $\frac{R}{Nil(R)}$ is G-regular. Furthermore, we characterize $\mathbb{Z}G$-clean rings. Also, this paper is involved with investigating $\mathbb{F}_2C_2$ as a social group and measuring influence a member of it's rather than others.

1. Introduction

Recall an element x of R is called regular (unit regular) if there exists $y \in R$ (a unit $u \in R$) such that $xyx = x$ ($xux = x$). Some properties of regular rings and strongly regular has been studied in [8, 11].

An element $x \in R$ is said to be π-regular if there exist $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. An element $x \in R$ is said to be strongly π-regular if $x^n = x^{2n} y$. The ring

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R is π-regular if every element of R is π-regular and is strongly π-regular if every element of R strongly π-regular. By a result of Azumaya [3] and Dischinger [10], the element x can be chosen to commute with y. In particular, this definition is left-right symmetric. π-regular and strongly π-regular rings, are studied in particular in [3, 4, 5, 6]. A group action (or just action) of G on X is a binary operation:

$$\mu : X \times G \longmapsto X$$

(If there is no fear of confusion, we write $\mu(x, g)$ simply as by x^g) such that
(I) $(x^g)^h = x^{gh}$ for all $x \in X$ and $g, h \in G$,
(II) $x^1 = x$ for all $x \in X$.
Let I be a two sided ideal of R. Then G can acts naturally on $\dfrac{R}{I}$ by the rule $\mu(r + I, g) = \mu(r, g) + I$

An element $x \in R$ is said to be G-regular (resp. strongly G-regular) if there exist $y \in R$ and $g \in G$ such that $x^g = x^g y x^g$ ($x^g = x^{2g} y$). The ring R is G-regular (resp. strongly G-regular) if each element of R is G-regular (resp. strongly G-regular). Basic properties of G-regular (resp. strongly G-regular) rings have been studied in [19]. An element $x \in R$ is said to be $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) if there exist $y \in R$ and $g \in G$, $n \in \mathbb{Z}$ such that $x^{ng} = x^{ng} y x^{ng}$(resp. $x^{(n+1)g} = x^{ng} y$). The ring R is $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) if each element of R is $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular). For example $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ are $\mathbb{Z}G$-regular rings. In [18], we defined the notion of these rings. An element of a ring is called clean if it can be written as the sum of a unit and an idempotent. A ring is clean if each of its element is clean. This notion was introduced by Nicholson in [17] as a sufficient condition for a ring to have the exchange property. In recent years, there have been many investigations concerning variants of the clean properties, see [1, 16]. An element of a ring R is called r-clean if it is the sum of an idempotent and a regular element. A ring R is called r-clean if each of its element is r-clean. Clearly regular rings and clean rings are r-clean. r-clean rings have been studied in [2]. $Id(R)$, $C(R)$, $Nil(R)$, $U(R)$, $J(R)$, $N(R)$ denote the set of all idempotent elements of R, the center of R, the set of all nilpotent elements of R, the set of all unit elements of R, the radical Jacobson of R and the prime radical of R, respectively. Also, $\mathbb{Z}G - Reg(R)$ denotes the set of $\mathbb{Z}G$-regular elements of R. A ring R is called an abelian ring if $Id(R)$ is a subset of $C(R)$.

We use $M_n(R)$ to stand in the ring of all $n \times n$ matrix over a ring R.
Also we define:

$$[a_{ij}]^{ng} = \left[a_{ij}^{ng}\right], a^{g_1+g_2} = a^{g_1} a^{g_2}$$

For each $g, g_1, g_2 \in G$, $n \in \mathbb{Z}$.
For each $n \in \mathbb{Z}$, we mean a^{ng} by $(a^g)^n$. also

$$(x_i)_{i\in I}^g = ({x_i}^g)_{i\in I}$$

Furthermore, if $S \subseteq R$, then we have:

$$S^g = \{x^g | x \in S\}$$

A ring R is said to be boolean in case every element in R is idempotent. Let R be a ring and M be a (left) R-module. M is said to be a simple R-module if $M \neq 0$, and M has no R-submodules other than 0 and M and also M is said to be a semisimple R-module if every R-submodule of M is an R-module direct summand of M. A ring R is simple (semisimple) if it is simple (semisimple) as a (left) module over itself. A ring R is said semiperfect if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. An element x in a ring R is said to be quasiregular if there is some element $y \in R$ such that $x + y = xy = yx$. It is not difficult to see that an element x in a ring R is quasiregular if and only if $1 - x$ is a unit of R. A ring R is said to be orthogonally finite (or sometimes I-finite) there exists no infinite set of orthogonal idempotents in the ring.

2. Abelian $\mathbb{Z}G$-regular rings

In this section, we show that in an abelian $\mathbb{Z}G$-regular ring R, the $Nil(R)$ is a two-sided ideal of R and $R/Nil(R)$ is G-regular.

Lemma 2.1. *Let $x \in R$ be unit regular. Then $x = eu$, for some $e \in Id(R)$ and $u \in U(R)$.*

Proof. Suppose x is unit regular. Then for some $v \in U(R)$ we have $xvx = x$. Let $e = xv \in Id(R)$ and $u = v^{-1}$. Then $x = eu$. □

Lemma 2.2. *Let R be an abelian regular ring. Then R is unit regular.*

Proof. See [[5], Theorem 2]. □

Theorem 2.3. *Let R be an abelian ring and $x \in R$. Then x is $\mathbb{Z}G$-regular if and only if there exists $e \in Id(R)$ such that ex^g is regular and $(1 - e)x^g \in Nil(R)$.*

Proof. If x be $\mathbb{Z}G$-regular, then there exist $n \in \mathbb{Z}$ and $g \in G$ such that x^{ng} is regular. Hence by Lemmas 2.2 and 2.1, $x^{ng} = eu$ for some $e \in Id(R)$ and $u \in U(R)$. Then

$$\begin{aligned} ex^g(x^{(n-1)g}u^{-1})ex^g &= (ex^{ng}u^{-1})ex^g \\ &= (eeuu^{-1})ex^g \\ &= (euu^{-1})ex^g \\ &= e^2x^g \\ &= ex^g \end{aligned}$$

Hence, ex^g is regular. Now $((1-e)x^g)^n = (1-e)x^{ng} = (1-e)eu = 0$, since $(1-e) \in C(R)$ and $x^{ng} = eu$.

Conversely, suppose for some $e \in Id(R)$, ex^g is regular and $(1-e)x^g \in Nil(R)$. For some $n \in \mathbb{Z}$, $((1-e)x^g)^n = (1-e)x^{ng} = 0$. Hence, $ex^{ng} = x^{ng}$. Since ex^g is regular, by Lemma 2.1, $ex^g = cu$ for some $c \in Id(R)$ and $u \in U(R)$. Hence $x^{ng} = ex^{ng} = (ex^g)^n = (cu)^n = cu^n$, since $c \in C(R)$.

Let $y = cu^{-n}$. Then

$$\begin{aligned} x^{ng}yx^{ng} &= x^{ng}cu^{-n}x^{ng} \\ &= cu^n cu^{-n}x^{ng} \\ &= c^2u^n u^{-n}x^{ng} \\ &= cx^{ng} \\ &= ccu^n \\ &= cu^n = x^{ng} \end{aligned}$$

and hence x is $\mathbb{Z}G$-regular. □

Lemma 2.4. *Let R be an abelian ring. Let $x \in R$ such that x is $\mathbb{Z}G$-regular. Then for some $e \in Id(R)$ and $u \in U(R)$ we have $ex^g = eu$.*

Proof. Let R be an abelian ring and $x \in R$ such that x is $\mathbb{Z}G$-regular. Then by [[18], Theorem 2.9] and proof of the Theorem 2.3 for some $e \in Id(R)$ and $v \in U(R)$ and $m \in \mathbb{Z}$ and $g \in G$, we have $x^{mg} = ev$ and ex^g is regular. Hence, by Lemmas 2.1 and 2.2, $ex^g = cw$ for some $c \in Id(R)$ and $w \in U(R)$. In fact, $e = c$. For $e(ex^g) = e(cw)$. But $e(ex^g) = ex^g = cw$. Thus $ecw = cw$ and therefore $ec = c$. Since $e, c \in C(R)$, then $((ex^g))^m = ex^{mg} = cw^m$. Since $x^{mg} = ev$ and $ex^{mg} = eev = ev = cw^m$, hence $e = cw^m v^{-1}$. Thus $ec = cw^m v^{-1}c = cw^m v^{-1} = e$. Since $c \in C(R)$. Hence, $ec = e$. Since $ec = c$ and $ec = e$, then $e = c$. Thus, $ex^g = ew$. □

Theorem 2.5. *Let R be an abelian $\mathbb{Z}G$-regular ring. Then $Nil(R)$ is a two-sided ideal of R.*

Proof. Let $r, w \in R$. Since R is $\mathbb{Z}G$-regular. Thus, there exists $g \in G$ such that $r^g, w^g \in R$. Let $w^g \in Nil(R)$ and $r^g \in R$. Suppose $r^g w^g = (rw)^g$ is not in $Nil(R)$. By Lemma 2.4, there exist $e \in Id(R)$ and $u \in U(R)$ such that $e(rw)^g = r^g e w^g = eu$. Observe that $e \neq 0$. For if $e = 0$ then $(1-e)(rw)^g = (rw)^g \in Nil(R)$ by Theorem 2.3 and this contradicts the assumption that $(rw)^g \notin Nil(R)$. Since $ew^g \in Nil(R)$, let n be the smallest integer such that $(ew^g)^n = 0$. Then $n \neq 2$, since $e \neq 0$. Thus, $0 = r^g ew^g (ew^g)^{n-1} = eu(ew^g)^{n-1} = u(ew^g)^{n-1}$. Hence $(ew)^{n-1} = 0$ a contradiction. Thus, for any $w^g \in Nil(R)$ and $r^g \in R$, we have $r^g w^g \in Nil(R)$. A simillar argument will show that for any $w^g \in Nil(R)$ and $r^g \in R$, we have $w^g r^g \in Nil(R)$. Now, let $w^g, z^g \in Nil(R)$ and suppose $w^g + z^g \notin Nil(R)$. Then, once again, there exist

$0 \neq c \in Id(R)$ and $v \in U(R)$ such that $c(w+z)^g = cv$. Hence $cw^g = cv - cz^g = cv(1 - v^{-1}z^g)$. Since $1 - v^{-1}z^g \in Nil(R)$, $1 - v^{-1}z^g = u \in U(R)$. Thus $cw^g = cvu$. But $cw^g \in Nil(R)$ and $cvu \notin Nil(R)$. Hence $w^g + z^g \in Nil(R)$. Thus $Nil(R)$ is a two-sided ideal of R. □

Before starting the second major result, the following two well-known lemmas are needed.

Lemma 2.6. *Let R be a ring with 1 and I be a two sided nil ideal of R. If $[c] \in Id(R/I)$, then there exists $e \in Id(R)$ such that $[e] = [c]$ in R/I.*

Proof. Since I is a two sided nil ideal of R, then $I \subseteq rad(R)$ [[15], Lemma 4.11] and then the lemma is clear by [[15], Theorem 21.28]. □

Lemma 2.7. *Let I be a two-sided nil ideal of R, $k = R/I$ and $v \in R$. Then $[u] \in U(k)$ if and only if $u \in U(R)$.*

Theorem 2.8. *Let R be an abelian ring. Then R is $\mathbb{Z}G$-regular if and only if $Nil(R)$ is a two-sided ideal of R and $R/Nil(R)$ is G-regular.*

Proof. Suppose R is $\mathbb{Z}G$-regular. By Theorem 2.5, $Nil(R)$ is a two-sided ideal of R. Let $[x] \in R/Nil(R)$. Then, there exist $y \in R$, and $n \in \mathbb{Z}$ and $g \in G$, $x^{ng}yx^{ng} = x^{ng}$. Thus $e = x^{ng}y \in Id(R)$ and therefore $1 - e \in Id(R)$. Since $1 - e \in C(R)$ then $((1-e)x^g)^n = (1-e)x^{ng} = (1 - x^{ng}y)x^{ng} = 0$. Thus $(1-e)x^g = (1-x^{ng}y)x^g \in Nil(R)$. Thus, $[x^g][x^{(n-1)g}][x^g] = [x^{ng}y][x^g] = [x^g]$. Then $R/Nil(R)$ is G-regular.
Suppose $Nil(R)$ is a two-sided ideal of R and $K = R/Nil(R)$ is G-regular. Let $x \in R$ such that $[x]$ is G-regular, then $[x^g]$ is regular, Since $Id(R) \subset C(R)$, by [[18], Theorem 2.8], $[x^g]$ is unit regular. Thus by Lemma 2.1, we have $[x^g] = [c][u]$ for some $[c] \in Id(K)$ and $[u] \in U(K)$. By Lemma 2.6, there exists $e \in Id(R)$ such that $[c] = [e]$, and by Lemma 2.7, $u \in U(R)$.
Thus, $x^g = eu + w$ for some $w \in Nil(R)$. Now, $ex^g = e(eu + w) = eu + ew = e(u + w)$. Since $w \in J(R)$, $u + w \in U(R)$. Thus, ex^g is regular. Further, $(1 - e)x^g = x^g - ex^g = (eu + w) - (eu + ew) \in Nil(R)$. Hence, $(1 - e)x^g \in Nil(R)$. Thus by Theorem 2.3, x is $\mathbb{Z}G$-regular. □

Theorem 2.9. *A ring R is abelian $\mathbb{Z}G$-regular if and only if $Id(R) \subset C(R)$, $Nil(R)$ is a two-sided ideal of R, and for every $x \in R$ there exist $e \in Id(R)$, $u \in U(R)$, $g \in G$ and $w \in Nil(R)$ such that $x^g = eu + w$.*

Proof. It is obvious by Theorem 2.8 and Lemma 2.1. □

Theorem 2.10. *Suppose $Id(R)$ is a subset of $C(R)$. Then R is $\mathbb{Z}G$-regular if and only if for some two-sided nil ideal I of R, $K = R/I$ is $\mathbb{Z}G$-regular.*

Proof. Suppose R is $\mathbb{Z}G$-regular. By Theorem 2.5, $I = Nil(R)$ is a two-sided ideal of R, and by Theorem 2.8, $K = R/I$ is G-regular and hence $\mathbb{Z}G$-regular.
For the converse, assume that R/I is $\mathbb{Z}G$-regular for some two-sided nil ideal I of R. Then $Nil(R/I) = Nil(R)/I$ is a two-sided ideal of R/I by Theorem 2.8. So $Nil(R)$ is a two-sided ideal of R. Since R/I is $\mathbb{Z}G$-regular, so is $R/Nil(R)$. Therefore, by Theorem 2.8, R is $\mathbb{Z}G$-regular. □

A consequence of the above theorem is following corollary.

Corollary 2.11. *Suppose R is an abelian ring. Then R is $\mathbb{Z}G$-regular if and only if $R/N(R)$ is $\mathbb{Z}G$-regular where $N(R)$ is the prime radical of R.*

3. $\mathbb{Z}G$-clean rings

Definition 3.1. An element $x \in R$ is called $\mathbb{Z}G$-clean if $x = e + r$, where e is an idempotent and r is a $\mathbb{Z}G$-regular element in R. A ring R is called $\mathbb{Z}G$-clean if every element of R is $\mathbb{Z}G$-clean.

For example $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_2C_2$ that $\mathbb{F}_2 = \{0, 1\}$ and $C_2 = \{1, x\}$, are $\mathbb{Z}G$-clean rings.

Proposition 3.2. *Every homomorphic image of a $\mathbb{Z}G$-clean ring is $\mathbb{Z}G$-clean.*

Proof. Since multiplication is preserved by every ring homomorphism, the homomorphic image of $\mathbb{Z}G$-regular (resp. idempotent) is $\mathbb{Z}G$-regular (resp. idempotent) of its ring. Since addition is also preserved by every ring homomorphism, the result follows. □

Remark 3.3. Inverse of above theorem may not be correct. For example $\mathbb{Z}_4 \cong \mathbb{Z}/4\mathbb{Z}$ ($\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$) is $\mathbb{Z}G$-clean, but $\mathbb{Z}$ is not $\mathbb{Z}G$-clean.

Proposition 3.4. *A finite direct product $\prod_{i\in I} R_i$ (I is a finite set) of $\mathbb{Z}G$-clean rings $\{R_i\}_{i\in I}$ is $\mathbb{Z}G$-clean.*

Proof. Since multiplication in a direct product of rings is defined componentwize, an element in a direct product of rings is a $\mathbb{Z}G$-regular (resp. idempotent) of that ring if and only the entry in each of its components is a $\mathbb{Z}G$-regular (resp. idempotent) of its ring. Since addition in a direct product of rings is also defined componentwise, the result follows from a simple computation. □

Proposition 3.5. *Let R be an orthogonally finite ring and $J \leq K$ be two sided ideals in a ring R. If J and K/J are both $\mathbb{Z}G$-clean, then K is $\mathbb{Z}G$-clean.*

Proof. Suppose that J and K/J are both $\mathbb{Z}G$-clean. Given $x \in K$, it follows from the $\mathbb{Z}G$-cleaness of K/J that $x - e - r \in J$ for some $e \in Id(K)$, $r \in \mathbb{Z}G - Reg(K)$, consequently, $x - e - r = f + s$, $f \in Id(J)$, $s \in \mathbb{Z}G - Reg(J)$, then $x = (e + f) + (r + s)$. Since R is orthogonally finite, then $(e + f) \in Id(K)$ and also $(r + s) \in \mathbb{Z}G - Reg(K)$. □

Proposition 3.6. *Let R be a ring, if e is a central idempotent element of R and eRe and $(1 - e)R(1 - e)$ are both $\mathbb{Z}G$-clean, then so is R.*

Proof. For any idempotent e in a ring, we have the Peirce decomposition:

$$R = eRe \oplus eRf \oplus fRe \oplus fRf$$

Where $f = 1 - e$ is the complementary idempotent to e. Since e, f are central idempotents, then we have the Peirce decomposition:

$$R = eRe \oplus fRf \cong \begin{bmatrix} eRe & 0 \\ 0 & fRf \end{bmatrix}$$

So each $A \in R$ is the from $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where a, b belong to eRe, fRf respectively. a, b are $\mathbb{Z}G$-clean by hypothesis thus $a = r_1 + e_1$, $b = r_2 + e_2$, where r_1, r_2 are $\mathbb{Z}G$-regular and e_1, e_2 are idempotent. So

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} r_1 + e_1 & 0 \\ 0 & r_2 + e_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} + \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$$

Since r_1, r_2 are $\mathbb{Z}G$-regular, so there exist $y_1, y_2 \in R$, $g_1, g_2 \in G$ and $n_1, n_2 \in \mathbb{Z}$, such that, $r_1^{n_1g_1} y_1 r_1^{n_1g_1} = r_1^{n_1g_1}$, $r_2^{n_2g_2} y_2 r_2^{n_2g_2} = r_2^{n_2g_2}$, therefore we have:

$$\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}^{n_1g_1+n_2g_2} \begin{bmatrix} r_1^{-n_2g_2} y_1 & 0 \\ 0 & r_2^{-n_1g_1} y_2 \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}^{n_1g_1+n_2g_2} =$$

$$\begin{bmatrix} r_1^{n_1g_1} y_1 & 0 \\ 0 & r_2^{n_2g_2} y_2 \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}^{n_1g_1+n_2g_2} =$$

$$\begin{bmatrix} r_1^{n_1g_1} y_1 r_1^{n_1g_1} r_1^{n_2g_2} & 0 \\ 0 & r_2^{n_2g_2} y_2 r_2^{n_2g_2} r_2^{n_1g_1} \end{bmatrix}^{n_1g_1+n_2g_2} =$$

$$\begin{bmatrix} r_1^{n_1g_1} r_1^{n_2g_2} & 0 \\ 0 & r_2^{n_2g_2} r_2^{n_1g_1} \end{bmatrix} =$$

$$\begin{bmatrix} r_1^{n_1g_1+n_2g_2} & 0 \\ 0 & r_2^{n_1g_1+n_2g_2} \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}^{n_1g_1+n_2g_2}$$

So $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ is $\mathbb{Z}G$-regular. □

Proposition 3.7. *Let R be a $\mathbb{Z}G$-clean ring and e be a central idempotent in R. Then eRe is also $\mathbb{Z}G$-clean.*

Proof. Since e is a central, it follows that eRe is homomorphic image of R. Hence, the result follows from Proposition 3.2. □

Corollary 3.8. *Let $e_1, e_2, ..., e_n$ be orthogonal central idempotents with $e_1 + e_2 + ... + e_n = 1$. Then e_iRe_i is $\mathbb{Z}G$-clean for each i, if and only if so is R.*

Proof. One direction of corollary follows Propositoin 3.6 by induction the other direction follows from Proposition 3.7. □

Theorem 3.9. *A full matrix ring $M_n(R)$ is $\mathbb{Z}G$-clean if the underlying ring R is $\mathbb{Z}G$-clean.*

Proof. Since R is a $\mathbb{Z}G$-clean ring by Collorary 3.8, e_iRe_i is $\mathbb{Z}G$-clean for each i and since the set of matrix units $\{E_{ii}\}_{i=1}^n$ is a complete set of orthogonal idempotents for $M_n(R)$ with each corner ring $E_{ii}M_n(R)E_{ii}$ isomorphic to R, the result follows. □

Proposition 3.10. *Let A,B be two rings, ${}_AC_B$ a bimodule and $R = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$. Then R is $\mathbb{Z}G$-clean if and only if both A and B are $\mathbb{Z}G$-clean.*

Proof. If R is $\mathbb{Z}G$-clean, then

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix} = \begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix} + \begin{bmatrix} e_a & 0 \\ e_c & e_b \end{bmatrix}$$

Where

$$(\begin{bmatrix} e_a & 0 \\ e_c & e_b \end{bmatrix})^2 = \begin{bmatrix} e_a & 0 \\ e_c & e_b \end{bmatrix}$$

and

$$\begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix} \in \mathbb{Z}(G) - reg(R)$$

Then there exists

$$\begin{bmatrix} y_1 & 0 \\ y_2 & y_3 \end{bmatrix} \in R$$

Where

$$\begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix}^{ng} \begin{bmatrix} y_1 & 0 \\ y_2 & y_3 \end{bmatrix} \begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix}^{ng} = \begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix}^{ng}$$

Then

$$\begin{bmatrix} r_a^{ng} y_1 r_a^{ng} & 0 \\ r_c^{ng} y_1 r_a^{ng} + r_b^{ng} y_2 r_a^{ng} + r_b^{ng} y_3 r_c^{ng} & r_b^{ng} y_3 r_b^{ng} \end{bmatrix} = \begin{bmatrix} r_a & 0 \\ r_c & r_b \end{bmatrix}^{ng}$$

So $r_a, r_b \in \mathbb{Z}G - reg(R)$ and $e_a, e_b \in Id(R)$ and we have $a = r_a + e_a, b = r_b + e_b$. □

In particular, induction shows that for each $n \geq 1$, a ring R is $\mathbb{Z}G$-clean if and only if the ring of all $n \times n$ lower triangular matrixs over R is $\mathbb{Z}G$-clean.

Lemma 3.11. *Let R be a commutative ring and $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be $\mathbb{Z}G$-regular. Then a_0 is $\mathbb{Z}G$-regular and for each i, a_i^{ng} is nilpotent for some $n \in \mathbb{Z}$ and $g \in G$.*

Proof. Since $f(x)$ is $\mathbb{Z}G$-regular, thus there exists $h(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ such that $f^{ng} h f^{ng} = f^{ng}$. So $a_0^{ng} b_0 a_0^{ng} = a_0^{ng}$. Therefore a_0 is $\mathbb{Z}G$-regular. Now we show that for each i, a_i^{ng} is nilpotent for some $n \in \mathbb{Z}$ and $g \in G$. It is enough to show that for each prime ideal P of R; every $a_i \in P$. Since P is prime, thus $R/P[x]$ is an integral domain. Define $\lambda : R[x] \longrightarrow R/P[x]$ by $\lambda(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} (a_i + P) x^i$. λ is an epimorphism. Then $\lambda(f^{ng})\lambda(h)\lambda(f^{ng}) = \lambda(f^{ng})$, so $(\lambda(f))^{ng})\lambda(h)(\lambda(f))^{ng}) = (\lambda(f))^{ng}$, therefore $deg((\lambda(f))^{ng})\lambda(h)(\lambda(f))^{ng})) = deg((\lambda(f))^{ng}))$. Thus $deg((\lambda(f))^{ng}) + \lambda(h) + deg((\lambda(f))^{ng}) = deg((\lambda(f))^{ng})$. Then $deg((\lambda(f))^{ng}) + \lambda(h) = 0$. So $deg((\lambda(f))^{ng}) = 0$, thus $\lambda(f^{ng})$. Therefore $a_1^{ng} + P = a_2^{ng} + P = ... = a_m^{ng} + P$. □

Theorem 3.12. *If R is a commutative ring. Then $R[x]$ is not $\mathbb{Z}G$-clean.*

Proof. Suppose that $x = r + e$, where $r \in \mathbb{Z}G - Reg(R)$ and $e \in Id(R)$. Since $Id(R) = Id(R[x])$, then $x - e = r$ is $\mathbb{Z}G$-regular. Hence, by Lemma 3.11, $1 = 1^{ng}$ should be nilpotent for some $n \in \mathbb{Z}$ and $g \in G$, which is a contradiction. □

Theorem 3.13. *Let R be a ring. Then ring $R[[x]]$ is $\mathbb{Z}G$-clean if and only if so is R.*

Proof. If $R[[x]]$ is $\mathbb{Z}G$-clean, then $R \cong R[[x]]/(x_i)$ is $\mathbb{Z}G$-clean. Conversly, suppose that R is $\mathbb{Z}G$-clean. $R[[x]] \cong \{(a_i) : a_i \in R, \forall i \geq 0\} = \prod_{i\geq 0} R_i$. Therefore by Theorem 3.4, $R[[x]]$ is a $\mathbb{Z}G$-clean ring. □

Lemma 3.14. *The $\mathbb{Z}G$-regular elements of any ring are $\mathbb{Z}G$-clean.*

Proof. Since 0 is idempotent, any $\mathbb{Z}G$-regular r can be written as the sum of a $\mathbb{Z}G$-regular and an idempotent by written $r = r + 0$. □

Proposition 3.15. *Every $\mathbb{Z}G$-regular ring is $\mathbb{Z}G$-clean.*

Proof. This follows immediately from Lemma 3.14. □

Remark 3.16. $\mathbb{Z}G$-clean rings may not be $\mathbb{Z}G$-regular.

Example 3.17. $\mathbb{Z}_4$ is not $\mathbb{Z}G$-regular, because 2 is not $\mathbb{Z}G$-regular in $\mathbb{Z}_4$, but it is easy to check that $\mathbb{Z}_4$ is $\mathbb{Z}G$-clean.

Corollary 3.18. *Every strongly π-reguler ring is $\mathbb{Z}G$-clean.*

Proof. It follows from Azumaya [[3] ,Theorem 3] that a ring R is strongly π-regular if and only if for every element $r \in R$ there is some $z \in R$ such that $r^n = r^n z r^n$ with $rz = zr$ for some positive integer n. It is clear that if G is a trivial group (group with only one element), then R is a $\mathbb{Z}G$-regular ring. Then by Proposition 3.15, R is $\mathbb{Z}G$-clean. □

Example 3.19. The converse of the Corollary 3.18 is false. Let F be a field, and $F(X)$ the field of fractions of the polynomial ring $F[X]$. Extend $\{X^n | n \in \mathbb{Z}\}$ to a basis β of $F(X)$ over F. Let T be the free prouduct of $F(X)$ with the (unital) free algebra on two elements $F\langle A, B\rangle$. Let $V = \{BwA | w \in \beta - \{X^n | n < 0\}\} \subseteq T$, and P be the ideal of T generated by A^2, B^2, AwA, AwB, BwB, for all $w \in \beta - \{1\}$, V, and $\bigcup_{k=1}^{\infty}\{(BX^{-1}A)^{n_k}, (BX^{-2}A)^{n_k}, ..., (BX^{-k}A)^{n_k}\}$, where $n_k > 2^k + 1$. Set $R = T/P$ and $S = M_2(R)$. By [[9], Example 3.4], S is $\mathbb{Z}G$-clean. But this ring is not strongly π-reguler.

Lemma 3.20. *The unit elements of any ring are $\mathbb{Z}G$-clean.*

Proof. Since any unit element is regular and any regular element is $\mathbb{Z}G$-regular so from Lemma 3.14 is trivial. □

Lemma 3.21. *Every quasiregular element is $\mathbb{Z}G$-clean.*

Proof. Any quasiregular element x can be written as the sum of a unit and an idempotent by writing $x = (x-1)+1$ since an element x is quasiregular if and only if $x-1$ a unit. also from proof of lemma 3.20, any unit element is $\mathbb{Z}G$-regular, therefore the quasiregular element is $\mathbb{Z}G$-clean. □

Lemma 3.22. *The idempotent elements of any ring are $\mathbb{Z}G$-clean.*

Proof. Let $e \in Id(R)$, then $1-e \in Id(R)$. We can write $e = (2e-1)+(1-e)$ and since $(2e-1)(2e-1) = 1$, then $(2e-1) \in U(R)$ and also $(2e-1)$ is $\mathbb{Z}G$-regular. Therefore e is $\mathbb{Z}G$-clean. □

Proposition 3.23. *Every division ring, local ring, boolean ring is $\mathbb{Z}G$-clean.*

Proof. This result follows immediately from lemmas 3.20 and 3.21 and 3.22 since all division rings, local rings and boolean rings consist entirely of units, quasiregulars and idempotent elements. □

Theorem 3.24. *Let R be a ring. Then $x \in R$ is $\mathbb{Z}G$-clean if and only if $1-x$ is $\mathbb{Z}G$-clean.*

Proof. If is $\mathbb{Z}G$-clean. Then we have $x = r+e$, where r is $\mathbb{Z}G$-regular and e is an idempotent. Thus $1-x = -r+(1-e)$. In $\mathbb{Z}G$-regular rings there exist $n \in \mathbb{Z}$ and $y \in R$ and $g \in G$ such that $r^{ng} = r^{ng}yr^{ng}$. Hence when n is odd we have

$$(-r)^{ng} = (-r)^{ng}(-y)(-r)^{ng}$$

And when n is even,

$$(-r)^{ng} = (-r)^{ng}y(-r)^{ng}$$

Then $-r$ is $\mathbb{Z}G$-regular. Since $1-e$ is idempotent, we have $1-x$ is $\mathbb{Z}G$-clean. Conversely, since $x = 1-(1-x)$ and $(1-x)$ is $\mathbb{Z}G$-clean by the first part of the proof, we deduce x is $\mathbb{Z}G$-clean. □

Corollary 3.25. *Let R be a ring and $x \in J(R)$. Then x is $\mathbb{Z}G$-clean.*

Proof. Let $x \in J(R)$. Then $1-x \in U(R)$. Hence, by Lemme 3.20, $1-x$ is $\mathbb{Z}G$-clean. Therefore, by Theorem 3.24, x is $\mathbb{Z}G$-clean. □

Proposition 3.26. *Every semisimple ring is $\mathbb{Z}G$-clean.*

Proof. Since every semisimple ring is isomorphic to a finite direct product of full matrix rings over division rings, this is the Wedderburn-Artin Theorem, Since every division ring is $\mathbb{Z}G$-clean by Proposition 3.23, every full matrix ring with entries from a division ring is $\mathbb{Z}G$-clean by Theorem 3.9. Since every direct product of $\mathbb{Z}G$-clean rings are $\mathbb{Z}G$-clean by Theorem 3.4, the result now follows. Therefore, every semisimple ring is $\mathbb{Z}G$-clean. □

Theorem 3.27. *Every clean ring is $\mathbb{Z}G$-clean.*

Proof. Since the ring is clean, then every element in its can be written as the sum of a unit and an idempotent of the ring. And since every unit element is a $\mathbb{Z}G$-regular element, then every element in its can be written as the sum of a $\mathbb{Z}G$-regular and an idempotent of the ring, so the ring is $\mathbb{Z}G$-clean. □

Corollary 3.28. *$\mathbb{Z}G$-clean rings may not be clean.*

Example 3.29. Let F be a field with $char(F) \neq 2$, $A = F[[x]]$ and K be the field of fractions of A. All the ideals of A are generated by power of x, denote by (x^n). Define:

$$R = \{r \in End(A_F) : \text{there exist } q \in K \text{ and a positive integer } n, \text{ with } r(a) = qa \text{ for all } a \in (x^n)\}$$

By [[12], Example 1], R is a regular ring and then $\mathbb{Z}G$-regular. So by Proposition 3.15, R is a $\mathbb{Z}G$-clean ring. But by [2], R is not clean.

Corollary 3.30. *Every semiperfect ring is $\mathbb{Z}G$-clean.*

Proof. By [7] any semiperfect ring is clean and by Theorem 3.27, every clean ring is $\mathbb{Z}G$-clean. □

Example 3.31. The ring $\mathbb{Z}_{(2)}$ of all rational numbers with odd denominators (when written in lowest terms) is semiperfect, and that the infinite direct product $\mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times ...$ is clean (see [14]), and then is $\mathbb{Z}G$-clean, but not semiperfect.

We have follow diagram:

$$\text{semiperfect ring} \longrightarrow \text{clean ring} \longrightarrow r\text{-clean ring} \longrightarrow \mathbb{Z}G\text{-clean ring}$$

Theorem 3.32. *If $R \neq 0$ is a reduced directly finite $\mathbb{Z}G$-clean ring and 0 and 1 are the only idempotent in R. Then R is clean.*

Proof. Since R is $\mathbb{Z}G$-clean, each $x \in R$ has the form $x = r + e$, where r is $\mathbb{Z}G$-clean and e is an idempotent. If $r = 0$, then $x = e = (2e - 1) + (1 - e)$ and since $2e - 1$ is a unit and $1 - e$ is an idempotent, we have x is clean. But if $r \neq 0$, then there exist a positive integer n and $y \in R$ and $1 \in G$ such that $r^n = r^n y r^n$. Thus $r^n y$ is an idempotent, and so by hypothesis $r^n y = 0$ or $r^n y = 1$. If $r^n y = 0$, then $r^n = r^n y r^n = 0$, and therefore $r = 0$, since R is reduced, which is a contradiction, hence $r^n y = 1$ and since R is directly finite so r is invertible. Thus x is clean. Hence R is a clean ring. □

Corollary 3.33. *Let R be a reduced directly finite ring and 0 and 1 are the only idempotent in R. Then the following statements are equivalent:*
(I) R is semiperfect.
(II) R is clean.
(III) R is $\mathbb{Z}G$-clean.

Proof. This result follows immediately from [7] and Theorem 3.32. □

4. Application of $\mathbb{Z}G$-clean and strongly $\mathbb{Z}G$-clean elements in social groups

These days graph models are finding many applications in different fields of science and technology such as computer science, topology, operation research, biological and social sciences.

In a social group, it is observed that some people can influence others and it can happen only when there is a strong relationship between them. Now Let $\mathbb{F}_2C_2$ to be a social group with a, b, c, d such that $a = 0$, $b = 1$, $c = x$, $d = 1 + x$. Let $R = F_2 = \mathbb{Z}_2$ and $G = C_2$ ($\mathbb{Z}_2$ is the ring of order 2, which is a field). Writing down the elements:

$\mathbb{F}_2 = \{0, 1\}$ and $C_2 = \{1, x\}$

$\mathbb{F}_2C_2 = \{\sum_{g\in G} a_g g \mid a_g \in \mathbb{F}_2\}$

$= \{0_{\mathbb{F}_2}.1_{C_2} + 0_{\mathbb{F}_2}.x, 1_{\mathbb{F}_2}.1_{C_2} + 0_{\mathbb{F}_2}.x, 0_{\mathbb{F}_2}.1_{C_2} + 1_{\mathbb{F}_2}.x, 1_{\mathbb{F}_2}.1_{C_2} + 1_{\mathbb{F}_2}.x\}$

$= \{0_{\mathbb{F}_2C_2}, 1_{\mathbb{F}_2C_2}, 1_{\mathbb{F}_2}.x, 1_{\mathbb{F}_2}.1_{C_2} + 1_{\mathbb{F}_2}.x\}$

$= \{0, 1, x, 1 + x\}$

Not that . is $\mathbb{F}_2$ module multiplication. Now let us construct the Cayley tables for $\mathbb{F}_2C_2$.

+	0	1	x	1+x
0	0	1	x	1+x
1	1	0	1+x	x
x	x	1+x	0	1
1+x	1+x	x	1	0

.	0	1	x	1+x
0	0	0	0	0
1	0	1	x	1+x
x	0	x	1	1+x
1+x	0	1+x	1+x	1+x

Table 1. Cayley tables

$Id(\mathbb{F}_2C_2) = \{0, 1\}$, since by Table 1, we have:
$0^2 = 0$ and $1^2 = 1$
An element $a \in R$ is said to be $\mathbb{Z}G$-regular if there exist a positive integer n and $b \in R$ and $g \in G$ such that $a^{ng} = a^{ng}ba^{ng}$. Then by Table 2, we have:

a	b
0	0,1,x,1+x
1	1
x	1,x
1+x	0,1,x,1+x

Table 2. Element ZG-regular (g=1)

$\mathbb{Z}G - Reg(\mathbb{F}_2C_2) = \{0, 1, x, 1 + x\}$
Where $g = 1$. By Definition 3.1, and the relation of idempotent and $\mathbb{Z}G$-regular elements of $\mathbb{F}_2C_2$ (see Table 3), so:

ZG-clean elements	Idempotent elements +	ZG-regular elements
0	1	1
	0	0
1	0	1
	1	0
x	0	x
	1	$1 + x$
$1 + x$	1	x
	0	$1 + x$

Table 3. Relation of idempotent and ZG-regular elements (g=1)

$\mathbb{Z}G$-Clean elements of $\mathbb{F}_2C_2$ will be follows:

$\mathbb{F}_2C_2 = \{0, 1, x, 1 + x\}$

Then $\mathbb{F}_2C_2$ is $\mathbb{Z}G$-clean. Let us consider a directed graph of this social group.

The graph of this social group will be as follow:

$G = (V, E)$

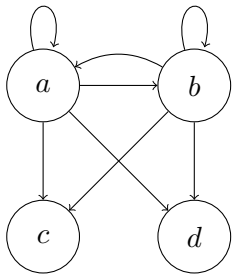
$V = \{a, b, c, d\}$, $E = \{aa, bb, ab, ba, ac, bd, bc, ad\}$

The Table 4 show degree of vertices in this graph:

Vertices	Input deg	Output deg	Deg
a	2	4	6
b	2	4	6
c	2	0	2
d	2	0	2

Table 4. Degree of vertices

Graph shape shows members a, b in this social group are more influence than others, because by Table 4, degree of vertices a and b are 6, and if members a and b was removed from graph of social group, then none of the members of this social group did not have the property $\mathbb{Z}G$-clean.



5. Conclusions

In this paper, abelian $\mathbb{Z}G$-regular and abelian strongly $\mathbb{Z}G$-regular rings have been investigated. Results have contained a description of these rings. Also in this research, some properties of $\mathbb{Z}G$-clean and $\mathbb{Z}G$-clean rings have been introduced. Measurement theory has been considered as a useful mean to study the kinds of things that can be measured. Furthermore, measure of influence of a member rather than the others in a group ring as a social group has been proposed.

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