



Research Paper

**LIMITS AND COLIMITS IN THE CATEGORY OF  
 PRE-DIRECTED COMPLETE PRE-ORDERED SETS**

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**ABSTRACT.** In this paper, some categorical properties of the category  $\text{Pre-Dcpo}$  of all pre-dcpo; pre-ordered sets which are also pre-directed complete, with pre-continuous maps between them is considered. In particular, we characterize products and coproducts in this category. Furthermore, we show that this category is neither complete nor cocomplete. Also, epimorphisms and monomorphisms in  $\text{Pre-Dcpo}$  are described. Finally, some adjoint relations between the category  $\text{Pre-Dcpo}$  and others are considered. More precisely, we consider the forgetful functors between this category and some well-known categories, and study the existence of their left and right adjoints.

1. INTRODUCTION AND PRELIMINARIES

Directed complete partially ordered sets play an important role in domain theory was introduced in the 1970s by D. S. Scott as a foundation for program semantics. The categorical,

DOI: 10.29252/as.2020.1833

MSC(2010): 06A11, 06F99, 08A02, 18A30,18A40.

Keywords: Pre-dcpo, Product, Coproduct, Free, Cofree.

Received: 30 Oct 2019, Accepted: 13 Jun 2020.

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algebraic, and logical properties of dcpos, directed complete partially ordered sets, with directed join-preserving maps between them have been studied in some papers and books, see for example [1], [4], [3], [10].

It has been shown that all colimits of dcpos exist but their description is rather complicated. In order to describe colimits one needs to have information about dcpo congruences, but there was no theory of congruences in the literature. So in [5, 6] we have studied and characterized the congruences of directed complete partially ordered sets. In there, we defined the concept of a pre-dcpo which has a major role in characterizing dcpo congruences. Thus to get more information about pre-dcpos, we will study some categorical properties of the category **Pre-Dcpo** of pre-dcpos in this paper. In particular, we describe products and coproducts in this category. Among other things, we show that equalizers and pullbacks do not exist necessarily in this category and then so it is not a complete category. Moreover, it is also proved that this category is not cocomplete. Furthermore, epimorphisms and monomorphisms in **Pre-Dcpo** are characterized. Finally, we take the forgetful functors between this category and the categories of dcpos, posets, pre-ordered sets and sets, and study the existence of their left and right adjoints.

Let us now give some preliminaries needed in the sequel.

A relation  $\leq$  on a set  $P$  is called a *pre-order* if  $\leq$  is reflexive and transitive (but not necessary anti-symmetric). In this case  $(P, \leq)$  is said to be a pre-ordered set. The category of all pre-ordered sets with pre-order preserving maps between them will be denoted by **Pre-Set**.

Let  $(P, \leq)$  be a pre-ordered set. Then we define the relation  $\sim_{\leq}$  (or simply  $\sim$ ) on  $P$  as  $\leq \cap \geq$ . In other words,  $x \sim y$  if and only if  $x \leq y$  and  $y \leq x$ , for  $x, y \in P$ . Recall that  $\sim_{\leq}$  is an equivalence relation on  $P$ . This is the largest equivalence relation on  $P$  which is contained in the pre-order  $\leq$ .

It is a well-known fact that  $(P / \sim, \leq / \sim)$  is a poset, where for every  $X, Y \in P / \sim$ ,  $X(\leq / \sim)Y$  if and only if  $x \leq y$  for some  $x \in X$  and  $y \in Y$  if and only if  $x \leq y$  for each  $x \in X$  and  $y \in Y$ .

Next, let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset  $D$  of a partially ordered set is called *directed*, denoted by  $D \subseteq^d P$ , if for every  $a, b \in D$  there exists  $c \in D$  such that  $a, b \leq c$ ; and  $P$  is called *directed complete*, or briefly a *dcpo*, if for every  $D \subseteq^d P$ , the directed join  $\bigvee^d D$  exists in  $P$ .

A *dcpo map* or a *continuous map*  $f: P \rightarrow Q$  between dcpos is a map with the property that for every  $D \subseteq^d P$ ,  $f(D)$  is a directed subset of  $Q$  and  $f(\bigvee^d D) = \bigvee^d f(D)$ . Thus we have the category **Dcpo** of all dcpos with continuous maps between them.

In the following, we recall from [5, 6] the following simple generalization of dcpos, namely, pre-dcpo.

**Definition 1.1.** Let  $(P, \leq)$  be a pre-ordered set,  $A \subseteq P$  and  $p \in P$ .

- (a) We say that  $p$  is an *upper pre-bound* of  $A$  if  $a \leq p$  for each  $a \in A$ .
- (b) An upper pre-bound  $p$  of  $A$  is called a *pre-supremum* (or a *pre-join*) if  $p \leq b$  for each upper pre-bound  $b$  of  $A$ .

In contrast to the case of the supremum in posets, there may be more than one pre-supremum for a given subset of a pre-ordered set, in general.

The set of all pre-supremums of a subset  $A$  of  $P$ , will be denoted by  $\text{presup}_P A$  or simply  $\text{presup} A$ . Note that  $p \sim q$  for each two  $p, q \in \text{presup} A$ . Moreover, if  $p \in \text{presup} A$  and  $r \sim p$ , then  $r \in \text{presup} A$ .

**Definition 1.2.** Let  $(P, \leq)$  be a pre-ordered set.

- (a) A non-empty subset  $D \subseteq P$  is called *pre-directed* (denoted by  $D \subseteq^{pd} P$ ) if for every  $a, b \in D$  there exists  $c \in D$  such that  $a, b \leq c$ .
- (b)  $(P, \leq)$  is called *pre-directed complete*, or briefly a *pre-dcpo*, if each pre-directed subset  $D \subseteq^{pd} P$  has at least one pre-supremum.

**Definition 1.3.** Let  $P$  and  $Q$  be pre-dcpo. We say that a map  $f: P \rightarrow Q$  is a *pre-dcpo map* if, for each pre-directed set  $D \subseteq^{pd} P$  and every pre-supremum  $s$  of  $D$ , the set  $f(D)$  is a pre-directed subset of  $Q$  and  $f(s)$  is a pre-supremum of  $f(D)$ .

The category of all pre-dcpo and all pre-dcpo maps between them will be denoted by **Pre-Dcpo**.

## 2. PRODUCTS AND COPRODUTS IN THE CATEGORY **Pre-Dcpo**

In this section, we give the description of products, coproducts and terminal object in the category **Pre-Dcpo**. We also show that pullbacks and equalizers do not exist necessarily in this category and so it is not a complete category.

**Remark 2.1.** In the category **Pre-Dcpo**, the terminal object is the one-element object.

**Lemma 2.2.** The Cartesian product of a family of pre-dcpo, with the componentwise pre-order, is a pre-dcpo.

*Proof.* Let  $\{A_i : i \in I\}$  be a family of pre-dcpo's and  $A =: \prod_{i \in I} A_i$ . We show that  $A$  with the componentwise pre-order is a pre-dcpo. To see this, let  $D \subseteq^{pd} A$ . We prove that  $(s_i)_{i \in I}$  is a pre-supremum of  $D$ , where each  $s_i$  is a pre-supremum of the pre-directed set  $D_i = \{a \in A_i : \exists d = (d_k)_{k \in I} \in D, a = d_i\}$ , for all  $i \in I$ . First, we show that  $D_i$  is a pre-directed subset of  $A_i$ , for all  $i \in I$ . Note that, for all  $i \in I$ ,  $D_i = \pi_i(D)$ , where  $\pi_i : A \rightarrow A_i$  is the  $i$ -th projection. Next, for all  $i \in I$ ,  $\pi_i$  is a pre-order preserving map and so maps a pre-directed subset of  $A$  to a pre-directed subset of  $A_i$ . Hence, for all  $i \in I$ ,  $D_i = \pi_i(D)$  is a pre-directed subset of  $A_i$ . On the other hand, each  $A_i$  is a pre-dcpo and so each  $D_i$  has at least a pre-supremum  $s_i$  in  $A_i$ . Now, we show that  $(s_i)_{i \in I}$  is a pre-supremum of  $D$  in  $A$ . To see this, let  $(d_i)_{i \in I} \in D$ . Then we have  $d_i \in D_i$  and so  $d_i \leq s_i$ , for all  $i \in I$ . Hence  $(d_i)_{i \in I} \leq (s_i)_{i \in I}$ . Now let  $(c_i)_{i \in I}$  be any upper pre-bound of  $D$  and  $a_i$  be an arbitrary element in  $D_i$ . Thus, there exists  $(d'_i)_{i \in I} \in D$  with  $d'_i = a_i$ . We also have  $(d'_i)_{i \in I} \leq (c_i)_{i \in I}$  and then  $a_i = d'_i \leq c_i$ . This gives  $c_i$  is an upper pre-bound of  $D_i$  and so  $s_i \leq c_i$ , for all  $i \in I$ . Consequently  $(s_i)_{i \in I} \leq (c_i)_{i \in I}$ , as required.  $\square$

**Lemma 2.3.** The projection maps  $\pi_i : A \rightarrow A_i$ ,  $i \in I$ , are pre-dcpo maps, where  $A =: \prod_{i \in I} A_i$  and  $\{A_i : i \in I\}$  is a family of pre-dcpo's.

*Proof.* Let  $D \subseteq^{pd} A$  and  $c := (c_i)_{i \in I}$  be a pre-supremum of  $D$ . It is enough to show that  $c_i = \pi_i(c)$  is a pre-supremum of  $D_i := \pi_i(D) = \{a \in A_i : \exists d = (d_k)_{k \in I} \in D, a = d_i\}$ , for all  $i \in I$ . To show this, take an arbitrary element  $a \in D_i$ . Then there exists  $(d_i)_{i \in I} \in D$  with  $d_i = a$ . Moreover,  $(d_i)_{i \in I} \leq (c_i)_{i \in I}$  and so  $a = d_i \leq c_i$ . This shows  $c_i$  is an upper pre-bound of  $D_i$ . Now let  $z \in A_i$  be any upper pre-bound of  $D_i$ . First we see that  $(x_i)_{i \in I}$  is an upper pre-bound of  $D$ , where  $x_j = c_j$ , for  $j \neq i$ , and  $x_i = z$ . To prove this, let  $(y_i)_{i \in I} \in D$ . Then since  $(c_i)_{i \in I}$  is a pre-supremum of  $D$ ,  $y_i \leq c_i$ , for all  $i \in I$ . We also have  $y_i \in D_i$  and so  $y_i \leq z$  (because  $z$  is an upper pre-bound of  $D_i$ ). Hence,  $(y_i)_{i \in I} \leq (x_i)_{i \in I}$ . This gives that  $c = (c_i)_{i \in I} \leq (x_i)_{i \in I}$  (because  $c$  is a pre-supremum of  $D$ ). Thus  $c_i \leq x_i = z$ , as required.  $\square$

**Proposition 2.4.** The product of a family of pre-dcpo's is their Cartesian product with the componentwise pre-order.

*Proof.* Let  $\{A_i : i \in I\}$  be a family of pre-dcpo's and  $A := \prod_{i \in I} A_i$ . By Lemma 2.2,  $A$  with the componentwise pre-order is a pre-dcpo. Also, by Lemma 2.3, the projection maps  $\pi_i : A \rightarrow A_i$  are pre-dcpo maps. To see the universal property of products, notice that for every pre-dcpo  $B$  with pre-dcpo maps  $f_i : B \rightarrow A_i$ ,  $i \in I$ , the unique map  $f : B \rightarrow A$  given by  $f(b) = (f_i(b))_{i \in I}$ ,  $b \in B$  exists and satisfies  $\pi_i \circ f = f_i$ , for all  $i \in I$ . Also, it is straightforward to see that  $f$  is a pre-dcpo map.

To show the uniqueness of  $f$ , suppose that  $h: B \rightarrow A$  is also a pre-dcpo map with  $\pi_i \circ h = f_i$ , for all  $i \in I$ . Then for every  $b \in B$ ,

$$f(b) = (f_i(b))_{i \in I} = (\pi_i(h(b)))_{i \in I} = h(b).$$

□

Now, we consider coproducts.

**Remark 2.5.** It is clear that the initial object in the category **Pre-Dcpo** is the empty set.

**Theorem 2.6.** *The coproduct of a family of pre-dcpo's is their disjoint union.*

*Proof.* Let  $\{A_i: i \in I\}$  be a family of pre-dcpo's and  $A := \dot{\bigcup}_{i \in I} A_i$  be the disjoint union of  $A_i$ ,  $i \in I$ . The disjoint union  $A$  with the pre-order inherited from  $A_i$ ,  $i \in I$ ; that is

$$x \leq y \text{ in } A \text{ if and only if } x \leq y \text{ in } A_i, \text{ for some } i \in I$$

is a pre-dcpo. Indeed, if  $D$  is a pre-directed subset of  $A$  then by the definition of the pre-order on  $A$ ,  $D \subseteq^{pd} A_i$  for some  $i \in I$ , and the pre-suprema of  $D$  in  $A$  are exactly the pre-suprema of  $D$  in  $A_i$ . Now, since for each  $i \in I$ ,  $A_i$  is a pre-dcpo, we get that  $A$  is a pre-dcpo too.

Moreover, the injection maps  $u_i: A_i \rightarrow A$ , defined by  $u_i = id_A|_{A_i}$ ,  $i \in I$  are pre-dcpo maps. Finally, since  $A$  satisfies the universal property of the coproduct of  $\{A_i \mid i \in I\}$  in **Set**, for every pre-dcpo  $B$  and pre-dcpo maps  $f_i: A_i \rightarrow B$ ,  $i \in I$ , the mapping  $f: A \rightarrow B$  given by  $f(a) = f_i(a)$  for  $a \in A_i$ , is the unique map with  $f \circ u_i = f_i$ , for all  $i \in I$ . This map is also a pre-dcpo map, because if  $D$  is a pre-directed subset of  $A$  and  $s$  is a pre-supremum of  $D$ , then by the definition of the pre-order on  $A$ ,  $D \subseteq^{pd} A_i$  for some  $i \in I$  and  $s \in A_i$ . Hence by the fact that for all  $i \in I$ ,  $f_i$  is a pre-dcpo map and by the definition of  $f$ , we observe that  $f$  is a pre-dcpo map. □

In the following, we study equalizers in the category **Pre-Dcpo**.

**Theorem 2.7.** *Let  $f, g: P \rightarrow Q$  be pre-dcpo maps. If  $E = \{p \in P \mid f(p) = g(p)\}$  is a pre-dcpo with the pre-order induced from  $P$ , then  $(E, i)$  is the equalizer of  $f$  and  $g$ , where  $i$  is the inclusion map from  $E$  to  $P$ .*

*Proof.* The inclusion map satisfies  $f \circ i = g \circ i$ . Also, if  $e: K \rightarrow P$  is a pre-dcpo map with  $f \circ e = g \circ e$  then the map  $\gamma: K \rightarrow E$  given by  $\gamma(x) = e(x)$  is the unique pre-dcpo map such that  $i \circ \gamma = e$ . □

In the following example, we see that the set  $E = \{p \in P \mid f(p) = g(p)\}$  with the pre-order induced by  $P$  where  $f, g: P \rightarrow Q$  are pre-dcpo maps need not to be a pre-dcpo, in general.

**Example 2.8.** Take the pre-dcpo  $P = \mathbb{N} \cup \{\infty_0, \infty_1\}$  with the pre-order  $\preceq := \{(n, m) \mid n, m \in \mathbb{N}, n \leq m\} \cup \{(\infty_0, \infty_1), (\infty_1, \infty_0)\} \cup \{(n, \infty_0) \mid \mathbb{N}\} \cup \{(n, \infty_1) \mid \mathbb{N}\} \cup \Delta_P$ , where  $\leq$  denotes the natural linear order on  $\mathbb{N}$ . Next, we consider the pre-dcpo maps  $f := Id_P$  and  $g: P \rightarrow P$  where  $g(n) = n$  for all  $n \in \mathbb{N}$ ,  $g(\infty_0) = \infty_1$ , and  $g(\infty_1) = \infty_0$ . Then we have  $E = \{p \in P \mid Id_P(p) = g(p)\} = \{p \in P \mid p = g(p)\} = \mathbb{N}$  which is not a pre-dcpo with the induced pre-order from  $P$ . In fact, the pre-directed set  $\mathbb{N}$  does not have a pre-supremum.

Now, we show that the equalizers in the category **Pre-Dcpo** do not necessarily exist.

**Example 2.9.** Let  $P$ ,  $f$  and  $g$  be pre-dcpo and pre-dcpo maps introduced in Example 2.8, respectively. Now, we show that the equalizer of  $f$  and  $g$  does not exist. On the contrary, let  $(E, e)$  be the equalizer of  $f$  and  $g$ . First of all, we prove three facts:

Fact (1):  $\text{Im}(e) = \mathbb{N}$ . First notice that  $\infty_0, \infty_1 \notin \text{Im}(e)$  because  $f \circ e = g \circ e$ . In other words,  $\text{Im}(e) \subseteq \mathbb{N}$ . Next, let  $k: \mathbb{N} \rightarrow P$  be a pre-dcpo map with  $k(n) = n$ , for all  $n \in \mathbb{N}$ , where  $\mathbb{N}$  is considered with the discrete order. Also we have  $f \circ k = g \circ k$ , so by the universal property of equalizers there exists a unique pre-dcpo map  $\bar{k}: \mathbb{N} \rightarrow E$  satisfying  $e \circ \bar{k} = k$ . This gives that  $\text{Im}(e) = \mathbb{N}$ .

Fact (2): The pre-order on  $E$  is also antisymmetric and so is an order. On the contrary, if there are two distinct elements  $x$  and  $y$  in  $E$  with  $x < y < x$ . Then  $e(x) \neq e(y)$  (because  $e$  is a monomorphism and so is injective). Now, applying Fact (1), take  $e(x) = n$  and  $e(y) = m$ , for some  $m, n \in \mathbb{N}$ . Since  $e$  is pre-order preserving, we have  $n \leq m \leq n$  and so  $n = m$  which is a contradiction.

Fact (3):  $E$  is a chain: To see this, let  $x$  and  $y$  be two distinct elements of  $E$ . Applying Fact (1) and without loss of generality, take  $e(x) = n < e(y) = m$ , for some  $m, n \in \mathbb{N}$ . Now, we define the pre-dcpo map  $q: \mathbf{2} \rightarrow P$  with  $q(0) = n$  and  $q(1) = m$ , where  $\mathbf{2}$  is the two-element chain  $\{0 < 1\}$ . Thus by the universal property of equalizers there exists a unique pre-dcpo map  $\bar{q}: \mathbf{2} \rightarrow E$  satisfying  $e \circ \bar{q} = q$ . This gives  $e(\bar{q}(0)) = q(0) = n = e(x)$  and  $e(\bar{q}(1)) = q(1) = m = e(y)$ . Since  $e$  is injective,  $\bar{q}(0) = x$  and  $\bar{q}(1) = y$ . Moreover,  $x = \bar{q}(0) < \bar{q}(1) = y$  because  $\bar{q}$  is pre-order preserving. Consequently,  $E$  is a chain.

Finally, by Fact (2), the pre-dcpo  $E$  is indeed a dcpo which is also a chain by Fact (3). Thus  $\bigvee^d E$  exists and take  $n = e(\bigvee^d E)$  for some  $n \in \mathbb{N}$ . So  $e(x) \leq n$ , for all  $x \in E$ . This gives that  $n + 1 \notin \text{Im}(e) = \mathbb{N}$ , which is a contradiction. Consequently, the equalizer of  $f$  and  $g$  does not exist.

To see that pullbacks do not necessarily exist in the category **Pre-Dcpo**, first we recall from [2] the following theorem.

**Theorem 2.10.** *For each category  $\mathcal{C}$  the following conditions are equivalent:*

- (1)  $\mathcal{C}$  is finitely complete,

- (2)  $\mathcal{C}$  has finite products and equalizers,
- (3)  $\mathcal{C}$  has finite products and finite intersections,
- (4)  $\mathcal{C}$  has pullbacks and the terminal object.

**Theorem 2.11.** *The pullbacks do not necessarily exist in **Pre-Dcpo**.*

*Proof.* The category **Pre-Dcpo** has products and the terminal object but does not have equalizers, so by the above Theorem, it does not have pullbacks, too.  $\square$

So we get the following theorem:

**Theorem 2.12.** *The category **Pre-Dcpo** is not complete.*

**2.1. Monomorphisms, epimorphisms, and cocompleteness.** In this subsection, we first prove that monomorphisms and epimorphisms in the category **Pre-Dcpo** are exactly injective and surjective pre-dcpo maps, respectively. Finally, we show that this category can not be cocomplete.

**Theorem 2.13.** *A pre-dcpo map is a monomorphism if and only if it is injective.*

*Proof.* Using the fact that the free dcpo on a one-element set exists (see the next section), we conclude, by a general fact from category theory (see [2, Proposition 8.29], that a pre-dcpo map is a monomorphism if and only if it is injective.  $\square$

**Theorem 2.14.** *A pre-dcpo map is an epimorphism if and only if it is surjective.*

*Proof.* It is well-know that in any concrete category every surjective morphism is an epimorphism. So, it is sufficient to show that any epimorphism in **Pre-Dcpo** is a surjective map. To see this, let  $f: A \rightarrow B$  be a pre-dcpo map which is not surjective. Thus there exists an element  $b_0 \in B$  which is not in  $\text{Im}(f)$ . Now, we define two pre-dcpo maps  $h, g: B \rightarrow X$  as  $h(b) = x_1 = g(b)$ , for all  $b \in \text{Im}(f)$ ,  $h(b) = x_2$  and  $g(b) = x_3$  otherwise, where  $X = \{x_1, x_2, x_3\}$  is a pre-dcpo with the total pre-order  $X \times X$ . Notice that every subset  $Y$  of  $X$  is pre-directed and every element of  $X$  is a pre-supremum of  $Y$ . Therefore  $h$  and  $g$  are pre-dcpo maps. We also have  $h \circ f = g \circ f$  while  $h \neq g$  since  $h(b_0) \neq g(b_0)$ . This gives that  $f$  is not an epimorphism, as required.  $\square$

In the following, we show that the category **Pre-Dcpo** is not cocomplete.

Recall that an object  $S$  of a category  $\mathcal{C}$  is called a separator if for any two distinct arrows  $f, g: A \rightarrow B$ , there exists an arrow  $h: S \rightarrow A$  such that  $f \circ h \neq g \circ h$ .

Also, recall from [2, Theorem 12.13] that a co-wellpowered cocomplete category which has a separator is complete. Therefore, we first show that this category has a separator and co-wellpowered and then since this category is not complete, we conclude that this category also is not cocomplete.

**Theorem 2.15.** *Every non-empty pre-dcpo is a separator in the category **Pre-Dcpo**.*

*Proof.* Let  $X$  be a non-empty pre-dcpo and  $f, g: P \rightarrow Q$  two distinct pre-dcpo maps. Now, take  $p_0 \in P$  for which  $f(p_0) \neq g(p_0)$  and the constant pre-dcpo map  $h: X \rightarrow P$  defined by  $h(x) = p_0$ , for all  $x \in X$ . So we have  $f \circ h \neq g \circ h$ . This gives that  $X$  is a separator in the category **Pre-Dcpo**, as required.  $\square$

**Theorem 2.16.** *The category **Pre-Dcpo** is co-wellpowered.*

*Proof.* We must show that the class of pairwise non-isomorphic quotient objects of any pre-dcpo is a set. Let  $P$  be a pre-dcpo and  $Q$  be a quotient object of  $P$ ; that is there exists an epimorphism  $f: P \rightarrow Q$ . By Theorem 2.14,  $f$  is surjective. So there exists an injective map  $g: Q \rightarrow P$  with  $f \circ g = \text{Id}$ . This implies that  $Q$  is isomorphic to a subset of  $P$  in **Set**. Hence the class of pairwise non-isomorphic quotient objects of  $P$  is a subset of the powerset of  $P$ , and therefore it is a set.  $\square$

**Theorem 2.17.** *The category **Pre-Dcpo** is not cocomplete.*

*Proof.* On the contrary, if the category **Pre-Dcpo** is cocomplete, then by Theorem 12.13 of [2], Theorem 2.15 and Theorem 2.16, we conclude that the category **Pre-Dcpo** is complete which is a contradiction.  $\square$

### 3. ADJOINT RELATIONS FOR **Pre-Dcpo**

In this section, we consider the following diagram of forgetful functors

$$\begin{array}{ccc}
 \mathbf{Dcpo} & \xrightarrow{U_1} & \mathbf{Pre-Dcpo} \\
 U_2 \downarrow & & \downarrow U_3 \\
 \mathbf{Pos} & \xrightarrow{U_4} & \mathbf{Pre-Set} \\
 & \searrow U_5 & \downarrow U_6 \\
 & & \mathbf{Set}
 \end{array}$$

and study the existence of the left and the right adjoints for these functors. We recall that the left adjoint to the left vertical forgetful functor  $U_2$ , in the above square has been found in



[8, 9]. Also, the forgetful functor  $U_5$  has been considered in [7] where it is shown that  $U_5$  has a left adjoint while it does not have a right adjoint.

Here, we show that the forgetful functors  $U_1$ ,  $U_4$  and  $U_6$  have left adjoints. We also prove that the functors  $U_1, U_4$  and  $U_3$ , except  $U_6$ , do not have right adjoints.

**Free dcpo over a pre-dcpo.** By a *free dcpo on a pre-dcpo*  $P$  we mean a dcpo  $F$  together with a pre-dcpo map  $\tau: P \rightarrow F$  with the universal property that given any dcpo  $A$  and a pre-dcpo map  $f: P \rightarrow A$  there exists a unique dcpo map  $\bar{f}: F \rightarrow A$  such that  $\bar{f} \circ \tau = f$ .

**Lemma 3.1.** Let  $(P, \leq)$  be a pre-ordered set. Then  $(P, \leq)$  is a pre-dcpo if and only if  $(P / \sim, \leq / \sim)$  is a dcpo.

*Proof.* Let  $(P, \leq)$  be a pre-dcpo. Recalling the definition of  $\leq / \sim$  from the introduction, we see that the canonical surjection  $\pi: P \rightarrow P / \sim$  preserves and reflects the relation; that is

$$p \leq_P q \Leftrightarrow [p]_{\sim} (\leq / \sim) [q]_{\sim}.$$

From this, it is straightforward to prove that  $D$  is a pre-directed subset of  $(P, \leq)$  if and only if  $\pi(D)$  is a directed subset of  $(P / \sim, \leq / \sim)$ . Moreover,  $\pi$  maps any pre-supremum of a pre-directed subset  $D$  of  $P$  to the supremum of  $\pi(D)$ ; and conversely the supremum of any  $E \subseteq^d P / \sim$  is of the form  $[p]_{\sim}$ , where  $p$  is a pre-supremum of  $D = \pi^{-1}(E)$ .  $\square$

**Theorem 3.2.** For a given pre-dcpo  $(P, \leq)$ , the free dcpo on  $P$  is  $F = (P / \sim, \leq / \sim)$ .

*Proof.* Recall from Lemma 3.1, that  $(P / \sim, \leq / \sim)$  is a dcpo. Moreover,  $(P / \sim, \leq / \sim)$ , with the canonical map  $\pi: P \rightarrow P / \sim$ ,  $x \mapsto [x]_{\sim}$  as a universal map, is the free dcpo on  $P$ . Now, by the proof of Lemma 3.1,  $\pi$  is a pre-dcpo map.

To prove the universal property of  $\pi: P \rightarrow P / \sim$  for dcpos, take a pre-dcpo map  $f: P \rightarrow B$  to a dcpo  $B$ . Then the map  $\bar{f}: P / \sim \rightarrow B$  defined by  $\bar{f}([x]_{\sim}) = f(x)$ , is the unique dcpo map satisfying  $\bar{f} \circ \pi = f$ . First, we show that  $\bar{f}$  is a dcpo map. To prove this, let  $D \subseteq^d P / \sim$  be a directed subset of  $P / \sim$  with the supremum  $[s]_{\sim}$ . So by the proof of Lemma 3.1,  $\pi^{-1}(D)$  is a pre-directed subset of  $P$  and  $s$  is a pre-supremum of it. This concludes that  $f(s)$  is the supremum of  $f(\pi^{-1}(D))$  in  $B$ , because  $f$  is a pre-dcpo map. Hence by two facts  $\bar{f}([s]_{\sim}) = f(s)$  and  $f(\pi^{-1}(D)) = \bar{f}(D)$ , we have just proved that  $\bar{f}(\bigvee^d D) = \bigvee^d \bar{f}(D)$ , as required.

To establish the uniqueness of  $\bar{f}$ , suppose that  $h: P / \sim \rightarrow B$  is also a dcpo map such that  $h \circ \pi = f$ . Then  $h \circ \pi = f = \bar{f} \circ \pi$  gives  $h = \bar{f}$ , because  $\pi$  is epic.  $\square$

**Corollary 3.3.** The forgetful functor  $U_1: \mathbf{Dcpo} \rightarrow \mathbf{Pre-Dcpo}$  has a left adjoint.

In the following, we see that the cofree dcpo over a pre-dcpo does not necessarily exist.

**Cofree dcpo over a pre-dcpo.** By a *cofree dcpo on a pre-dcpo*  $P$  we mean a dcpo  $K$  together with a pre-dcpo map  $\sigma: K \rightarrow P$  with the universal property that given any dcpo  $A$  and a pre-dcpo map  $g: A \rightarrow P$  there exists a unique dcpo map  $\bar{g}: A \rightarrow K$  such that  $\sigma \circ \bar{g} = g$ .

**Theorem 3.4.** *If  $P$  is a pre-dcpo in which there are  $x, y \in P$  with  $x < y < x$ , then the cofree dcpo over  $P$  does not exist.*

*Proof.* Let  $P$  be a pre-dcpo in which there are  $x, y \in P$  with  $x < y < x$  and let  $K(P)$  be the cofree dcpo over  $P$ . Take  $\sigma: K(P) \rightarrow P$  to be the cofree pre-dcpo map.

First we see that  $\sigma$  is injective. This is because, otherwise there exist  $a \neq b \in K(P)$  such that  $\sigma(a) = \sigma(b) = p_0$ . Then, considering the dcpo map  $f: \{\theta\} \rightarrow P$  from the singleton dcpo  $\{\theta\}$ , defined by  $f(\theta) = p_0$ , we see that there exist two dcpo maps  $f_1, f_2: \{\theta\} \rightarrow K(P)$ , given by  $f_1(\theta) = a$  and  $f_2(\theta) = b$ , such that  $\sigma \circ f_1 = f$  and  $\sigma \circ f_2 = f$ . This contradicts the universal property of the cofree map  $\sigma$ .

Now, by hypothesis there exist  $x, y \in P$  with  $x < y < x$ . Define the pre-dcpo map  $f: \mathbf{3} \rightarrow P$  by  $f(0) = f(2) = x$  and  $f(1) = y$ , where  $\mathbf{3}$  is the three-element chain  $0 < 1 < 2$ . Then, by the universal property of cofree maps, there exists a unique dcpo map  $\bar{f}: \mathbf{3} \rightarrow K(P)$  with  $\sigma \circ \bar{f} = f$ . Take  $\bar{f}(0) = q_0$ ,  $\bar{f}(1) = q_1$  and  $\bar{f}(2) = q_2$ . Since  $\bar{f}$  is a dcpo map, in particular order-preserving, so  $q_0 \leq q_1 \leq q_2$ . Moreover,  $\sigma(q_0) = f(0) = x$ ,  $\sigma(q_2) = f(2) = x$  and  $\sigma(q_1) = f(1) = y$ . This gives  $q_0 = q_2$ , because  $\sigma$  is injective. Hence  $q_0 = q_2 = q_1$  and so  $x = \sigma(q_0) = \sigma(q_1) = y$ , which is a contradiction.  $\square$

**Corollary 3.5.** *The forgetful functor  $U_1: \mathbf{Dcpo} \rightarrow \mathbf{Pre-Dcpo}$  does not have a right adjoint.*

**Remark 3.6.** 1. It is easily to show that the free poset over a pre-orderd set  $(P, \leq)$  is  $(P / \sim, \leq / \sim)$  with the canonical surjective map  $\pi: P \rightarrow P / \sim, p \mapsto [p]_{\sim}$  as a universal map. Consequently, the forgetful functor  $U_4: \mathbf{Pos} \rightarrow \mathbf{Pre-Set}$  has a left adjoint.

2. In the same way that we proved Lemma 3.4, one can show that the cofree poset over a pre-ordered set does not necessarily exist. Consequently, the forgetful functor  $U_4: \mathbf{Pos} \rightarrow \mathbf{Pre-Set}$  does not have a right adjoint.

3. Also, one can show that the free pre-ordered set on a set  $X$  is  $(X, =)$  with the identity map  $Id_X: X \rightarrow X$  as a universal map. Consequently, the forgetful functor  $U_6: \mathbf{Pre-Set} \rightarrow \mathbf{Set}$  has a left adjoint.

4. Also, one can show that the cofree pre-ordered set on a set  $X$  is  $(X, \nabla = X \times X)$ , with the identity map as a cofree map. Consequently, the forgetful functor  $U_6: \mathbf{Pre-Set} \rightarrow \mathbf{Set}$  has a right adjoint.

5. Also, in the same way that we proved Lemma 3.4, one can easily show that the cofree pre-dcpo over a pre-ordered set  $P$  in which there are two elements  $x, y$  with  $x < y < x$  does not exist. Consequently, the forgetful functor  $U_3: \mathbf{Pre-Dcpo} \rightarrow \mathbf{Pre-Set}$  does not have a right adjoint.

### Open problems:

- (a) Is the category  $\mathbf{Pre-Dcpo}$  Cartesian closed?
- (b) Does the functor  $U_3$  have a left adjoint?

## 4. ACKNOWLEDGEMENT

The author gratefully acknowledge the referees for the very valuable comments which improved the paper.

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