Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 7 No. 2 (2020) pp 115-134.

TOPICS IN TOPOLOGICAL MI-GROUPS

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Abstract. A many identities group (MI-group, for short) is an algebraic structure which is generalized a monoid with cancellation laws and is endowed with an invertible anti-automorphism representing inversion. In other words, an MI-group is an algebraic structure generalizing the group concept, except most of the elements have no inverse element. The concept of a topological MI-group, as a preliminary study, in the paper " Topological MI-group: Initial study" was introduced by M. Holčapek and N. Škorupová, and we have given a more comprehensive study of this concept in our two recent papers. This article is a continuation of the effort to develop the theory of topological MI-groups and is focused on the study of separation axioms and the isomorphism theorems for topological MI-groups. Moreover, some conditions under which a MI-subgroup is closed will be investigated, and finally, the existence of nonnegative invariant measures on the locally compact MI-groups are introduced.

1. Introduction

A many identities group (MI-group, in short) is a special algebraic structure in which certain elements (called pseudoidentities) behave like the identity element and having a monoidal substructure. The concept of MI-group, in the paper "MI-algebras: A new framework for

DOI:10.29252/as.2020.1801

MSC(2010): Primary:05C50

Keywords: MI-groups, Monoids, Pseudoidentity elements, Topological MI-groups, Full MI-subgroups.

Received: 11 March 2020, Accepted: 13 Jun 2020.

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arithmetics of (extensional) fuzzy number" has already been introduced. In this new algebraic structure, the set of pseudoidentities play an essential role. These elements generalize the role of the identity element, by which we can derive various properties of groups in a weaker form. In the second section, we recall some basic definitions, examples, propositions and theorems related to MI-groups and topological MI-groups. In the third section, under some special conditions, the separation axioms in topological MI-groups are investigated. In the fourth section the isomorphism theorems for topological MI-groups are expressed. The fifth section present some results about closed MI-subgroups. The final two sections is devoted to the existence of nonnegative invariant measures on the locally compact MI-groups and our conclusions.

2. **Preliminaries**

An MI-group is based on a generalization of the concept of monoid that satisfies the cancellation laws and is endowed with an invertible anti-automorphism representing inversion. Pseudoidentities elements play an important and undeniable role, i.e. elements that possess similar properties to the identity element. The most important types of such elements are the form xx^{-1}, where $x \in G$. In this section, we first discuss the definitions and important concepts of the MI-groups.

Definition 2.1. (Definition 2.1 [8]) A triplet $(G, \star, {}^{-1}, e)$ is said to be an MI-group if it satisfies the following axioms:

(1) $(G, \star)$ is a monoid,

(2) ${}^{-1} : G \to G$ is an involutive anti-automorphism, i.e., $\forall x, y \in G$, it holds

$(i)(x \star y)^{-1} = y^{-1} \star x^{-1}$,

$(ii)(x^{-1})^{-1} = x$,

(3) $x \star (y \star y^{-1}) = (y \star y^{-1}) \star x$ for any $x, y \in G$,

(4) the cancellation laws hold, i.e., $\forall x, y, z \in G$,

$x \star y = x \star z \Longrightarrow y = z$ (*left cancellation law*),

$y \star x = z \star x \Longrightarrow y = z$ (*right cancellation law*).

Typically, we write $(G, \star, {}^{-1}, e) = G$ and $x \star y = xy$. Let P_G be the least submonoid of G that contains the set $\{xx^{-1} : x \in G\}$. Elements of P_G are called pseudoidentity elements, e is called an (strong) identity element and the involutive anti-automorphism ${}^{-1}$ of G will be called the inversion of G. By Lemma 2.1 of [8], we have

$$P_G = \{x_1x_1^{-1}x_2x_2^{-1}...x_nx_n^{-1} \mid x_1, x_2, ..., x_n \in G, n \in \mathbb{N}\}.$$

Moreover, $sx = xs$ for every $x \in G$ and $s \in P_G$ (Axiom (3) of definition) . Also by Lemmas 2.2 and 2.3 of [8], we get

i) $xx^{-1} = x^{-1}x, \forall x \in G$

ii) $s = s^{-1}, \forall s \in P_G$.

The recent feature shows that the elements P_G are symmetric. If $P_G = \{e\}$, then G has a group structure. It should be noted that the above definition of an MI-group and P_G can be based on the definition 2.1 from [6].

For MI-groups G and H, a mapping $f : G \to H$ is a homomorphism of MI-groups, provided that

(1) $f(x \star_G y) = f(x) \star_H f(y),\ \forall x, y \in G$,

(2) $f(e_G) = e_H$,

(3) $f(x^{-1}) = f(x)^{-1}\ \forall x \in G$.

Let H be a non- empty subset of G. The set H is said to be closed in G, if $xs \in H$ implies $x \in H$ whenever $x \in G$ and $s \in P_G$. The set

$$\overline{H}^G = \bigcap \{K \subseteq G \mid K \textit{ is closed in } G \textit{ and } H \subseteq K\}$$

is called a closure of H in G. By theorem 3.1 of [2],

$$\overline{H}^G = \{x \in G \mid \exists s \in P_G : xs \in H\}.$$

Definition 2.2. (Definition 2.8 [6]) Let $G = (G, \star, {}^{-1}, e)$ be an MI-group, and $H \subseteq G$. If $H = (H, \star, {}^{-1}, e)$ is itself an MI-group under the product and inversion of G, then H is said to be an MI-subgroup of G, which is denoted by $H \leq G$.

According to Theorem 2.4 of [8], H is an MI-subgroup of G if and only if $e \in H$ and $xy^{-1} \in H$ for each $x, y \in H$. By theorem 2.3 of [6], P_G is an MI-subgroup of G. By Lemma 2.1 of [8], P_G is also an abelian MI-subgroup of G. An MI-subgroup H of an MI-group G that contains P_G is said to be full and is denoted by $H \leq_f G$. We say H is a non-full MI-subgroup, if H is not a full MI-subgroup.

Example 2.3. Let $G = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ be the set of all closed real intervals. By example 2.2 of [6], we know $(G, +, -, [0, 0])$ under algebraic actions

$$[a, b] + [c, d] = [a + c, b + d],$$

$$-[a, b] = [-b, -a],$$

is an additive abelian MI-group. Obviously, $P_G = \{[-x, x] \mid x \geq 0\}$.

Example 2.4. Let $G^+ = \{[a, b] \mid a, b \in \mathbb{R}^+, a \leq b\}$ be the set of all closed real intervals of positive real numbers. As in Example 2.3, it is easy to see that G^+ under algebraic actions .

and $^{-1}$ defined as follows is an abelian multiplicative MI-group:

$$[a, b].[c, d] = [a.c, b.d],$$

$$[a, b]^{-1} = [1/b, 1/a].$$

In this case we will write $G^+ = (G^+, ., ^{-1}, [1, 1])$, where [1, 1] is the identity element of G^+.

Now, like the topological groups, we have the following definition:

Definition 2.5. Suppose that G is an MI-group, whose underlying space is a topological space. Then G is called a topological MI-group if $(x, y) \to x \star y$ maps $G \times G$ onto G and $x \to x^{-1}$ maps G on G continuously.

For example, every MI-group $G = (G, \star, ^{-1}, e)$ endowed with the discrete topology is a topological MI-group.

Example 2.6. (Internal topology on MI-groups)(Definition 3.1 of [1])

Let G be an MI-group, and U subset of G. We say that U is open in G, if $\overline{U^c}^G = U^c$, i.e. U^c is closed in G from the MI-groups point of view, where U^c is complement of U in G.

According to this definition, we get

$$U^c = \{x \in G \mid \exists\ s \in P_G\ xs \in U^c\},$$

or

$$U = \{x \in G \mid \forall s \in P_G\ xs \in U\} = \{x \in G \mid xP_G \subseteq U\}.$$

So according to this relationship, to prove the openness of a set is enough to show that $U \subseteq \{x \in G \mid xP_G \subseteq U\}$. Obviously U is open if and only if $UP_G = U$. It is clear that the family of such subsets of G, including $\emptyset$ and G, has the properties of a topology. This topology is called the Internal topology on an MI-group.

It is obvious that P_G and every full MI-subgroup of G are open. The subset U of G is closed if and only if $\overline{U}^G = U$, that is, the topological closure and the MI-group closure of U are the same. Also every neighborhood of e contains P_G. By Propositions 3.2 and 3.3 of [1], each MI-group G under this topology, becomes a topological MI-group satisfying

(1) $\qquad$ *U is open in G and $x \in G$ imply xU is open in G.*

Also by Remark 3.5 of [1], P_G is the smallest open subset of G which contains e. Therefore for each $x \in G$, xP_G is the smallest and simplest open subset of G containing x. In fact, according to the definition of open sets in this topology, each open set of x, clearly including xP_G.

Throughout the text, we say U is closed in an MI-group G, if $\overline{U}^G = U$, while we say U is topologically closed in G, if $\overline{U} = U$.

3. Separation axioms in topological MI-groups

In the theory of topological groups, it is known that each topological group satisfying the T_0 separation axiom is also Hausdorff and hence regular. But in topological MI-groups, this is not necessarily true. Indeed, as will be seen , the topological MI-group $G = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ of real intervals under the internal topology is a T_0 space, but is not a Hausdorff space. Before that, we first recall the separation principles in topological spaces.

Definition 3.1. Let X be a topological space.

i) X is said to be T_0 space if it satisfies the T_0 axiom, i.e. for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subseteq X$ so that U contains one of x and y but not the other.

ii) A space X is a T_1 space or Frechet space if it satisfies the T_1 axiom, i.e. for each $x, y \in X$ such that $x \neq y$, there are two open subset U and V so that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

iii) A space X is a T_2 space or Hausdorff space if it satisfies the T_2 axiom, i.e. for each $x, y \in X$ such that $x \neq y$ there are two disjoint open subsets U and V of X so that $x \in U$ and $y \in V$.

iv) A space X is regular if for each $x \in X$ and each closed $C \subseteq X$ such that $x \notin C$, there are two disjoint open sets $U, V \subseteq X$ so that $x \in U$ and $C \subseteq V$. A regular T_1 space is called a T_3 space.

v) A space X is normal if for each pair A and B of disjoint closed subsets of X, there is a pair U and V of disjoint open subsets of X so that $A \subseteq U$, $B \subseteq V$. A normal T_1 space is called a T_4 space.

If a topological MI-group is a T_0 space, we say that it is a T_0 MI-group. We begin this section by stating and proving the above claim.

Proposition 3.2. *Let G be the topological MI-group $G = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ of real intervals under the internal topology. Then G is a T_0 MI-group, but it is not a Hausdorff topological MI-group.*

Proof. By example 2.3, $P_G = \{[-x, x] \mid x \geq 0\}$. Therefore, according to the last paragraph of the previous section, for each $[a, b] \in G$,

$$[a, b] + P_G = \{[a - x, b + x] \mid x \geq 0\}$$

is the smallest and simplest open subset of G containing $[a, b]$. So just enough, consider such open subsets of G. For every $[a, b], [c, d] \in G$ such that $[a, b] \neq [c, d]$, we have

$$\begin{aligned}[a,b]+P_G\cap[c,d]+P_G\neq\emptyset &\iff \exists x,y\geq 0, [a-x,b+x]=[c-y,d+y]\\ &\iff a+b=c+d.\end{aligned}$$

i.e. two intervals $[a, b]$ and $[c, d]$ have the same center. Therefore, non-centered intervals are separated by open subsets $[a, b] + P_G$ and $[c, d] + P_G$ of G . But for intervals $[a, b]$ and $[c, d]$ with the same center, one is inside the other. For example, if the interval $[a, b]$ is within the interval $[c, d]$, then for $x = a - c \geq 0$ we have $c = a - x$ and $d = b + x$. Therefore

$$[c, d] = [a, b] + [-x, x] \in [a, b] + P_G,$$

while $[a, b] \notin [c, d] + P_G$. Indeed, there is no real non-negative number x, as $[a, b] = [c, d] + [-x, x]$. Therefore $[c, d]$ has an open neighborhood that does not contain $[a, b]$. However, only one of these has an open neighborhood that does not contain another. Thus G is a T_0 MI-group. On the other hand, every open neighborhood of $[a, b]$, obviously includes $[a, b] + P_G$ and so it includes $[c, d]$. Therefore, the internal topology on G is not Hausdorff. □

In general, for the internal topology on MI-groups, we have the following important fact:

Theorem 3.3. *Every T_0 topological MI-group under the internal topology is a Hausdorff space if and only if $P_G = \{e\}$.*

Proof. By Remark 3.5 of [1], every neighborhood of e contains P_G. Therefore, identity element e can not be separated from any member of P_G unless $P_G = \{e\}$, i.e. the MI-group G has a group structure. □

Remark 3.4. By previous theorem, every topological MI-group G under the internal topology is Hausdorff if and only if it has a group structure. In other words, the existence of a topological property, like Hausdorff, has altered its algebraic structure. It should be noted that in this case each subset of G is open, so the internal topology changes to a discrete topology on G. Indeed, for each subset U of G, we have clearly $U = \{x \mid x\{e\} \subseteq U\}$.

From now on, we focus on topological MI-groups satisfying (1), i.e. if U be an open basis at e, then the families $\{xU\}$ and $\{Ux\}$, where x runs through all elements of G and U runs through all elements of U, are open bases for G in each $x \in G$. Therefore for each open subset U of G and every $x \in U$, there is a neighborhood V of e such that $xV \subseteq U$.

In view of the above discussion, it seems that an additional condition is necessary for a T_0 topological MI-group to be Hausdorff. Accordingly, we define the following condition on a topological MI-group and examine the separation axioms.

Definition 3.5. A topological MI-Group G is said to have the property $\star$, if for every open subset U of G and $x \in U$, there is a neighborhood V of x in G such that $\overline{V}^G \subseteq U$.

Based on this property, we will have the following important theorem in relation to the principles of separation in topological MI-groups under the desired topology.

Theorem 3.6. *Let G be a T_0 topological MI-group with property $\star$. Then G is Hausdorff and regular space.*

Proof. Since G is a T_0 space, for every $x, y \in G$ such that $x \neq y$ there is an open set $U \subseteq G$ so that U contains one of x and y but not the other. For example if $x \in U$ and $y \notin U$, then by property $\star$, there is a neighborhood V of x such that $\overline{V}^G \subseteq U$. Also there are neighborhoods W and V_0 of e such that $V = xW$ and $V_0^2 \subseteq W$. This implies that xV_0 and yV_0^{-1} are disjoint neighborhoods of x and y, Respectively. Indeed if $z \in xV_0 \cap yV_0^{-1}$, then $z = xv_0 = yv_1^{-1}$ for some $v_0, v_1 \in V_0$. Hence, we obtain

$$yv_1^{-1}v_1 = xv_0v_1 \in xV_0^2 \subseteq xW = V.$$

Since $v_1v_1^{-1} \in P_G$, we find that $y \in \overline{V}^G$, which implies that $y \in U$, that this is a contradiction. Therefore G is Hausdorff. In relation to regularity, let U be a neighborhood of e in G. Hence there are neighborhoods V and W of e such that $\overline{V}^G \subseteq U$ and $W^2 \subseteq V$. We can also assume that W is symmetric, i.e. $W = W^{-1}$. Then if $x \in \overline{W}$, we have $xW \cap W \neq \emptyset$. Therefore $xw_1 = w_2$ for some $w_1, w_2 \in W$, and so $xw_1w_1^{-1} = w_2w_1^{-1} \in WW^{-1} = W^2 \subseteq V$. Since $w_1w_1^{-1} \in P_G$, we find that $x \in \overline{V}^G \subseteq U$, and so $\overline{W} \subseteq U$, i.e. G satisfies the axiom of regularity at e. For other points of G, let U be a neighborhood of an arbitrary member $x \in G$. By assumption, there is a neighborhood V of x such that $\overline{V}^G \subseteq U$. Also there are neighborhoods W and V_0 of e such that $V = xW$ and $V_0^2 \subseteq W$, Where V_0 can be selected symmetrically. Now if $y \in \overline{xV_0}$, then $yV_0 \cap xV_0 \neq \emptyset$ and thus $yv_1 = xv_2$ for some $v_1, v_2 \in V_0$. Hence, we will have

$$yv_1v_1^{-1} = xv_2v_1^{-1} \in xV_0V_0^{-1} = xV_0^2 \subseteq xW = V.$$

Since $v_1v_1^{-1} \in P_G$, we find that $y \in \overline{V}^G \subseteq U$ and so $\overline{xV_0} \subseteq U$, where xV_0 is a neighborhood of x. Therefore G satisfies the axiom of regularity at every point. □

The following Lemma plays an important role in the study of topological MI-groups.

Lemma 3.7. *([1], Proposition 3.6) Let G be a topological MI-group, and U be a open subset of it. Then $\overline{U}^G$ is also open.*

Proof. Let $x \in \overline{U}^G$. Then there is $s \in P_G$ such that $xs \in U$. Since U is open, there exist a neighborhood V of e such that $xsV \subseteq U$ and so $xVs \subseteq U$. Hence $xV \subseteq \overline{U}^G$. Therefore x is an interior of $\overline{U}^G$. □

Theorem 3.8. *Let G be a topological MI-group with the property $\star$ and let $U = \{U_\alpha\}_{\alpha\in I}$ be an open basis at e. Then $U' = \{\overline{U_\alpha}^G\}_{\alpha\in I}$ is also an open basis at e.*

Proof. By previous Lemma, for every $\alpha \in I$, $\overline{U_\alpha}^G$ is also an open subset of G. By assumption, for each neighborhood U of e, there is a neighborhood V of e such that $\overline{V}^G \subseteq U$. Since U is an open basis at e, there exists $U_\alpha \in U$ such that $U_\alpha \subseteq V$ and so $\overline{U_\alpha}^G \subseteq \overline{V}^G \subseteq U$. Thus the proof is completed. □

Remark 3.9. Based on the previous theorem, for the topological MI-groups with property $\star$, we can consider the bases at e in which each member of the basis is closed in MI-group, i.e. $\overline{U_\alpha}^G = U_\alpha$.

Theorem 3.10. *Let G be a topological MI-group with property $\star$. Then for each subset K of G, $\overline{K}^G$ is compact if and only if K is compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $\overline{K}^G$. Then $\overline{K}^G \subseteq \underset{\alpha\in I}{\cup} U_\alpha$ and so for every $x \in \overline{K}^G$ there is $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. By assumption there is a neighborhood V_{α_x} of x such that $\overline{V_{\alpha_x}}^G \subseteq U_{\alpha_x}$. Therefore, it is clear that $\{V_{\alpha_x}\}_{\alpha_x\in I}$ is also an open cover of $\overline{K}^G$ and so K. Since K is compact, we may take a finite number of subsets $V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n}$ such that $K \subseteq \overset{n}{\underset{i=1}{\cup}} V_{\alpha_i}$. So for each $x \in \overline{K}^G$ there is $s \in P_G$ such that $xs \in K$ and hence $xs \in V_{\alpha_i}$ for some α_i. Therefore $x \in \overline{V_{\alpha_i}}^G$. Since $\overline{V_{\alpha_i}}^G \subseteq U_{\alpha_i}$, we have $x \in U_{\alpha_i}$ and so $\overline{K}^G \subseteq \overset{n}{\underset{i=1}{\cup}} U_{\alpha_i}$. Therefore every open cover of $\overline{K}^G$ has a finite subcover, i.e. $\overline{K}^G$ is compact. On the contrary, suppose that $\overline{K}^G$ is compact and $\{U_\alpha\}_{\alpha\in I}$ be an open cover of K. As before, we can choose an open cover $\{V_\alpha\}_{\alpha\in I}$ of K such that for each $\alpha \in I$, $\overline{V_\alpha}^G \subseteq U_\alpha$. Then we will have

$$\overline{K}^G \subseteq \overline{\underset{\alpha\in I}{\cup} V_\alpha}^G \subseteq \underset{\alpha\in I}{\cup} \overline{V_\alpha}^G,$$

where recent inclusion is simply proven. Since by Lemma 3.7, for every $\alpha \in I$, $\overline{V_\alpha}^G$ is also an open subset of G, family $\{\overline{V_\alpha}^G\}_{\alpha\in I}$ is an open cover of $\overline{K}^G$. Hence there are $\alpha_1, \alpha_2, ..., \alpha_n \in I$ such that $\overline{K}^G \subseteq \overset{n}{\underset{i=1}{\cup}} \overline{V_{\alpha_i}}^G$ and so we have $K \subseteq \overset{n}{\underset{i=1}{\cup}} \overline{V_{\alpha_i}}^G \subseteq \overset{n}{\underset{i=1}{\cup}} U_{\alpha_i}$, which implies that K is compact. □

Remark 3.11. If the internal topology has $\star$ property, then P_G is closed. Indeed P_G is an open subset of G that contains e and so there is a neighborhood V of e such that $\overline{V}^G \subseteq P_G$. Therefore $\overline{P_G}^G \subseteq \overline{V}^G \subseteq P_G$, i.e. $\overline{P_G}^G = P_G$. Hence P_G is an open and closed subset of G. If $G \neq P_G$, obviously G is unconnected under the internal topology. With the same argument, it is easy to see that for each $x \in G$, xP_G is also closed.

Theorem 3.12. *Let G be a topological MI-group with property $\star$. Then every open MI-subgroup H of G is closed. Also it is closed in MI-group G.*

Proof. Let $x \in \overline{H}^G$. Thus there is $s \in P_G$ such that $xs \in H$. Since H is open, by assumption there exists a neighborhood V of xs such that $\overline{V}^G \subseteq H$. But $xs \in V$ implies that $x \in \overline{V}^G$ and so $x \in H$. Therefore $\overline{H}^G = H$, i.e. H is closed in MI-group. On the other hand, for each $x \in G$, xH is open and so by Lemma 3.7, $\overline{xH}^G$ is also open. Hence by relationship $\overline{H}^G = (\bigcup_{\overline{xH}^G \neq \overline{H}^G} \overline{xH}^G)^c$, $\overline{H}^G$ and so H is closed. □

We finish this section by expressing a new separation principle for topological MI-groups.

Definition 3.13. A topological MI-group G is said to be T_0^* space, if for every $x, y \in G$ such that $x \neq y$ there is an open subset U which is closed in G(i.e. $\overline{U}^G = U$) so that U contains one of x and y but not the other.

If a topological MI-group is a T_0^* space, we say that it is a T_0^* MI-group. Apparently, a T_0^* MI-group is also T_0 MI-group. It's easy to see that every T_0 MI-group with $\star$-property is also a T_0^* MI-group. The importance of this separation principle is that under it, the topological MI-group G will be Hausdorff:

Theorem 3.14. *Let G be a T_0^* MI-group. Then G is also Hausdorff.*

Proof. Since G is a T_0^* space, for each $x, y \in G$ such that $x \neq y$ there is an open subset U of G which is closed in G so that U contains one of x and y but not the other. For example, suppose that $x \in U$ and $y \notin U$. Also there are neighborhoods V and W of e such that $U = Vx$ and $W^2 \subseteq V$. This implies that Wx and $W^{-1}y$ are disjoint neighborhoods of x and y, respectively. In fact if $z \in Wx \cap W^{-1}y$, then $z = w_1x = w_2^{-1}y$ for some $w_1, w_2 \in W$. Hence we obtain

$$w_2w_2^{-1}y = w_2w_1x \in W^2x \subseteq Vx = U.$$

Since $w_2w_2^{-1} \in P_G$, we find that $y \in \overline{U}^G = U$, that this is a contradiction. Therefore G is Hausdorff. □

Anyway, we'll have

$$(T_0, \star\ property) \Longrightarrow T_0^* \Longrightarrow T_0$$

But it seems that the principle of regularity can not be derived from this principle. However, this seems to be the weakest principle of separation that can be placed on any topological MI-group so that it is still Hausdorff. It is interesting to note that in the internal topology, the above principle is equivalent to Hausdorff.

4. The isomorphism Theorems in topological MI-groups

In this section we discuss about the isomorphism theorems in topological MI-groups.

Theorem 4.1. *Let G and $\widetilde{G}$ be topological MI-groups with identities e and $\widetilde{e}$, respectively, such that $P_{\widetilde{G}}$ is closed in MI-group $\widetilde{G}$, i.e $\overline{P_{\widetilde{G}}}^{\widetilde{G}} = P_{\widetilde{G}}$. Let f be an open, continuous homomorphism of G onto $\widetilde{G}$. Then $H = kerf$ is a normal MI-subgroup of G and the sets $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, $\widetilde{x} \in \widetilde{G}$, are exactly the distinct closure of cosets of H in G, so that the mapping $\widetilde{x} \to f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \Phi(\widetilde{x})$ is an open, continuous homomorphism of $\widetilde{G}$ onto the MI-group G/H with the quotient topology and $Ker\ \Phi = P_{\widetilde{G}}$. Also Φ induces an homeomorphism and isomorphism of $\widetilde{G}/P_{\widetilde{G}}$ onto G/H.*

Proof. By theorem 4.14 of [6], $kerf \lhd G$ and so G/H is a quotient MI-group. At first for every $y, y' \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, we show that $\overline{yH}^G = \overline{y'H}^G$. Since $f(y), f(y') \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$, there are $s', t' \in P_{\widetilde{G}}$ such that $f(y)s', f(y')t' \in \widetilde{x}P_{\widetilde{G}}$ and hence there are $s_1, t_1 \in P_{\widetilde{G}}$ such that $f(y)s' = \widetilde{x}s_1$ and $f(y')t' = \widetilde{x}t_1$. But f is surjective and so is strong, i.e. there are $s, t \in P_G$ such that $s' = f(s)$ and $t' = f(t)$, so that $f(ys) = \widetilde{x}s_1$ and $f(y't) = \widetilde{x}t_1$. Accordingly, we will see

$$f(yy'^{-1}st^{-1}) = f(yst^{-1}y'^{-1}) = f(ys(y't)^{-1}) = f(ys)f(y't)^{-1}$$

$$= \widetilde{x}s_1(\widetilde{x}t_1)^{-1} = \widetilde{x}s_1t_1^{-1}\widetilde{x}^{-1} = \widetilde{x}\widetilde{x}^{-1}s_1t_1^{-1} \in P_{\widetilde{G}}.$$

Therefore $yy'^{-1}st^{-1} \in Kerf = H$. Since $st^{-1} \in P_G$, $yy'^{-1} \in \overline{H}^G$ and so $\overline{yH}^G = \overline{y'H}^G$. On the other hand, for $y \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$ fixed , according to this relation for every $y' \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, we have $y' \in \overline{yH}^G$. Hence $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) \subseteq \overline{yH}^G$. Conversely, for each $u \in \overline{yH}^G$, there is $s \in P_G$ such that $us \in yH$ and so $us = yh$, for a member $h \in H$. Since $f(h), f(s) \in P_{\widetilde{G}}$ and $f(y) \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$, we obtain

$$f(u)f(s) = f(y)f(h) \subseteq \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}} P_{\widetilde{G}} \subseteq \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}.$$

Therefore $f(u) \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$, and consequently $u \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, i.e. $\overline{yH}^G \subseteq f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$. Thus $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{yH}^G$ and this is independent of the selection of each member of $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$. Let $\widetilde{U}$ be an open subset of $\widetilde{G}$. We will show that $\Phi(\widetilde{U})$ is open in G/H. It is sufficient to show that, $\varphi^{-1}(\Phi(\widetilde{U}))$ is open in G. But we notice that

$$\varphi^{-1}(\Phi(\widetilde{U})) = \varphi^{-1}\{f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) :\ \widetilde{x} \in \widetilde{U}\} = \bigcup_{\widetilde{x}\in\widetilde{U}} f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = f^{-1}(\overline{\widetilde{U}P_{\widetilde{G}}}^{\widetilde{G}}),$$

where the recent equality is simply obtained from the relation $f^{-1}(\bigcup_u A_u) = \bigcup_u f^{-1}(A_u)$. Accordingly, since $\widetilde{U}P_{\widetilde{G}}$ is open in $\widetilde{G}$, hence $\overline{\widetilde{U}P_{\widetilde{G}}}^{\widetilde{G}}$ is also open and so by continuity of f, $f^{-1}(\overline{\widetilde{U}P_{\widetilde{G}}}^{\widetilde{G}})$ is open in G. Therefore Φ is an open mapping. Let $\{\overline{uH}^G : u \in U\}$ be an open subset of G/H, where U is open in G. Hence we have

$$\Phi^{-1}\{\overline{uH}^G : u \in U\} = \{\widetilde{x} : \exists u \in U\ s.t\ f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{uH}^G\}.$$

But the recent set is equal to $\overline{f(UP_G)}^{\widetilde{G}}$. Since UP_G is open in G and f is an open mapping, the set $f(UP_G)$ is open in $\widetilde{G}$ and so by lemma 3.7, $\overline{f(UP_G)}^{\widetilde{G}}$ is open. About the last equality, for every $\widetilde{x}$ such that $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{uH}^G$ we see that $u \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$ and so $f(u) \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$. Therefore there are $s, t \in P_{\widetilde{G}}$ such that $f(u)s = \widetilde{x}t$. Since f is epimorphism, hence is strong and as a result there is $s_1 \in P_G$ such that $s = f(s_1)$ and

$$\widetilde{x}t = f(u)s = f(u)f(s_1) = f(us_1) \in f(UP_G),$$

i.e. $\widetilde{x} \in \overline{f(UP_G)}^{\widetilde{G}}$. Conversely if $\widetilde{x} \in \overline{f(UP_G)}^{\widetilde{G}}$, then there is $s \in P_{\widetilde{G}}$ such that $\widetilde{x}s \in f(UP_G)$. Therefore there are $u \in U$ and $t \in P_G$ such that $\widetilde{x}s = f(ut)$. Since $f(t) \in P_{\widetilde{G}}$, we can see

$$f(u)f(t) = f(ut) = \widetilde{x}s \in \widetilde{x}P_{\widetilde{G}} \implies f(u) \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}.$$

Hence $u \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, i.e. $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{uH}^G$. Thus Φ is continuous. Also Φ is surjective. In fact for every $\overline{aH}^G \in G/H$, it is sufficient that $\widetilde{x} = f(a)$. Then $f(a) = \widetilde{x} \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$ and so $a \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$, i.e. $\overline{aH}^G = f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \Phi(\widetilde{x})$. Now we will show that Φ is a homomorphism. For each $\widetilde{x}, \widetilde{y} \in \widetilde{G}$, We have to show that $\Phi(\widetilde{x}\widetilde{y}) = \varphi(\widetilde{x}) \star \Phi(\widetilde{y})$ or

$$f^{-1}(\overline{\widetilde{x}\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}) = f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) \star f^{-1}(\overline{\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}).$$

Let $x \in f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$ and $y \in f^{-1}(\overline{\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}})$. Then $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{xH}^G$ and $\overline{yH}^G = f^{-1}(\overline{\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}})$. Also we have $f(x) \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$ and $f(y) \in \overline{\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}$ and so there are $t', s' \in P_{\widetilde{G}}$ such that $f(x)t' \in \widetilde{x}P_{\widetilde{G}}$ and $f(y)s' \in \widetilde{y}P_{\widetilde{G}}$. Therefore there are $t^\star, s^\star \in P_{\widetilde{G}}$ such that $f(x)t' = \widetilde{x}t^\star$ and $f(y)s' = \widetilde{y}s^\star$. As a result

$$f(xy)t's' = f(x)f(y)t's' = \widetilde{x}\widetilde{y}t^\star s^\star \in \widetilde{x}\widetilde{y}P_{\widetilde{G}}.$$

Since $t's' \in P_{\widetilde{G}}$, we conclude that $f(xy) \in \overline{\widetilde{x}\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}$ or $xy \in f^{-1}(\overline{\widetilde{x}\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}})$. Hence $\overline{xyH}^G = f^{-1}(\overline{\widetilde{x}\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}})$. Finally since H is normal MI-subgroup of G, we can see that

$$f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) \star f^{-1}(\overline{\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{xH}^G \star \overline{yH}^G = \overline{xyH}^G = f^{-1}(\overline{\widetilde{x}\widetilde{y}P_{\widetilde{G}}}^{\widetilde{G}}).$$

Therefore Φ is an open, continuous homomorphism of $\widetilde{G}$ onto the MI-group G/H with the quotient topology.

Also by lemma 4.11 of [6], $P_{G/H} = \{\overline{H}^G\}$ and so we will have

$$Ker\ \Phi = \{\widetilde{x} : f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{H}^G\}.$$

By theorem 2.5 of [6], $P_{\widetilde{G}} \subseteq Ker\ \Phi$. Conversely if $\widetilde{x} \in Ker\ \Phi$, then $f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{H}^G$. Since $e \in \overline{H}^G$, we have $f(e) = \widetilde{e} \in \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$ and so $\overline{\widetilde{e}P_{\widetilde{G}}}^{\widetilde{G}} = \overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}$ or $\widetilde{x} \in \overline{P_{\widetilde{G}}}^{\widetilde{G}} = P_{\widetilde{G}}$. Therefore $Ker\ \Phi = P_{\widetilde{G}}$. Accordingly, since for every $\widetilde{x} \in P_{\widetilde{G}}$,

$$\Phi(\widetilde{x}) = f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}}) = \overline{H}^G,$$

hence Φ restricted to $P_{\widetilde{G}}$ is constant and so by theorem 2.7 of [2], Φ is not monomorphism. But by theorem 5.1 of [2], $\widetilde{G}/P_{\widetilde{G}} \cong G/H$. Finally, applying Φ instead of f in the first part, it is easy to see that there is an open, continuous homomorphism Φ' of G/H onto $\widetilde{G}/P_{\widetilde{G}}$ and $Ker\Phi' = P_{G/H} = \{\overline{H}^G\}$. Hence Φ' is injective and so is isomorphism. Also since Φ' is open, Φ'^{-1} is continuous and so Φ' is a homeomorphism of G/H onto $\widetilde{G}/P_{\widetilde{G}}$. □

It is necessary to mention that a homomorphism $f : G \to H$ of MI-groups that satisfies $f(P_G) = P_H$, is called the strong homomorphism of MI-groups. According to the theorem 2.6 of [6], every epimorphism is strong.

The second isomorphism theorem for groups has a complete analogue for topological MI-groups, as follows.

Theorem 4.2. *Let G and $\widetilde{G}$ be topological MI-groups with identities e and $\widetilde{e}$, respectively, such that $P_{\widetilde{G}}$ is closed in MI-group $\widetilde{G}$, i.e $\overline{P_{\widetilde{G}}}^{\widetilde{G}} = P_{\widetilde{G}}$. Let f be an open, continuous homomorphism of G onto $\widetilde{G}$. Let $\widetilde{H}$ be any normal full MI-subgroup of $\widetilde{G}$, $H = f^{-1}(\widetilde{H})$, and $H' = Ker(f)$. Then $H \lhd G$, $H' \lhd H$, and the MI-groups G/H, $\widetilde{G}/\widetilde{H}$, and $(G/H')/(H/H')$ are topologically isomorphic.*

Proof. We first show that H is a normal MI-subgroup of G. Let $y \in \overline{xHx^{-1}}^G$. Then there are $s \in P_G$ and $h \in H$ such that $ys = xhx^{-1}$. Since $\widetilde{H}$ is normal in $\widetilde{G}$, we will have

$$f(ys) = f(x)f(h)f(x)^{-1} \in f(x)\widetilde{H}f(x)^{-1} \subseteq \overline{f(x)\widetilde{H}f(x)^{-1}}^{\widetilde{G}} \subseteq \overline{\widetilde{H}}^{\widetilde{G}}.$$

Therefore there is $\widetilde{t} \in P_{\widetilde{G}}$ such that $f(ys)\widetilde{t} \in \widetilde{H}$. Since f is strong, $\widetilde{t} = f(t)$ for some $t \in P_G$. Hence $f(yst) \in \widetilde{H}$ and so $yst \in H$, i.e. $y \in \overline{H}^G$, as $st \in P_G$. Therefore $\overline{xHx^{-1}}^G \subseteq \overline{H}^G$ and so by Theorem 4.2 of [6], H is a normal MI-subgroup of G. Also for each $x \in H'$, we have $f(x) \in P_{\widetilde{G}} \subseteq \widetilde{H}$, so $x \in H$, i.e. $H' \leq H \leq_f G$. Since by theorem 4.14 of [6], $H' \lhd G$, by theorem 4.6 of [6], $H' \lhd H$. Also it is easy to see $H/H' \lhd G/H'$.

Now let ψ be the natural mapping of $\widetilde{G}$ onto $\widetilde{G}/\widetilde{H}$. It is known that ψ is an open, continuous homomorphism and hence $\psi \circ f$ is an open, continuous homomorphism of G onto $\widetilde{G}/\widetilde{H}$ with kernel $\overline{H}^G$. Therefore by the previous theorem, $G/\overline{H}^G$ is topologically isomorphic with $\widetilde{G}/\widetilde{H}$. But in the other hand, by the lemma 4.1 of [6] for each $x \in G$ we have $\overline{x\overline{H}^G}^G = \overline{xH}^G$. Hence $G/\overline{H}^G = G/H$ and so the MI-groups G/H and $\widetilde{G}/\widetilde{H}$ are topologically isomorphic.

According to the previous theorem, the mapping f will induce an open and continuous homomorphism $\Phi(\widetilde{x}) = f^{-1}(\overline{\widetilde{x}P_{\widetilde{G}}}^{\widetilde{G}})$ of $\widetilde{G}$ onto the MI-group G/H' with the quotient topology and $\Phi^{-1}(H/H') = \widetilde{H}$. Using the argument of the preceding paragraph, $\widetilde{G}/\widetilde{H}$ is topologically isomorphic to $(G/H')/(H/H')$. □

5. Some more results about MI-subgroups

In this section, we describe some other features of MI-subgroups of a topological MI-group. Specifically, these results show that under what conditions a MI-subgroup will be closed. To this end, first we will explain and prove the following important result which shows the connection between the algebraic and topological closures.

Proposition 5.1. *Suppose G is a topological MI-group satisfying (1) and U be a subset of G. Then $\overline{(\overline{U})}^G \subseteq \overline{(\overline{U}^G)}$.*

Proof. Suppose that $x \in \overline{(\overline{U})}^G$. Then $xs \in \overline{U}$ for some $s \in P_G$. Therefore, for each neighborhood xsV of xs we have $xsV \cap U \neq \emptyset$, where V is a neighborhood of e in G. Hence $xsv = u$ for some $u \in U$ and $v \in V$. Then $xvs = u \in U$ and so $xv \in \overline{U}^G$, i.e. $xV \cap \overline{U}^G \neq \emptyset$. Therefore $x \in \overline{(\overline{U}^G)}$. □

Under the condition $\star$, it is easy to see the converse of the previous proposition is also valid:

Proposition 5.2. *Let U be a subset of a topological MI-group G with property $\star$. Then $\overline{(\overline{U}^G)} \subseteq \overline{(\overline{U})}^G$.*

Proof. Let $x \in \overline{(\overline{U}^G)}$. Then for every neighborhood V of e, we have $xV \cap \overline{U}^G \neq \emptyset$ and so $xvs \in U$ for some $s \in P_G$ and $v \in V$. By property $\star$, there is a neighborhood W of xvs such that $\overline{W}^G \subseteq U$. On the other hand, since $xvs \in W$ we have $xv \in \overline{W}^G$ and so $xv \in U$. Therefore $xV \cap U \neq \emptyset$, i.e. $x \in \overline{U}$ and so $x \in \overline{(\overline{U})}^G$. □

Corollary 5.3. *Let G be a topological MI-group with property $\star$. Then for each subset U of G by the above propositions, we have $\overline{(\overline{U})}^G = \overline{(\overline{U}^G)}$. Therefore, for every subset U of G such that $\overline{U}^G = U$, we have $\overline{(\overline{U})}^G \subseteq \overline{U}$ and so we will have $\overline{(\overline{U})}^G = \overline{U}$, i.e. $\overline{U}$ is also closed in MI-group G. Similarly, if U is topologically closed then $\overline{U}^G$ is also topologically closed.*

Theorem 5.4. *Let G be a topological MI-group and H a MI-subgroup of G such that $\overline{H}^G = H$ and $U \cap \overline{H} = U \cap H$, for some neighborhood U of e in G and $U \cap H \neq \emptyset$. Then H is topologically closed in G.*

Proof. Since H is closed in MI-group G, by the previous corollary $\overline{H}$ is also closed in G, i.e. $\overline{(\overline{H})}^G = \overline{H}$. Now suppose that $x \in \overline{H}$. If $x \in U$ then $x \in U \cap \overline{H}$ and so by assumption $x \in U \cap H$, i.e. $x \in H$. If $x \notin U$ then for neighborhood xU of x, we have $xU \cap H \neq \emptyset$. Hence $xu = h$, for some $u \in U$ and $h \in H$. Therefore $x^{-1}xu = x^{-1}h \in \overline{H}$ and so $u \in \overline{(\overline{H})}^G = \overline{H}$, i.e. $u \in U \cap \overline{H}$. Hence by assumption $u \in U \cap H$ and so $u \in H$. Since $xuu^{-1} = hu^{-1} \in H$, we will have $x \in \overline{H}^G = H$. However $x \in H$ and hence $\overline{H} = H$. □

Theorem 5.5. *Let G be a topological MI-group and H a discrete MI-subgroup of G such that $\overline{H}^G = H$. Then H is closed.*

Proof. By assumption, since e is an isolated point of H, there is a neighborhood V of e such that $V \cap H = \{e\}$. If $x \in V \cap \overline{H}$, then $x \in \overline{H}$ and $x \in V$. Let x_α, $\alpha \in D$, be a net in H such that $x_\alpha \to x$. Therefore there is a $\alpha_0 \in D$ such that $x_\alpha \in V$ for each $\alpha \succeq \alpha_0$. Hence $x_\alpha \in V \cap H = \{e\}$, i.e. $x_\alpha = e$. Thus $x = e$ and so $V \cap \overline{H} = \{e\}$, i.e. $V \cap \overline{H} = V \cap H$. Therefore, by theorem 5.4, H is closed. □

Theorem 5.6. *Let G be a topological MI-group and H a MI-subgroup of G such that $\overline{H}^G = H$. If there is a neighborhood U of e including $P_{\overline{H}}$ such that $\overline{U} \cap H$ is topologically closed in G then H is also topologically closed.*

Proof. Let $x \in \overline{H}$. Hence $xx^{-1} \in P_{\overline{H}}$ and so by assumption $xx^{-1} \in U$. Thus there exist a neighborhood V of e such that $xx^{-1}V \subseteq U$. Let W be a symmetric neighborhood of e in G such that $W^2 \subseteq V$. Suppose that x_α, $\alpha \in D$, be a net in H such that $x_\alpha \to x$. Since $x^{-1} \in \overline{H}$, there is $y \in Wx^{-1} \cap H$. Also there is a $\alpha_0 \in D$ such that $x_\alpha \in xW$ for each $\alpha \succeq \alpha_0$. Therefore for every $\alpha \succeq \alpha_0$, we have

$$yx_\alpha \in (Wx^{-1})(xW) = xx^{-1}W^2 \subseteq xx^{-1}V \subseteq U.$$

Therefore $yx_\alpha \in \overline{U} \cap H$. Since $yx_\alpha \to yx$ and $\overline{U} \cap H$ is closed, we have $yx \in \overline{U} \cap H$, i.e. $yx \in H$. Finally, since $y^{-1}yx \in y^{-1}H \subseteq H$, we will have $x \in \overline{H}^G = H$, which implies that $\overline{H} \subseteq H$. Therefore H is closed. □

Theorem 5.7. *Let G be a topological MI-group such that $G = \bigcup_{n=1}^{\infty} V^n$, for every neighborhood V of e. If there is a discrete normal MI-subgroup H of G that is closed in G, then $P_G = \{e\}$ and $H \subseteq Z(G)$, where $Z(G)$ is the center of G.*

Proof. By hypothesis, for each $h \in H$ there is a neighborhood U of e such that $hU \cap H = \{h\}$. According to the continuity of the mapping $x \to x^{-1}hx$ at e, there is a neighborhood V of e such that $V^{-1}hV \subseteq hU$. Therefor $x^{-1}hx \in hU$ for every $x \in V$. On the other hand, since H is normal, we have $\overline{x^{-1}Hx}^G = \overline{H}^G = H$ for each $x \in V$. Consequently, $x^{-1}hx \in hU \cap H = \{h\}$, i.e. $x^{-1}hx = h$ for every $x \in V$. Since $G = \bigcup_{n=1}^{\infty} V^n$, this is valid for every $x \in G$. Indeed, for each $x \in G$ there are $v_1, v_2, ..., v_k \in V$ such that $x = v_1v_2...v_k$ and so after k steps, we will have

$$x^{-1}hx = \overbrace{v_k^{-1} ... \overbrace{v_2^{-1}\overbrace{v_1^{-1}hv_1}^{h}v_2}^{h}\overset{\vdots}{...}v_k}^{h} = h.$$

Therefore $x^{-1}hx = h$, for each $x \in G$ and $h \in H$. Especially, for $h = e$ we have $x^{-1}x = e$, i.e. $P_G = \{e\}$. Hence G have a group structure. Also by multiplying x on the sides of relationship $x^{-1}hx = h$, we will have $hx = xh$, for each $x \in G$ and $h \in H$. Hence $H \subseteq Z(G)$. □

Definition 5.8. Let G be a topological MI-group and let U be a neighborhood of e. U is said to be P_G-invariant, if $UP_G = U$.

For example, for each neighborhood U of e, $W = UP_G$ is obviously P_G-invariant. In fact, $WP_G = UP_GP_G = UP_G^2 \subseteq UP_G = W$.

Now we present another form of theorem 5.6 with a different hypothesis based on the new definition.

Theorem 5.9. *Let G be a topological MI-group and H a MI-subgroup of G such that $\overline{H}^G = H$. If there is a neighborhood P_G-invariant U of e such that $\overline{U} \cap H$ is topologically closed in G then H is also topologically closed.*

Proof. Let U be a neighborhood P_G-invariant of e and V be a neighborhood of e such that $V^2 \subseteq U$. Suppose that $x \in \overline{H}$ and x_α, $\alpha \in D$, be a net in H such that $x_\alpha \to x$. Since $x^{-1} \in \overline{H}$, there is $y \in Vx^{-1} \cap H$. According to the definition, there is a $\alpha_0 \in D$ such that $x_\alpha \in xV$ for each $\alpha \succeq \alpha_0$. That way for every $\alpha \succeq \alpha_0$, we will have

$$yx_\alpha \in (Vx^{-1})(xV) = xx^{-1}V^2 \subseteq P_GU = U.$$

Therefore $yx_\alpha \in \overline{U} \cap H$. Since $yx_\alpha \to yx$ and $\overline{U} \cap H$ is closed, we have $yx \in \overline{U} \cap H$, i.e. $yx \in H$. Now since $y^{-1}yx \in y^{-1}H \subseteq H$, for $y^{-1}y \in P_G$ we have $x \in \overline{H}^G = H$, which this complete the proof. □

Theorem 5.10. *Let G be a topological MI-group, let U be any neighborhood P_G-invariant of e and F be a compact subset of G. Then there is an open subset V of G such that $xVx^{-1} \subseteq U$ for each $x \in F$.*

Proof. Let W be a symmetric neighborhood of e such that $W^3 \subseteq U$. We can assume that W is $P_G - invariant$, otherwise we can replace it with WP_G. Since the family $\{Wx\}_{x \in F}$ is an open cover of compact subset F of G, there are $x_1, x_2, ..., x_k \in F$ such that $F \subseteq \bigcup_{i=1}^{k} Wx_i$. Let $V = \bigcap_{i=1}^{k} x_i^{-1}Wx_i$. Clearly V is an open subset of G by (1) and for each $i = 1, 2, ..., k$ since $V \subseteq x_i^{-1}Wx_i$, we have

$$x_iVx_i^{-1} \subseteq x_ix_i^{-1}Wx_ix_i^{-1} = (x_ix_i^{-1})^2W \subseteq P_GW = W.$$

Finally, for every $x \in F$ we have $x = wx_i$ for some $x_i \in F$ and so

$$xVx^{-1} = wx_iVx_i^{-1}w^{-1} \subseteq wWw^{-1} \subseteq W^3 \subseteq U.$$

□

It is necessary to remember that an anti-involution $\theta : G \to G$ is an automorphism on G such that $\theta^2 = I$ and $\theta(xy) = \theta(y)\theta(x)$. We end this section with a different result on this concept.

Theorem 5.11. *Let G be an MI-group such that $\{e\}$ is closed in G, i.e. $\overline{\{e\}}^G = \{e\}$. If there is an anti-involution $\theta : G \to G$ such that $\theta(x) \in P_G x^{-1}$ for each $x \in G$, Then $\theta(x) = x^{-1}$.*

Proof. By assumption for each $x \in G$, there is $s \in P_G$ such that $\theta(x) = sx^{-1}$ and hence

$$x = \theta(\theta(x)) = \theta(x^{-1})\theta(s) = (\theta(x))^{-1}\theta(s) = (sx^{-1})^{-1}\theta(s) = xs^{-1}\theta(s).$$

By left cancellation law in MI-group G we will have $s^{-1}\theta(s) = e$ and so $\theta(s) \in \overline{\{e\}}^G = \{e\}$, as $s^{-1} \in P_G$. Therefore $\theta(s) = e$ or $s = \theta^2(s) = \theta(e) = e$, i.e. $\theta(x) = x^{-1}$. □

Corollary 5.12. *By the previous theorem, inversion $^{-1}$ is only anti-involution on every MI-group G with the above property.*

6. Existence of nonnegative invariant measures on locally compact MI-groups

Suppose that G be a locally compact Abelian topological MI-group. We consider G as a commutative topological semigroup with identity e. A locally compact abelian semigroup G is embeddable in a locally compact group G', if there exists a bicontinuous semigroup monomorphism φ mapping G into G', i.e. if φ yields a homeomorphism between G and $\varphi(G)$. The following proposition shows that under what conditions a locally compact Abelian semigroup is embeddable as an open subsemigroup of a locally compact Abelian group.

Proposition 6.1. [17] *Let S be a locally compact abelian semigroup. The following conditions on S are eqivalent:*

i) S is a cancellation semigroup and satisfying (1).

ii) S is embeddable as an open subsemigroup of a locally compact group G.

Since each locally compact abelian topological MI-group G is a commutative topological semigroup with identity e and satisfying cancellation laws, by this proposition, G is embeddable as an open subsemigroup of a locally compact abelian group G' if and only if the translations $x \to xy$ are open maps for every $y \in G$. Finally, by the following theorem the restriction of the Haar measure of G' to G is a invariant nonnegative regular measure on G.

Theorem 6.2. [17] *Let S be a locally compact abelian semigroup and μ a nonnegative regular measure on S. Suppose that that S and μ satisfy the following condition.*

(*) For each open set U, xU is open for each $x \in S$ and $\mu(xU) = \mu(U) > 0$.

Then S is embeddable as an open subsemigroup in a locally compact abelian group G and μ is the restriction of the Haar measure of G to S. Conversely if S is an open subsemigroup of a locally compact abelian group G and if μ is the restriction of the Haar measure of G to S, then S is a locally compact abelian semigroup and S and μ satisfy condition $(*)$.

Definition 6.3. A semigroup G is said to be right reversible if $Gx \cap Gy \neq \emptyset$, for every $x, y \in G$.

Every MI-group G as a semigroup is right reversible. Indeed for each $x, y \in G$ by the axiom (3) of definition 1 and that $yy^{-1} = y^{-1}y$, we will have $x(y^{-1}y) = (y^{-1}y)x \in Gx$. On the other hand, $x(y^{-1}y) = (xy^{-1})y \in Gy$. Therefore $x(y^{-1}y) \in Gx \cap Gy$ which implies that $Gx \cap Gy \neq \emptyset$.

Generally, there is a similar theorem on locally compact topological semigroups as follows:

Theorem 6.4. [15] *A locally compact right reversible topological semigroup S having the translations open can support a right and left invariant measure if and only if G can be topologically embedded as an open subspace in a locally compact topological group G and the invariant measure on S is the restriction of an unimodular Haar measure on the group G.*

Since every topological MI-group as a topological semigroup is reversible and satisfying cancellation laws, by the preceding theorems we have:

Theorem 6.5. *Let G be a topological MI-group (abelian or nonabelian). Then G is embeddable as an open subsemigroup in a locally compact group G' if and only if G satisfy the condition (1). Moreover the restriction of a unimodular Haar measure on the group G' to G is an invariant nonnegative regular measure on G.*

In this way, by this embedding, the entire harmonic analysis can be moved to a topological MI-group satisfying (1).

Example 6.6. Every MI-group under the internal topology is a locally compact space. In fact for each $x \in G$, xP_G is a compact neighborhood of x. Also by example 2.6, the translations on G are open and by theorem 3.3, every MI-group G under the internal topology is Hausddorff if and only if $P_G = \{e\}$ and so the internal topology is discrete topology . Therefore every locally compact Hausdorff MI-group under the internal topology will have an invariant nonnegative regular measure, which will obviously be the counting measure.

Theorem 6.7. *Let G be a topological MI-group with property $\star$ and satisfying (1). Also let μ be an invariant nonnegative regular measure on G. If P_G be compact subset of G with $\mu(P_G) > 0$ and $\overline{P_G}^G = P_G$ then for every full noncompact MI-subgroup H of G which is closed in G ($\overline{H}^G = H$), we have $\mu(H) = +\infty$.*

Proof. By assumption, $P_G \subset H$ and so there is $x \in H - P_G$ which implies that $xP_G \cap P_G = \emptyset$. Since $xP_G \subseteq H$, we have $xP_G \cup P_G \subseteq H$ and hence

$$2\mu(P_G) = \mu(P_G) + \mu(xP_G) \leq \mu(H).$$

Since $P_G \cup xP_G$ is compact , by theorem 3.10, subset $\overline{P_G \cup xP_G}^G$ is also compact and $\overline{P_G \cup xP_G}^G \subset \overline{H}^G = H$. Hence there is $y \in H - \overline{P_G \cup xP_G}^G$, otherwise $H = \overline{P_G \cup xP_G}^G$ will be compact. It is obvious that $yP_G \cap P_G = \emptyset$ and $yP_G \cap xP_G = \emptyset$, otherwise we should have $y \in P_G$ or $y \in \overline{xP_G}^G$. Hence the sets P_G , xP_G and yP_G are pairwise disjoint and so

$$\mu(H) \geq \mu(P_G \cup xP_G \cup yP_G) = \mu(P_G) + \mu(xP_G) + \mu(yP_G) = 3\mu(P_G).$$

Continuing this method by induction for each $n \in \mathbb{N}$, we get $\mu(H) \geq n\mu(P_G)$ and so $\mu(H) = +\infty$. □

Corollary 6.8. *According to the recent theorem, if $\mu(G) < \infty$ then the topological MI-group G does not have any full noncompact MI-subgroups H which is closed in G and so G will be a compact topological MI-group.*

7. Conclusion

In this paper, we continued our development of topological MI-group theory focusing on separation axioms, the isomorphism theorems in topological MI-groups and the existence of nonnegative invariant measures on locally compact MI-groups that are well known in group theory. After a brief description of the basic concepts of the MI-groups and by introducing a particular condition on open sets in the T_0 MI-group G, we showed that the rest of the separation axioms are also valid. Under this new condition, called "the $\star$ property", each T_0 MI-group is also Hausdorff and regular. In addition, we have proved the isomorphism theorems for topological MI-groups. In fact, the line of proof, with appropriate changes, is similar to the case of topological groups. Furthermore, some interesting results about the closed MI-subgroups are presented. Finally, the existence of nonnegative invariant measures on locally compact MI-groups is verified. In fact, we have shown that under what conditions a topological MI-group possesses a nonnegative invariant measure. We will focus on other aspects of topological MI-groups in our next research.

8. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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