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FREE IDEALS AND REAL IDEALS OF THE RING OF FRAME MAPS FROM $\mathcal{P}(\mathbb{R})$ TO A FRAME

ALI AKBAR ESTAJI* AND AHMAD MAHMOUDI DARGHADAM

Abstract. Let $\mathcal{F}_{\mathcal{P}}(L)$ ($\mathcal{F}^*_{\mathcal{P}}(L)$) be the f-rings of all (bounded) frame maps from $\mathcal{P}(\mathbb{R})$ to a frame L. $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\uparrow f(-\frac{1}{n}, \frac{1}{n})$ is compact for any $n \in \mathbb{N}$ and the subring $\mathcal{F}_{\mathcal{P}_K}(L)$ is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\operatorname{coz}(f)$ is compact. We introduce and study the concept of real ideals in $\mathcal{F}_{\mathcal{P}}(L)$ and $\mathcal{F}^*_{\mathcal{P}}(L)$. We show that every maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ is real, and also we study the relation between the conditions "L is compact" and "every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real". We prove that for every nonzero real Riesz map $\varphi\colon \mathcal{F}_{\mathcal{P}}(L) \to \mathbb{R}$, there is an element p in ΣL such that $\varphi = \widetilde{p_{\text{coz}}}$ if L is a zero-dimensional frame for which $B(L)$ is a sub-σ-frame of L and every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real. We show that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is equal to the intersection of all free maximal ideals of $\mathcal{F}^*_{\mathcal{P}}(L)$ if $B(L)$ is a sub-σ-frame of a zero-dimensional frame L and also, $\mathcal{F}_{\mathcal{P}_K}(L)$ is equal to the intersection of all free ideals $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) if L is a zero-dimensional frame. Also, we study free ideals and fixed ideals of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_K}(L)$.

1. Introduction

We state from the start that, throughout, by the term ring we mean a commutative ring with identity and a subring of a commutative ring with identity does not imply the identity must belong to the subring.

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*Corresponding author

The ring of all real-valued continuous functions on a frame L is denoted by $\mathcal{R}L$ (see [6] for details). In [20] the authors introduced $\mathcal{F}_{\mathcal{P}}(L) := Frm(\mathcal{P}(\mathbb{R}), L)$ and showed that $R^X \cong \mathcal{F}_{\mathcal{P}}(\mathcal{P}(X))$. Also they proved that $\mathcal{F}_{\mathcal{P}}(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}L$ and showed that the inclusion may be strict.

Let $C(X, \mathbb{R}_d)$ denote the set of continuous functions from a space X into the discrete space of real-numbers $\mathbb{R}_d$. It is known that $C(X, \mathbb{R}_d) \leq C(X)$. If X is discrete, then

$$C(X, \mathbb{R}_d) = C(X) = \mathbb{R}^X \cong \mathcal{F}_{\mathcal{P}}(\mathcal{P}(X)).$$

In this manner, $\mathcal{F}_{\mathcal{P}}(L)$ is the generalization of the f-ring $C(X, \mathbb{R}_d)$.

In [7] an element $\alpha \in \mathcal{R}L$ is called *locally constant* if there exists a partition P of L, meaning P is a cover of L and its elements are pairwise disjoint, such that $\alpha|a$ is constant for each $a \in P$, where $\alpha|a \colon \mathcal{L}(\mathbb{R}) \to \downarrow a$ given by $\alpha|a(v) = \alpha(v) \wedge a$ for every $v \in \mathcal{L}(\mathbb{R})$. The set of all locally constant elements of $\mathcal{R}L$ is denoted by $\mathfrak{S}L$. In [7], Banaschewski showed that $\mathcal{F}_{\mathcal{P}}(L) \cong \mathfrak{S}L$ as f-rings.

For any completely regular Hausdorff space X, $C_\infty(X)$, the subring of all functions $C(X)$ which vanish at infinity, was introduced by Kohls in [22] (also, see [2, 5, 4, 3]). Also, $\mathcal{R}_\infty L$, the ring of real continuous functions vanishing at infinity on a frame L, was first discussed by Dube [10] (also, see [1, 15, 17]).

In this paper, we introduce the subring $\mathcal{F}_{\mathcal{P}_\infty}(L)$ which is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\uparrow f(-\frac{1}{n}, \frac{1}{n})$ is compact for any $n \in \mathbb{N}$ and the subring $\mathcal{F}_{\mathcal{P}_K}(L)$ which is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\operatorname{coz}(f)$ is compact. In Section 3, we show that every ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is an absolutely convex z-ideal and $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ is totally ordered ring if and only if I is a prime ideal of $\mathcal{F}_{\mathcal{P}}(L)$. In Section 4, we introduced real ideals in $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) and we show that a maximal ideal P of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) is real if and only if $\frac{\mathcal{F}_{\mathcal{P}}(L)}{P}$ (resp., $\frac{\mathcal{F}^*_{\mathcal{P}}(L)}{P}$) is archimedean (see Proposition 4.2). Proposition 4.10 contains a complete description of the residue class fields of $\mathcal{F}^*_{\mathcal{P}}(L)$ and also, shows a relation between the conditions "L is compact" and "every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real". In Proposition 4.13, we give a characterization of compact frames in terms of fixed ideals of $\mathcal{F}_{\mathcal{P}}(L)$. In Proposition 4.14, we give a characterization of real maximal ideals of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) in terms of the countable meet property. In Section 5, we show that for every zero-dimensional frame L, $\mathcal{F}_{\mathcal{P}_K}(L)$ is equal to the intersection of all free ideals of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) (see Proposition 5.2) and also every ideal of $\mathcal{F}_{\mathcal{P}_K}(L)$ is fixed (see Corollary 5.8). Next we prove that if $B(L)$ is a sub-σ-frame of a zero-dimensional frame L, then $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is equal to the intersection of all free maximal ideals of $\mathcal{F}^*_{\mathcal{P}}(L)$ and also every ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is fixed (see Proposition 5.4 and Corollary 5.7). In Section 6, we study nonzero bounded Riesz maps on $\mathcal{F}_{\mathcal{P}}(L)$.

2. Preliminaries

We begin by briefly recounting the familiar notions involved here. For this, we recall some necessary definitions and results on frames, $C(X)$, $\mathcal{F}_{\mathcal{P}}(L)$, *real-trail* and f-rings. Interested readers are referred to standard textbook on frames as [23], on f-rings such as [8], and on $C(X)$ such as [16].

Throughout this paper, L will represent a frame. The *pseudocomplement* of an element a of L is denoted by a^* and an element a in L is said to be *complemented* if $a \vee a^* = \top$ and in this case $a' = a^*$. Set $B(L) := \{ a \in L\colon\ a \text{ is complemented} \}$, the sublattice of all complemented elements of L. We easily see that $B(L)$ is a Boolean algebra. A frame L is *zero-dimensional* if it is join-generated by $B(L)$.

Regarding the f-ring $\mathcal{F}_{\mathcal{P}}(L)$ ($\mathcal{F}^*_{\mathcal{P}}(L)$) of all (bounded) frame maps from $\mathcal{P}(\mathbb{R})$ to a frame L, we use the notation of [20]. In [12], it is shown that for any frame L there is a zero-dimensional frame M such that $\mathcal{F}_{\mathcal{P}}L$ and $\mathcal{F}_{\mathcal{P}}M$ are isomorphic.

The properties of the zero map $z\colon \mathcal{F}_{\mathcal{P}}(L) \to L$, given by $z(f) = f(\{0\})$ that we shall frequently use are listed in the following proposition:

Proposition 2.1. [25] *For every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$, we have*

(1) *for every $n \in \mathbb{N}$, $z(f) = z(-f) = z(|f|) = z(f^n)$,*
(2) $z(fg) = z(f) \vee z(g)$,
(3) $z(f+g) \geq z(f) \wedge z(g)$,
(4) $z(f+g) = z(f) \wedge z(g)$, *while* $f, g \geq \mathbf{0}$,
(5) $z(f) = \top$ *if and only if* $f = \mathbf{0}$, *and*
(6) $z(f) = \bot$ *if and only if f is a unit element of* $\mathcal{F}_{\mathcal{P}}(L)$.

For every $f \in \mathcal{F}_{\mathcal{P}}(L)$ and every $A \subseteq \mathcal{F}_{\mathcal{P}}(L)$, we put $\operatorname{coz}(f) := f(\mathbb{R} \setminus \{0\})$, $\operatorname{Coz}(A) := \{ \operatorname{coz}(f)\colon f \in A \}$ and $Z(A) := \{ z(f)\colon f \in A \}$. Then $\operatorname{coz}(f) = (z(f))' = (z(f))^*$, which implies that for every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$, $z(g) \leq z(f)$ if and only if $\operatorname{coz}(f) \leq \operatorname{coz}(g)$ and this is equivalent to the fact that $coz(f) \prec\!\!\prec_{\operatorname{Coz}(L)} coz(g)$.

On the other hand, Estaji el at. in [13] proved that for every complemented a in L, the map $f_a\colon P(\mathbb{R}) \to L$ given by

$$f_a(X) = \begin{cases} \top & \text{if } 0, 1 \in X \\ a' & \text{if } 0 \in X \text{ and } 1 \notin X \\ a & \text{if } 0 \notin X \text{ and } 1 \in X \\ \bot & \text{if } 0 \notin X \text{ and } 1 \notin X, \end{cases}$$

belongs to $\mathcal{F}_{\mathcal{P}}(L)$, $f_a^2 = f_a$, $z(f_a) = a'$, $\operatorname{coz}(f_a) = a$ and for every $f \in \mathcal{F}_{\mathcal{P}}(L)$ and every $X \in P(\mathbb{R})$,

$$ff_a(X) = \begin{cases} a' \vee f(X) & \text{if } 0 \in X, \\ a \wedge f(X) & \text{if } 0 \notin X. \end{cases}$$

Therefore,

$$B(L) = Z(\mathcal{F}_{\mathcal{P}}(L)) = \operatorname{Coz}(\mathcal{F}_{\mathcal{P}}(L)) = \{x \in L : \ x \in \operatorname{Im}(f) \text{ for some } f \in \mathcal{F}_{\mathcal{P}}(L) \ \}.$$

An *real-trail* on L is a function $t \colon \mathbb{R} \longrightarrow L$ such that $\bigvee_{x \in \mathbb{R}} t(x) = \top$ and $t(x) \wedge t(y) = \bot$ for every $x, y \in \mathbb{R}$ with $x \neq y$. For every real-trail t on a frame L,

$$\begin{aligned} \varphi_t \colon \mathcal{P}(\mathbb{R}) \ &\longrightarrow \ L \\ X \ &\longmapsto \ \textstyle\bigvee_{x \in X} t(x) \end{aligned}$$

is a frame map. Throughout this paper, this notation will be used. In [12], it is shown that for every frame L, there is a zero-dimensional frame M such that $\mathcal{F}_{\mathcal{P}}(L) \cong \mathcal{F}_{\mathcal{P}}(M)$ (see [7]).

We recall from [19, 14] that for every f-ring A with bounded inversion, $F(A, L)$ is the set of all functions from A to L and for every element a of an f-ring A and every $r, s \in \mathbb{Q}$,

$$\delta_{rs}^a := (a - r)^+ \wedge (s - a)^+$$

is nominated as *interval projection* and a lattice-valued map $c \in F(A, L)$ is called

(1) *cozero lattice-valued map* if it satisfies
   - ($c1$) $c(0) = \bot$,
   - ($c2$) If $x \in A$ is a unit, then $c(x) = \top$,
   - ($c3$) If $x, y \geq 0$, then $c(x \vee y) = c(x) \vee c(y)$,
   - ($c4$) If $x, y \geq 0$, then $c(x \wedge y) = c(x) \wedge c(y)$.

(2) *continuous*, if $c(\delta_{pq}^x) = \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p<r<s<q}} c(\delta_{rs}^x)$ for any $p, q \in \mathbb{Q}$ and any $x \in A$.

(3) *bounded* if $\bigvee_{p,q \in \mathbb{Q}} c(\delta_{pq}^a) = \top$, for all $a \in A$.

(4) $\mathbb{Q}$-*compatible* if for every $\diamond \in \{+, \cdot, \vee, \wedge\}$, $a, b \in A$, and $r, s, w, z, p, q \in \mathbb{Q}$,

$$< r, s > \diamond < w, z > \subseteq < p, q > \Rightarrow c(\delta_{rs}^a) \wedge c(\delta_{wz}^b) \leq c(\delta_{pq}^{a \diamond b}).$$

We also will need the following propositions which appear in [14]. They are counterparts of Lemmas 3.1 and 3.5 in [21].

Proposition 2.2. *Let $c \in F(A, L)$ be a bounded cozero lattice-valued map, and let*

$$L_c(p, a) = \{s \in \mathbb{Q} \mid \ r < s \Rightarrow c(\delta_{rs}^a) \leq p \text{ for all } r \in \mathbb{Q}\},$$

and

$$U_c(p, a) = \{r \in \mathbb{Q} \mid \ r < s \Rightarrow c(\delta_{rs}^a) \leq p \text{ for all } s \in \mathbb{Q}\}$$

for every $(p, a) \in \Sigma L \times A$. Then $(L_c(p, a), U_c(p, a))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p_c}(a)$ for any $(p, a) \in \Sigma L \times A$.

Proposition 2.3. *Let $c \in F(A, L)$ be a $\mathbb{Q}$-compatible bounded continuous cozero lattice-valued map. For each nonzero bounded Riesz map $\phi \colon A \to \mathbb{R}$, if $p \in \Sigma L$ with $\bigvee c\big(\ker(\phi)\big) \le p$, then $\phi = \widetilde{p_c}$.*

Let A be an ordered ring. An A-module M is called an ordered module if $x, y \ge 0$ and $a \ge 0$ imply that $x + y \ge 0$ and $ax \ge 0$; and it is called an ℓ-module if it is a lattice with its order; an f-module if for every $a \ge 0$ with $a \in A$, $x, y \in M$, $a(x \wedge y) = ax \wedge ay$. An ℓ-module over $\mathbb{Q}$ is called a Riesz space; note that every Riesz space is an f-module. A submodule I of M is called an ℓ-ideal if $|a| \le |b|$ and $b \in I$ imply $a \in I$, where $|a| = a \vee (-a)$. A module homomorphism which preserves the lattice operations is called ℓ-module homomorphism. The ℓ-module homomorphisms between Riesz spaces are called Riesz maps. For more information see [11, 18].

3. Residue class ring of $\mathcal{F}_{\mathcal{P}}(L)$ or $\mathcal{F}^*_{\mathcal{P}}(L)$ modulo an ideal

The notion of restriction of $\alpha \in \mathcal{R}(L)$ to some $a \in L$ is introduced by Banaschewski in [6] corresponding to the topological notion of restricting continuous maps on a space to some open subspace: $\alpha|a$ is the homomorphism $\mathcal{L}(\mathbb{R}) \to {\downarrow}a$ such that $\alpha|a(p, q) = \alpha(p, q) \wedge a$, that is, the composite of α with the quotient map $L \to {\downarrow}a$ which takes x to $x \wedge a$. Similarly, we define $f|a \colon \mathcal{P}(\mathbb{R}) \to {\downarrow}a$ by $f|a(X) = f(X) \wedge a$ for every $a \in L$ and every $f \in \mathcal{F}_{\mathcal{P}}(L)$.

Remark 3.1. For every $f \in \mathcal{F}_{\mathcal{P}}(L)$, $f \in \mathcal{F}^*_{\mathcal{P}}(L)$ if and only if $f|\operatorname{coz}(f) \in \mathcal{F}^*_{\mathcal{P}}({\downarrow}\operatorname{coz}(f))$.

If X is a completely regular topological space and $f, g \in C(X)$ such that $z(g) \subseteq int(z(f))$, then there exists an element $h \in C(X)$ such that $f = hg$. Also if $\operatorname{coz}(f)$ is compact then there exists an $h \in C^*(X)$, such that $f = hg$ see [16]. If L is a completely regular frame and $\alpha, \beta \in \mathcal{R}L$ such that $\operatorname{coz}(\alpha) \prec\!\!\prec \operatorname{coz}(\beta)$ there exists an element $\gamma \in \mathcal{R}L$ such that $\alpha = \gamma\beta$ (see [9]). These facts lead us to the following result.

Lemma 3.2. *Let $f, g \in \mathcal{F}_{\mathcal{P}}(L)$ such that $z(g) \le z(f)$, then there exists an element $h \in \mathcal{F}_{\mathcal{P}}(L)$ such that $f = hg$. Also if $\operatorname{coz}(f)$ is compact, then there exists an element $h \in \mathcal{F}^*_{\mathcal{P}}(L)$ such that $f = hg$.*

Proof. We define the real-trail $t \colon \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} z(g) & \text{if } x = 0 \\ g(\{\frac{1}{x}\}) & \text{if } x \neq 0, \end{cases}$$

which implies that

$$g\varphi_t(\{x\}) = \begin{cases} z(g) & \text{if } x = 0 \\ \operatorname{coz}(g) & \text{if } x = 1 \\ \bot & \text{if } x \neq 1, 0. \end{cases}$$

Consider $h := f\varphi_t$. Then $\operatorname{coz}(h) = \operatorname{coz}(f)$, and for every $x \in \mathbb{R}$,

$$\begin{aligned} hg(\{x\}) &= f\varphi_t g(\{x\}) \\ &= \bigvee\{f(\{y\}) \wedge \varphi_t g(\{y'\})\colon yy' = x\} \\ &= \begin{cases} z(f) & \text{if } x = 0 \\ f(\{x\}) & \text{if } x \neq 0 \end{cases} \\ &= f(\{x\}), \end{aligned}$$

which implies that $f = hg$. Also, if $\operatorname{coz}(f)$ is compact, then $h|\operatorname{coz}(h)$ is bounded and, by Remark 3.1, $h \in \mathcal{F}^*_{\mathcal{P}}(L)$. □

As an immediate consequence of Lemma 3.2, we can state the following proposition.

Proposition 3.3. *Let I be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$). For every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$ (resp., $f, g \in \mathcal{F}^*_{\mathcal{P}}(L)$), if $z(f) \leq z(g)$ and $f \in I$, then $g \in I$.*

Let $a \in L$ and $f \in \mathcal{F}_{\mathcal{P}}(L)$, we say f on a is nonpositive (resp., nonnegative) if $f|a \leq 0$ (resp., $f|a \geq 0$) and equivalently, if $f((-\infty, 0]) \geq a$ (resp., $f([0, +\infty)) \geq a$). Also, we say f on a does not change sign if $f|a \leq 0$ or $f|a \geq 0$. The next proposition clarifies to some extent the relation between prime ideals and zero elements.

Proposition 3.4. *Let I be a proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Then the following statements are equivalent.*

(1) *I is a prime ideal.*
(2) *I contains a prime ideal.*
(3) *For every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$, if $fg = 0$, then $f \in I$ or $g \in I$.*
(4) *For every $f \in \mathcal{F}_{\mathcal{P}}(L)$, there is a zero element belonging to $Z[I]$ on which f does not change sign.*
(5) *For every $f \in \mathcal{F}_{\mathcal{P}}(L)$, there is an element g belonging to I such that $f(0, +\infty) \leq \operatorname{coz}(g)$ or $f(-\infty, 0) \leq \operatorname{coz}(g)$.*
(6) *I is a maximal ideal.*

Proof. (1) ⇒ (2) ⇒ (3). Trivial.

(3) $\Rightarrow$ (4). Consider $f \in \mathcal{F}_{\mathcal{P}}(L)$. Since $(f \vee \mathbf{0})(f \wedge \mathbf{0}) = \mathbf{0}$, we conclude that $f \vee \mathbf{0} \in I$ or $f \wedge \mathbf{0} \in I$. If $f \vee \mathbf{0} \in I$, then $f(0, +\infty) \leq (f \vee \mathbf{0})(0, +\infty) = \operatorname{coz}(f \vee \mathbf{0})$, which implies that $f(-\infty, 0] \geq z(f \vee \mathbf{0})$, that is $f|z(f \vee \mathbf{0}) \leq 0$. Similarly, if $f \wedge \mathbf{0} \in I$, then $f|z(f \wedge \mathbf{0}) \geq 0$.

(4) $\Rightarrow$ (5). Let $f \in \mathcal{F}_{\mathcal{P}}(L)$ be given. Then there is an element $g \in I$ such that $f|z(g) \leq 0$ or $f|z(g) \geq 0$, which implies that $z(g) \leq f(-\infty, 0]$ or $z(g) \leq f[0, +\infty)$, and so $f(0, +\infty) \leq \operatorname{coz}(g)$ or $f(-\infty, 0) \leq \operatorname{coz}(g)$.

(5) $\Rightarrow$ (1). Let $f, g \in \mathcal{F}_{\mathcal{P}}(L)$ with $fg \in I$ be given. Then for the element $h = |f| - |g|$ in $\mathcal{F}_{\mathcal{P}}(L)$, by hypothesis, there exists an element $\alpha \in I$ such that $h(0, +\infty) \leq \operatorname{coz}(\alpha)$ or $h(-\infty, 0) \leq \operatorname{coz}(\alpha)$. If $h(0, +\infty) \leq \operatorname{coz}(\alpha)$ we have

$$\operatorname{coz}(|f|) = (h + |g|)(0, +\infty) \leq h(0, +\infty) \vee |g|(0, +\infty) \leq \operatorname{coz}(\alpha) \vee |g|(0, +\infty) = \operatorname{coz}(|\alpha| \vee |g|).$$

Therefore,

$$\operatorname{coz}(f) = \operatorname{coz}(|f|) \wedge \operatorname{coz}(|\alpha| \vee |g|) = \operatorname{coz}(|f\alpha| + |fg|) \in \operatorname{coz}(I),$$

which implies that $f \in I$. We note similarly that if $h(-\infty, 0) \leq \operatorname{coz}(\alpha)$, then $g \in I$. Hence, I is a prime ideal.

(1) $\Leftrightarrow$ (6). By Proposition 3.3 in [12], it is clear, since $\mathcal{F}_{\mathcal{P}}(L)$ is a regular ring. □

Remark 3.5. We recall that an ideal I of an f-ring A is an ℓ-ideal if $|x| \leq |y|$ and $y \in I$ imply $x \in I$. Hence, by Proposition 3.3, every ideal $\mathcal{F}_{\mathcal{P}}(L)$ is an ℓ-ideal and also if I is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ then I is a convex ideal, that is, if whenever $0 \leq x \leq y$, and $y \in I$, then $x \in I$. Hence, by Theorem 5.2 in [16], $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ is a partially ordered ring, according to the definition:

$$f + I \geq 0 \text{ if there exists an element } g \in \mathcal{F}_{\mathcal{P}}(L) \text{ such that } g \geq 0 \text{ and } f - g \in I.$$

Throughout this paper, this notation will be used. Also, by Theorem 5.3 in [16], the following statements hold for every ideal I of $\mathcal{F}_{\mathcal{P}}(L)$ and every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$.

(1) $f \in I$ if and only if $|f| \in I$.
(2) $f, g \in I$ implies $f \vee g \in I$.
(3) $(f \vee g) + I = (f + I) \vee (g + I)$.
(4) $f + I \geq 0$ if and only if $f - |f| \in I$.

The above results are true for $\mathcal{F}^*_{\mathcal{P}}(L)$.

To establish that A is totally ordered, it is enough to show that every element is comparable with 0. Therefore, in the following proposition, we have determined the elements of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ which are the nonnegative elements.

Proposition 3.6. *Let I be a proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Then for every $f \in \mathcal{F}_{\mathcal{P}}(L)$, the following statements hold.*

(1) $f + I \geq 0$ *if and only if* $f|a$ *is nonnegative on at least one* a *of* $Z[I]$.

(2) *If* $f(0, +\infty) \geq a$ *on at least one* a *of* $Z[I]$, *then* $f + I > 0$, *and if* I *is maximal, the converse holds as well.*

Proof. (1). *Necessity.* By Remark 3.5, $z(f - |f|) = f([0, +\infty)) \in Z[I]$. Therefore, f and $|f|$ have the same sign on $z(f - |f|)$, and hence f is nonnegative on $z(f - |f|)$.

Sufficiency. Let $f|z(g) \geq 0$ for some $g \in I$, then $z(|f| - f) = f([0, +\infty)) \geq z(g)$ for some $g \in I$. By Lemma 3.2, there exists an element $k \in \mathcal{F}_{\mathcal{P}}(L)$ such that $|f| - f = gk \in I$. We infer that $f + I \geq 0$.

(2) Let $f \in \mathcal{F}_{\mathcal{P}}(L)$ and $f(0, +\infty) \geq a$ for some $a \in Z[I]$, then $z(f) \wedge a = \perp$, and hence $f \notin I$. By the first statement, $f + I > 0$.

Let I be a maximal ideal and $f + I > 0$. Then, by the first statement, there exists an element $a \in Z[I]$ such that $f|a \geq 0$. Since $f \notin I$, we conclude that there exists an element $b \in Z[I]$ such that $b \wedge z(f) = \perp$, which implies that $b \wedge a \in Z[I]$ and $f(0, +\infty) \geq a \wedge b$. □

The following example shows that the maximal condition on I is necessary in the inverse of the second statement of the above proposition.

Example 3.7. Suppose that I and J are ideals of $\mathcal{F}_{\mathcal{P}}(L)$ such that $I \subsetneq J$. If $f \in J \setminus I$, then $I + f^2 > 0$. Since $z(f) = z(f^2) \in Z[J]$, we infer that $z(f) \wedge a \neq \perp$, for any $a \in Z[J] \supseteq Z[I]$, which implies that $z(f) \wedge a \neq \perp$, for any $a \in Z[I]$. Therefore, there is not an element $a \in Z[I]$ such that $f(0, +\infty) \geq a$.

The relation between prime ideals of $\mathcal{F}_{\mathcal{P}}(L)$ and the residue class fields of $\mathcal{F}_{\mathcal{P}}(L)$ is clarified by the next proposition.

Proposition 3.8. *For every proper ideal* I *of* $\mathcal{F}_{\mathcal{P}}(L)$, $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ *is a totally ordered ring if and only if* I *is a prime ideal.*

Proof. Necessity. Consider $f \in \mathcal{F}_{\mathcal{P}}(L)$. Since $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ is a totally ordered ring, we conclude that $f + I \leq 0$ or $f + I \geq 0$, which by the first statement implies that there is a zero element belonging to $Z[I]$ on which f does not change sign, and so, by Proposition 3.4, I is a prime ideal.

Sufficiency. Consider $f + I$ and $g + I$ are two elements of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$. By Proposition 3.4, there is a zero element belonging to $Z[I]$ on which $f - g$ does not change sign, which implies that $f + I \leq g + I$ or $f + I \geq g + I$. Therefore, $\frac{\mathcal{F}_{\mathcal{P}}(L)}{I}$ is totally ordered ring. □

It is well known that a subring R' of a partially ordered ring R is called an absolutely convex if $f \in R'$ and $|g| \leq |f|$, then $g \in R'$ for every $f, g \in R$. It is clear that $\mathcal{F}^*_{\mathcal{P}}(L)$ is an absolutely convex subring of $\mathcal{F}_{\mathcal{P}}(L)$.

Proposition 3.9. *Let P be an ideal of an absolutely convex subring R of $\mathcal{F}_{\mathcal{P}}(L)$. If P is a semiprime ideal of R, then P is an absolutely convex ideal of R.*

Proof. Let P be a prime ideal of R, and let $(f, g) \in R \times P$ with $|f| \leq |g|$ be given. We define the real-trail $t\colon \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} z(f) & \text{if } x = 0 \\ \bigvee\{\, f(\{y\}) \wedge g(\{z\}) \colon y, z \in \mathbb{R} \setminus \{0\},\ \frac{y^2}{z} = x \,\} & \text{if } x \neq 0. \end{cases}$$

Then $f^2 = g\varphi_t \in P$, which implies that $f \in P$. □

4. Real ideals in $\mathcal{F}_{\mathcal{P}}(L)$

For every proper ideal P of $\mathcal{F}_{\mathcal{P}}(L)$, it is clear that $\theta\colon \mathbb{R} \to \frac{\mathcal{F}_{\mathcal{P}}(L)}{P}$ given by $r \mapsto \mathbf{r} + P$ is a monomorphism, which implies that $\frac{\mathcal{F}_{\mathcal{P}}(L)}{P}$ has a copy of $\mathbb{R}$. This fact leads to the following definition.

Definition 4.1. Let R be a subring of $\mathcal{F}_{\mathcal{P}}(L)$. A maximal ideal P of R is called real if $\frac{R}{P} \cong \mathbb{R}$, otherwise it is called hyper-real.

We recall from [16] that a totally ordered field F is said to be *archimedean* if for every element $a \in F$, there exists $n \in \mathbb{N}$ such that $n \geq a$. We will also need the following result which appears in [16, Theorem 0.21].

Proposition 4.2. *A maximal ideal P of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$) is real if and only if $\frac{\mathcal{F}_{\mathcal{P}}(L)}{P}$ (resp., $\frac{\mathcal{F}^*_{\mathcal{P}}(L)}{P}$) is archimedean.*

We recall from [16] that a nonarchimedean field is characterized (among all totally ordered fields) by the presence of infinitely large elements, that is, elements a such that $a > n$ for every $n \in \mathbb{N}$.

The following proposition relates unbounded functions of $\mathcal{F}_{\mathcal{P}}(L)$ with infinitely large elements modulo maximal ideals.

Proposition 4.3. *Let M be a maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Then for every $f \in \mathcal{F}_{\mathcal{P}}(L)$, the following statements are equivalent.*

(1) *$|f + M|$ is an infinitely large element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$.*
(2) *$f|a$ is unbounded for every $a \in Z[M]$.*
(3) *$|f|[n, +\infty) \in Z[M]$ for every $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ be given. Then, by Proposition 3.6, $|f + M| \geq n$ if and only if there exists an element $a \in Z[M]$ such that $|f||a \geq n$ if and only if $|f|[n, +\infty)$ is greater than or equal to a member of $Z[M]$ if and only if $|f|[n, +\infty) \in Z[M]$. □

Definition 4.4. An ideal I of a subring of $\mathcal{F}_{\mathcal{P}}(L)$ is *fixed* if $\bigvee_{f\in I}\operatorname{coz}(f)\neq\top$ and it is a *free ideal* if $\bigvee_{f\in I}\operatorname{coz}(f)=\top$.

The following proposition relates compact elements of L with proper free ideals of $\mathcal{F}_{\mathcal{P}}(L)$ or $\mathcal{F}^*_{\mathcal{P}}(L)$.

Lemma 4.5. *Let L be a zero-dimensional frame and $a\in L$. Then the following statements hold.*

(1) *If a is a compact element of L, then $a\in B(L)$.*

(2) *a is a compact element of L if and only if $a\in \operatorname{Coz}[I]\setminus Z[I]$ for every proper free ideal I of $\mathcal{F}_{\mathcal{P}}(L)$.*

(3) *a is a compact element of L if and only if $a\in \operatorname{Coz}[I]\setminus Z[I]$ for every proper free ideal I of $\mathcal{F}^*_{\mathcal{P}}(L)$.*

(4) *a is a compact element of L if and only if $a\in \operatorname{Coz}(M)$ for every free maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$.*

Proof. (1). It is clear.

(2). *Necessity.* By the first statement, $a=\operatorname{coz}(f_a)$. Then

$$a=\operatorname{coz}(f_a)\wedge\top=\operatorname{coz}(f_a)\wedge\bigvee_{f\in I}\operatorname{coz}(f)=\bigvee_{f\in I}\big(\operatorname{coz}(f_a)\wedge\operatorname{coz}(f)\big)=\bigvee_{f\in I}\operatorname{coz}(f_a f),$$

which implies that there are $f_1,f_2,\dots,f_n\in I$ such that

$$a=\bigvee_{i=1}^{n}\operatorname{coz}(f_a f_i)=\operatorname{coz}\Big(\sum_{i=1}^{n}(f_a f_i)^2\Big).$$

The proof is now complete, because $\sum_{i=1}^{n}(f_a f_i)^2\in I$. The rest is obvious.

Sufficiency. Let a be a noncompact element of L. Therefore, there is a subset S of L such that $\bigvee S=a$ and $\bigvee F\neq a$ for every finite subset F of S. We assume that I is the ideal of $\mathcal{F}_{\mathcal{P}}(L)$ generated by

$$\{f_{a'}\}\cup\{f\in\mathcal{F}_{\mathcal{P}}(L):\ \operatorname{coz}(f)\leq\bigvee F \text{ for some finite subset } F \text{ of } S\ \}.$$

Then I is a proper free ideal and $a\notin\operatorname{coz}[I]$, which is a contradiction. The proof is now complete.

(3). The proof is similar to the proof of the second statement. □

By the following proposition, we show that whenever $B(L)$ is a sub-σ-frame of L, then the equality $\mathcal{F}_{\mathcal{P}}(L)=\mathcal{F}^*_{\mathcal{P}}(L)$ implies that L is compact.

Proposition 4.6. *The following statements hold for every zero-dimensional frame L.*

(1) *If L is compact, then $\mathcal{F}_{\mathcal{P}}(L)=\mathcal{F}^*_{\mathcal{P}}(L)$.*

(2) *If $B(L)$ is a sub-σ-frame of L and $\mathcal{F}_{\mathcal{P}}(L) = \mathcal{F}^*_{\mathcal{P}}(L)$, then L is compact.*

Proof. (1). It is evident.

(2). Let L be not compact, and let $S \subseteq L$ such that $\bigvee S = \top$ and $\bigvee F \neq \top$ for every finite subset F of S. For every $a \in S$, there is a subset C_a of $B(L)$ such that $a = \bigvee C_a$. Consider $C := \bigcup_{a\in A} C_a$. Then $\bigvee F \neq \top$ for every finite subset F of C. Therefore, without losing generality we may assume that $\bigvee(C \setminus \{c\}) \neq \top$ for every $c \in C$. Let $B := \{c_{n+1} \in C \colon n \in \mathbb{N}\}$ be an infinite countable subset of C. Since $B(L)$ is a σ-frame, we conclude that $\bigvee B$ has a complement in L, say c_1. We put $b_n := \bigvee_{i=1}^{n} c_i$ for every $n \in \mathbb{N}$, and define the real-trail $t \colon \mathbb{R} \to L$ on L by

$$t(x) = \begin{cases} b_1 & \text{if } x = 1 \\ b_x \wedge b'_{x-1} & \text{if } x \in \mathbb{N} \setminus \{1\} \\ \bot & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_t \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$, which is a contradiction. □

Here, we show by an example that the condition "$B(L)$ is a sub-σ-frame of L" is necessary in Proposition 4.6.

Example 4.7. We recall from [16] that the set of all ordinals less than the first uncountable ordinal is denoted by $W(\omega_1)$, where ω_1 is the first uncountable ordinal. The topology on $W(\omega_1)$ is the interval topology. The space $W(\omega_1)$ is pseudocompact but not compact, which implies that $\mathcal{F}_{\mathcal{P}}\big(W(\omega_1)\big) = \mathcal{F}^*_{\mathcal{P}}\big(W(\omega_1)\big)$.

Remark 4.8. Let a be an element of a frame L. If $x \in B(L) \cap {\downarrow}a$, then $x \wedge (x' \wedge a) = \bot$ and $x \vee (x' \wedge a) = a$. Hence, $B(L) \cap {\downarrow}a \subseteq B({\downarrow}a)$. Therefore, if L is a zero-dimensional frame, then ${\downarrow}a$ is a zero-dimensional frame. Also, if $a \in B(L)$, then $B(L) \cap {\downarrow}a = B({\downarrow}a)$.

We recall that a subsete $\mathcal{F}$ of L is called a $z_{\mathcal{F}_{\mathcal{P}}}$**-filter** on L if the following statements hold:

(1) $0 \notin \mathcal{F}$,
(2) for every $a, b \in \mathcal{F}$, there exists a $c \in \mathcal{F}$ that $c \leq a \wedge b$, and
(3) if $b \in \mathcal{F}$, $a \in L$, and $b \leq a$, then $a \in \mathcal{F}$.

It is evident that $\mathcal{F} \subseteq L$ is a $z_{\mathcal{F}_{\mathcal{P}}}$-filter (resp., $z_{\mathcal{F}_{\mathcal{P}}}$-ultrafilter) if and only if a proper filter (resp., an ultrafilter) of $B(L)$. Therefore, a subsete I of $\mathcal{F}_{\mathcal{P}}(L)$ is a proper ideal (resp., maximal ideal) of $\mathcal{F}_{\mathcal{P}}(L)$ if and only if the family $z[I] = \{z(f) \mid f \in I\}$ is a filter (resp., ultrafilter) on $B(L)$. Also, a subsete $\mathcal{F}$ of $B(L)$ is a proper filter (resp., an ultrafilter) of $B(L)$ if and only if the family $z^{-1}[\mathcal{F}] = \{f \mid z(f) \in \mathcal{F}\}$ is a proper ideal (resp., maximal ideal) of $\mathcal{F}_{\mathcal{P}}(L)$.

Proposition 4.9. *The following statements hold for every zero-dimensional frame L and every $f \in \mathcal{F}_{\mathcal{P}}(L)$.*

(1) *$f \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$ if and only if there exists a maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that $|f+M|$ is an infinitely large element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$ and M is not real.*

(2) *Suppose that $a \in B(L)$ such that $B(\downarrow a)$ is a sub-σ-frame of $\downarrow a$. Then a is a compact element of L if and only if $a \notin Z[I]$ for every proper free ideal I in $\mathcal{F}_{\mathcal{P}}(L)$.*

(3) *$|f+M|$ is infinitely large for every free maximal ideal M in $\mathcal{F}_{\mathcal{P}}(L)$ if and only if for every $a \in B(L)$, a is non-compact element of L implies $f|a \in \mathcal{F}_{\mathcal{P}}(\downarrow a) \setminus \mathcal{F}^*_{\mathcal{P}}(\downarrow a)$.*

Proof. (1) *Necessity.* Let $f \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$ be given. We put $a_n := |f|[n, +\infty)$ for any $n \in \mathbb{N}$. Since for every finite subset S of $\mathbb{N}$, we have $\bot \neq \bigwedge_{i \in S} a_i \in B(L)$, we conclude that there exists an ultrafilter $\mathcal{F}$ of $B(L)$ containing $\{a_n : n \in \mathbb{N}\}$. Hence, $M := z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ and, by Proposition 4.3, $|f+M|$ is an infinitely large element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$, and also, by Proposition 4.2, M is not real.

Sufficiency. It is obvious.

(2) *Necessity.* By Lemma 4.5, it is clear.

Sufficiency. Let $a \in B(L)$ be not a compact element of L, then $\downarrow a$ is not a compact frame, which from Proposition 4.6 and Remark 4.8 imply that there is an element $f \in \mathcal{F}_{\mathcal{P}}(\downarrow a) \setminus \mathcal{F}^*_{\mathcal{P}}(\downarrow a)$. We define the real-trail $t \colon \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} f(\{x-1\}) & \text{if } x > 1 \\ z(f) \vee a' & \text{if } x = 0 \\ f(\{x+1\}) & \text{if } x < -1 \\ \bot & \text{if } 0 < x \leq 1 \text{ or } -1 \leq x < 0. \end{cases}$$

Hence, $\varphi_t \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$. For any $n \in \mathbb{N}$, we put $a_n := |\varphi_t|([n, -))$. Now, similar to the proof of the first statement, there exists a maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that it is not real and $\{a_n : n \in \mathbb{N}\} \subseteq Z[M]$. Therefore, M is a free maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ and $a \in Z[M]$, which is a contradiction.

(3) *Necessity.* Let $a \in B(L)$ be a non-compact element of L. Then, by Lemma 4.5, there exists a proper free ideal I of $\mathcal{F}_{\mathcal{P}}(L)$ such that $a \in Coz[I] \setminus Z[I]$. Let M be a free maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ with $I \subseteq M$. If $a' \in Z[M]$, then $\bot = a \wedge a' \in Z[M]$, which is a contradiction and this implies that $a \in Z[M]$. Therefore, by Proposition 4.3, $f|a \in \mathcal{F}_{\mathcal{P}}(\downarrow a) \setminus \mathcal{F}^*_{\mathcal{P}}(\downarrow a)$.

Sufficiency. We argue by contradiction. Let us assume that there exists a free maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that $|f+M|$ is not an infinitely large element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$. Then, by Proposition 4.3, there exists an element $a \in Z[M]$ such that $f|a$ is bounded. The hypothesis now implies that $a \in B(L)$ is a compact element of L, and so, by Lemma 4.5, $a \in Coz[M] \setminus Z[M]$, which is a contradiction. □

The following proposition shows the connection between real maximal ideals and compact frames.

Proposition 4.10. *The following statements hold for every zero-dimensional frame L.*

(1) *Every maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ is real.*
(2) *If L is compact, then every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real.*
(3) *If every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real and $B(L)$ is a sub-σ-frame of L, then L is compact.*

Proof. (1). If M is a maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ and $f \in \mathcal{F}^*_{\mathcal{P}}(L)$, then $|f + M| \leq n$ for some $n \in \mathbb{N}$, which from Proposition 4.2 implies that M is real.

(2). By Proposition 4.6 and by the first statement, it is evident.

(3). By Proposition 4.6, there exists an element $f \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$. By Proposition 4.9, there exists a maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that $|f + M|$ is an infinitely large element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$ and M is not real, which is a contradiction. □

Example 4.11. We recall from [16] that the set of all ordinals less than the first uncountable ordinal is denoted by $W(\omega_1)$, where ω_1 is the first uncountable ordinal. The topology on $W(\omega_1)$ is the interval topology. $W(\omega_1)$ is pseudocompact but not compact, which implies that $\mathcal{F}_{\mathcal{P}}\big(W(\omega_1)\big) = \mathcal{F}^*_{\mathcal{P}}(W\big(\omega_1)\big)$.

Example 4.12. Let $a, b \in \mathbb{R} \setminus \mathbb{Q}$ with $a < b$ and $X := \{r \in \mathbb{Q} \colon a < r < b\}$ be given. If $L := \{O \cap X \colon O \in \mathfrak{O}(\mathbb{R})\}$, then the following statements hold.

(1) L is a zero-dimensional frame.
(2) $B(L)$ is not a sub-σ-frame of L.
(3) L is not a compact frame.
(4) Every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real.

The next result is a new characterization of compact frames in terms of fixed ideals of $\mathcal{F}_{\mathcal{P}}(L)$.

Proposition 4.13. *The following statements are equivalent for every zero-dimensional frame L.*

(1) *L is compact.*
(2) *Every proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$ $(\mathcal{F}^*_p(L))$ is fixed.*
(3) *Every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ $(\mathcal{F}^*_p(L))$ is fixed.*

Proof. (1) $\Rightarrow$ (2). Let I be a free proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Since, by Lemma 4.5, $\top \in \operatorname{coz}(I)$, we conclude that $I = \mathcal{F}_{\mathcal{P}}(L)$, which is a contradiction.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq L$ such that $\top = \bigvee_{\lambda\in\Lambda} a_\lambda$. Without loss of generality, we can assume that $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq B(L)$. It is clear that $I =< f_{a_\lambda} \colon \lambda \in \Lambda >$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$. If $I \neq \mathcal{F}_{\mathcal{P}}(L)$, then there exists a maximal ideal M such that $I \subseteq M$, and so

$$\top = \bigvee_{\lambda\in\Lambda} a_\lambda = \bigvee Coz(I) \leq \bigvee Coz(M),$$

which is a contradiction. Therefore, $I = \mathcal{F}_{\mathcal{P}}(L)$ and there exists a finite subset Λ' of Λ such that $\top = \mathrm{coz}\,(\mathbf{1}) = \bigvee_{\lambda\in\Lambda'} a_\lambda$. The proof is now complete □

We recall that a subset C of a frame L is said to have the *countable meet property* provided that the meet of any countable number of members of C is not the bottom.

We recall from [16] that a nonarchimedean field is characterized (among all totally ordered fields) by the presence of infinitely small elements, that is, elements a such that $a < \frac{1}{n}$ for every $n \in \mathbb{N}$.

The next result is a characterization of real maximal ideals of $\mathcal{F}_{\mathcal{P}}(L)$.

Proposition 4.14. *Let M be a maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ (resp., $\mathcal{F}^*_{\mathcal{P}}(L)$). Consider the following conditions on M.*

1. *M is real.*
2. *$Z[M]$ is closed under countable meet.*
3. *$Z[M]$ has the countable meet property.*

Then (2)$\Rightarrow$(3)$\Rightarrow$(1) and if $B(L)$ is a sub-σ-frame of a zero-dimensional frame L, then three conditions are equivalent.

Proof. (2) $\Rightarrow$ (3). It is clear, because $\bot \notin Z[M]$.

(3) $\Rightarrow$ (1). If M is not real, then there is an element $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $f + M$ is infinitely large, which, by Proposition 4.3, implies that $z_n := |f|[n, +\infty) \in Z[M]$ for every $n \in \mathbb{N}$. It is clear that $\bigwedge_{n\in\mathbb{N}} z_n = \bot$, which is a contradiction.

(1) $\Rightarrow$ (2). Let $B(L)$ be a sub-σ-frame of a zero-dimensional frame L. Consider $\{z_n\}_{n\in\mathbb{N}}$ in $Z[M]$ with $z := \bigwedge_{n\in\mathbb{N}} z_n \notin Z[M]$. We put $b_1 = z'_1$ and $b_i = z'_i \wedge (\bigwedge_{j=1}^{i-1} z_j)$ for every $i \geq 2$. Consider $g_n = f_{b_n} \wedge 2^{-n}$ for every $n \in \mathbb{N}$. Hence, $\{g_n\}_{n\in\mathbb{N}} \subseteq M$. We define the real-trail $t \colon \mathcal{P}(\mathbb{R}) \to L$ on the frame L by

$$t(r) = \begin{cases} \bigwedge_{n\in\mathbb{N}} z_n & \text{if } r = 0 \\ b_n & \text{if } r = 2^{-n}, \text{ for some } n \in \mathbb{N} \\ \bot & \text{otherwise.} \end{cases}$$

Then the following statements hold.

(1) $\sum_{i=1}^{n} g_i = \bigvee_{i=1}^{n} g_i \in M$.
(2) $\bigvee_{i\in\mathbb{N}} g_i = \varphi_t$.
(3) $\operatorname{coz}(\bigvee_{i=1}^{n} g_i) = \bigvee_{i\in\mathbb{N}} b_i$.
(3) $z(\varphi_t) = \bigwedge_{n\in\mathbb{N}} z_n \notin Z[M]$ and $\varphi_t \notin M$.

Then $0 \neq \varphi_t + M = \varphi_t + \sum_{i=1}^{n} g_i + M \leq 2^{-n}$ for every $n \in \mathbb{N}$, which implies that $\varphi_t + M$ is infinitely small element of $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$, and so $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$ is not archimedean. Therefore, by Proposition 4.2, M is not real, which is a contradiction. □

Definition 4.15. An ultrafilter $\mathcal{F}$ of $B(L)$ is called a *real ultrafilter* if $Z^{\leftarrow}(\mathcal{F})$ is a real maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$.

Corollary 4.16. *Let $\mathcal{F}$ be an ultrafilter $\mathcal{F}$ of $B(L)$. Then the following statements hold.*

(1) *If $\mathcal{F}$ is closed under countable meet, then $\mathcal{F}$ is a real ultrafilter of $B(L)$.*
(2) *If $B(L)$ is a sub-σ-frame of a zero-dimensional frame L and $\mathcal{F}$ is a real ultrafilter of $B(L)$, then*
   (a) *$\mathcal{F}$ is closed under countable meet, and*
   (b) *if $\{f_n \colon n \in \mathbb{N}\} \subseteq \mathcal{F}_{\mathcal{P}}(L)$ such that $\bigwedge_{n\in\mathbb{N}} z(f_n) \in \mathcal{F}$, then $z(f_n) \in \mathcal{F}$ for some $n \in \mathbb{N}$.*

Proof. (1) and 2(a) follow from Proposition 4.14.

(b). We argue by contradiction. Let us assume that $\{z(f_n)\}_{n\in\mathbb{N}} \cap \mathcal{F} = \emptyset$. Then for every $n \in \mathbb{N}$, there is an element $z(g_n) \in \mathcal{F}$ such that $z(f_n) \wedge z(g_n) = \bot$. By the statement (a), $\bigwedge_{n\in\mathbb{N}} z(g_n) \in \mathcal{F}$ and, by hypothesis, $\bot = (\bigwedge_{n\in\mathbb{N}} z(g_n)) \wedge (\bigwedge_{n\in\mathbb{N}} z(f_n)) \in \mathcal{F}$, which is a contradiction. □

By Proposition 4.13, Example 4.12 shows that $B(L)$ is a sub-σ-frame of L is necessary in the following proposition.

Proposition 4.17. *Let $B(L)$ be a sub-σ-frame of a zero-dimensional frame L. Then every fixed maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is real.*

Proof. Consider M is a fixed maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq M$ with $\bigwedge_{n\in\mathbb{N}} z(f_n) = \bot$ be given. Since $B(L)$ is the sub-σ-frame of L, we conclude that

$$\bigvee_{f\in M} \operatorname{coz}(f) \geq \bigvee_{n\in\mathbb{N}} \operatorname{coz}(f_n) = (\bigvee_{n\in\mathbb{N}} \operatorname{coz}(f_n))'' = (\bigwedge_{n\in\mathbb{N}} z(f_n))' = \top,$$

which is a contradiction. Therefore, by Proposition 4.14, M is a real maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$. □

The following example shows that the above proposition does not hold if $B(L)$ is not a sub-σ-frame of the zero-dimensional frame L.

Example 4.18. Let $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ and $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ such that for every $n \in \mathbb{N}$, $a_n < a_{n+1} < b_{n+1} < b_n$ and $\lim_{n\to\infty} a_n = \sqrt{2} = \lim_{n\to\infty} b_n$. Consider $L := \{ O \cap (\mathbb{R} \setminus \mathbb{Q}) \colon O \in \mathfrak{O}(\mathbb{R}) \}$, $c := \{ x \in \mathbb{R} \setminus \mathbb{Q} \colon x \neq \sqrt{2} \} \in L$ and $c_n := \{ x \in \mathbb{R} \setminus \mathbb{Q} \colon x < a_n \text{ or } b_n < x \} \in L$ for every $n \in \mathbb{N}$. Then $\{ f_{c_n} \}$ is a subset of the fixed maximal ideal of $M_c := \{ f \in F_{\mathcal{P}}(L) \colon \operatorname{coz}(f) \leq c \}$. By Proposition 4.14, since $\bigwedge_{n\in\mathbb{N}} z(f_{c_n}) = \perp$, we conclude that M_c is not a real maximal. It is clear that $B(L)$ is not a sub-σ-frame of the zero-dimensional frame L.

Now, it is normal to ask what are the frames for which every real maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is fixed. This leads us to the following definition.

Definition 4.19. A frame L is said to be a *$\mathcal{F}_{\mathcal{P}}$-realcompact* provided that every real maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ is fixed.

We recall that a frame L is said to be a *Lindelöf* frame provided that every subset of S, $\bigvee S = \top$ implies there exists a countable subset S' of S such that $\bigvee S' = \top$.

Proposition 4.20. *If $B(L)$ is a sub-σ-frame of a Lindelöf zero-dimensional frame L, then L is an $\mathcal{F}_{\mathcal{P}}$-realcompact frame.*

Proof. Consider M is a real maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$. We show that M is fixed. If not, there exists a family $\{f_n\}_{n\in\mathbb{N}} \subseteq M$ such that $\bigvee_{n\in\mathbb{N}} \operatorname{coz}(f_n) = \top$, because L is Lindelöf. By Propositions 4.14 and 4.17, $\perp = \bigwedge_{n\in\mathbb{N}} z(f_n) \in Z[M]$, which is a contradiction. □

In view of Propositions 4.2 and 4.13, we obtain the following proposition.

Proposition 4.21. *Let L be a zero-dimensional frame. L is a compact frame if and only if L is an $\mathcal{F}_{\mathcal{P}}$-realcompact frame, and $\frac{\mathcal{F}_{\mathcal{P}}(L)}{M}$ is archimedean for every maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$.*

5. Free ideals of $\mathcal{F}_{\mathcal{P}}(L)$

It is well known that if X is a completely regular topological space, then

(1) $C_K(X)$ is equal to the intersection of all free ideals $C(X)$ and this is true for $C^*(X)$.
(2) $C_\infty(X)$ is equal to the intersection of all free maximal ideals $C^*(X)$.

In this section, we show a counterpart of above result in $\mathcal{F}_{\mathcal{P}}(L)$ and $\mathcal{F}^*_{\mathcal{P}}(L)$. We begin with the following definition.

Definition 5.1. For every frame L, $\mathcal{F}_{\mathcal{P}_K}(L)$ is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\operatorname{coz}(f)$ is a compact element of L.

The following proposition is a counterpart of (1) in $\mathcal{F}_{\mathcal{P}}(L)$ and $\mathcal{F}^*_{\mathcal{P}}(L)$.

Proposition 5.2. *Let L be a zero-dimensional frame. Then $\mathcal{F}_{\mathcal{P}_K}(L)$ is equal to the intersection of all free ideals $\mathcal{F}_{\mathcal{P}}(L)$ and this is true for $\mathcal{F}^*_{\mathcal{P}}(L)$.*

Proof. Consider $f \notin \mathcal{F}_{\mathcal{P}_K}(L)$. Since $\operatorname{coz}(f)$ is not compact, we conclude from Lemma 4.5 that there exists a free maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that $\operatorname{coz}(f) \in Z[M]$, which implies that $f \notin M$. Let I be an arbitrary free ideal of $\mathcal{F}_{\mathcal{P}}(L)$ and $f \in \mathcal{F}_{\mathcal{P}_K}(L)$. Since $\operatorname{coz}(f)$ is a compact element of L, we conclude from Propositions 3.3 and 4.9 that $f \in I$. Therefore, $\mathcal{F}_{\mathcal{P}_K}(L)$ is equal to the intersection of all free ideals $\mathcal{F}_{\mathcal{P}}(L)$. The rest is similar. □

Definition 5.3. For every frame L, $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ such that $\uparrow f(-\frac{1}{n}, \frac{1}{n})$, ordered by the relation of L, is a compact frame for every $n \in \mathbb{N}$.

The following proposition is a counterpart of (2) in $\mathcal{F}^*_{\mathcal{P}}(L)$.

Proposition 5.4. *If $B(L)$ is a sub-σ-frame of a zero-dimensional frame L, then $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is equal to the intersection of all free maximal ideals of $\mathcal{F}^*_{\mathcal{P}}(L)$.*

Proof. Let M be a free maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ and $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. Since for every $n \in \mathbb{N}$, $f(\mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}))$ is a compact element of L, we conclude from Lemma 4.5 that for every $n \in \mathbb{N}$, $f(\mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n})) \in Coz[M]$, which from Propositions 3.3, 4.10 and 4.14 implies $f \in M$.

Now, let $f \notin \mathcal{F}_{\mathcal{P}_\infty}(L)$ be given. Then $z_n := |f|([\frac{1}{n}, -)$ is not compact for some $n \in \mathbb{N}$, which from Lemma 4.5 implies that there exists a free maximal ideal M of $\mathcal{F}_{\mathcal{P}}(L)$ such that $z_n \in Z(M)$ and so $|f| + M \geq 1/n$. Thus, there exists a nonnegative element $g \in \mathcal{F}^*_{\mathcal{P}}(L)$ such that $|f| - 1/n - g \in M$. Now, supposing M^* the unique maximal ideal in $\mathcal{F}^*_{\mathcal{P}}(L)$ containing M, we have $|f| - 1/n - g \in M^*$ and so $|f| - 1/n + M^* \geq 0$. Hence, $|f| \notin M^*$, so $f \notin M^*$ and clearly M^* is a fee maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$. □

The following example shows that the above proposition does not hold if $B(L)$ is not a sub-σ-frame of the zero-dimensional frame L.

Example 5.5. Let $X = \mathbb{N} \cup \{n + \frac{1}{m+1} : m, n \in \mathbb{N}\}$ be a topological space with relative topology of $\mathbb{R}$, and $L = \mathfrak{O}(X)$. It is clear that $\{1 + \frac{1}{n+1}\}$ is a complemented element of L for every $n \in \mathbb{N}$, and $\bigvee_{n \in \mathbb{N}} \{1 + \frac{1}{n+1}\}$ is not a complemented element of L, then $B(L)$ is not sub-σ-frame of L. Also, since L is not compact, we conclude that $\mathcal{F}_{\mathcal{P}} L \neq \mathcal{F}_{\mathcal{P}_\infty} L$.

The function $\alpha : \mathbb{R} \to L$ given by

$$\alpha(x) = \begin{cases} [n, n + \frac{1}{2}] \cap X & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

is a real-trail on L, and $f_\alpha \in \mathcal{F}_{\mathcal{P}\infty}L \setminus \mathcal{F}_{\mathcal{P}K}L$ is a unit element of $\mathcal{F}_{\mathcal{P}}L$. Therefore, $\mathcal{F}_{\mathcal{P}K}L$ is a proper ideal of $\mathcal{F}_{\mathcal{P}\infty}L$.

The function $\alpha_n : \mathbb{R} \to L$ given by

$$\alpha_n(x) = \begin{cases} a_n = [n, n+\frac{1}{2}] \cap X & \text{if } x = 1 \\ a'_n = ([n, n+\frac{1}{2}] \cap X)' & \text{if } x = 0 \\ \perp & \text{otherwise} \end{cases}$$

is a real-trail on L for any $n \in \mathbb{N}$, and $\operatorname{coz}(f_{\alpha_n}) = a_n$. Since $f_{\alpha_n} \in \mathcal{F}_{\mathcal{P}K}L$, and $\bigvee_{n\in\mathbb{N}} a_n = \top$, we conclude that $\mathcal{F}_{\mathcal{P}K}L$ is a free ideal of rings $\mathcal{F}_{\mathcal{P}\infty}L$, $\mathcal{F}^*_{\mathcal{P}}L$ and $\mathcal{F}_{\mathcal{P}}L$.

If M is a free maximal ideal of $\mathcal{F}^*_{\mathcal{P}}L$ such that $\mathcal{F}_{\mathcal{P}K}L \subseteq M$, then $\mathcal{F}_{\mathcal{P}\infty} \nsubseteq M$ and

$$\mathcal{F}_{\mathcal{P}\infty} \neq \bigcap \{M \colon M \text{ is a free maximal ideal of } \mathcal{F}^*_{\mathcal{P}}L\}.$$

For the proofs of the following corollaries, we need the following proposition which is proved in [24, Corollary 3.6].

Proposition 5.6. *Let A be a commutative algebra over the rational numbers with unity. Let I be an ideal of A. Then an ideal D of I is a maximal ideal of I if and only if $D = M \cap I$ for some maximal ideal M in A, with $I \nsubseteq M$.*

Corollary 5.7. *If $B(L)$ is a sub-σ-frame of a zero-dimensional frame L, then every ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is fixed.*

Proof. Let N be a free maximal ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$. Since $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is an ideal of a commutative algebra over the rational numbers with unity $\mathcal{F}^*_{\mathcal{P}}(L)$, we conclude from Proposition 5.6 that there exists a maximal ideal M of $\mathcal{F}^*_{\mathcal{P}}(L)$ such that $\mathcal{F}_{\mathcal{P}_\infty}(L) \nsubseteq M$ and $N = M \cap \mathcal{F}_{\mathcal{P}_\infty}(L)$, which implies that M is free maximal ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ such tat $\mathcal{F}_{\mathcal{P}_\infty}(L) \nsubseteq M$, which is a contradiction by Proposition 5.4. Since every ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is contained in a maximal ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$, we conclude that every ideal of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is fixed. □

Corollary 5.8. *If L is a zero-dimensional frame, then every ideal of $\mathcal{F}_{\mathcal{P}_K}(L)$ is fixed.*

Proof. The proof is similar to the proof of Corollary 5.7. □

6. REAL RIESZ MAPS ON $\mathcal{F}_p(L)$

We recall from [16, p. 142] that for every real compact space X, $\varphi\colon C(X) \to \mathbb{R}$ is a nonzero Riesz map if and only if there exists a unique $p \in X$ such that $\varphi(f) = f(p)$ for each $f \in C(X)$.

In this section, we show that Proposition 6.2 is a counterpart of above result.

Lemma 6.1. *The map* $\operatorname{coz} : \mathcal{F}_{\mathcal{P}}(L) \to L$ *given by* $f \mapsto \operatorname{coz}(f)$ *is a* $\mathbb{Q}$*-compatible bounded continuous cozero lattice-valued map.*

Proof. By Proposition 2.1, the map coz is a cozero lattice-valued map.

For every $\diamond \in \{+, \cdot, \vee, \wedge\}$, $f, g \in \mathcal{F}_{\mathcal{P}}(L)$, and $r, s, w, z, p, q \in \mathbb{Q}$, if

$$< r, s > \diamond < w, z > \subseteq < p, q >$$

then we have

$$\begin{aligned}
\operatorname{coz}(\delta^f_{rs}) \wedge \operatorname{coz}(\delta^g_{wz}) &= \operatorname{coz}((f - r)^+ \wedge (s - f)^+) \wedge \operatorname{coz}((g - w)^+ \wedge (z - g)^+) \\
&= f(r, s) \wedge g(w, z) \\
&\leq f \diamond g(p, q) \\
&= \operatorname{coz}((f \diamond g - \mathbf{p})^+) \wedge (\mathbf{q} - (f \diamond g))^+ \\
&= \operatorname{coz}(\delta^{f \diamond g}_{pq}),
\end{aligned}$$

which implies that the lattice-valued map coz is $\mathbb{Q}$-compatible. For every $f \in \mathcal{F}_{\mathcal{P}}(L)$, we have

$$\bigvee_{p,q \in \mathbb{Q}} \operatorname{coz}(\delta^f_{pq}) = \bigvee_{p,q \in \mathbb{Q}} \operatorname{coz}((f - \mathbf{p})^+ \wedge (\mathbf{q} - f)^+) = \bigvee_{p,q \in \mathbb{Q}} f(p, q) = \top.$$

Hence, the lattice-valued map coz is bounded. Also, for any $p, q \in \mathbb{Q}$ and any $f \in \mathcal{F}_{\mathcal{P}}(L)$,

$$\begin{aligned}
\operatorname{coz}(\delta^f_{pq}) &= \operatorname{coz}((f - \mathbf{p})^+ \wedge (\mathbf{q} - f)^+) \\
&= f(p, q) \\
&= \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p < r < s < q}} f(r, s) \\
&= \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p < r < s < q}} \operatorname{coz}((f - \mathbf{r})^+ \wedge (\mathbf{s} - f)^+) \\
&= \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p < r < s < q}} \operatorname{coz}(\delta^f_{rs}),
\end{aligned}$$

which implies that the lattice-valued map coz is continuous. The proof is now complete. □

Proposition 6.2. *Let* $B(L)$ *be a sub-*σ*-frame of a zero-dimensional frame* L*, and let* $\varphi\colon \mathcal{F}_{\mathcal{P}}(L) \to \mathbb{R}$ *be a function such that* $\varphi \neq \mathbf{0}$ *and* $\varphi(rf + gh) = r\varphi(f) + \varphi(g)\varphi(h)$ *for every* $r \in \mathbb{R}$

and $f, g, h \in \mathcal{F}_{\mathcal{P}}(L)$. *If every maximal ideal of* $\mathcal{F}_{\mathcal{P}}(L)$ *is real, then there exists an element* $p \in L$ *such that* $\varphi = \widetilde{p_{\text{coz}}}$.

Proof. By Lemma 6.1, $\text{coz} \in F(\mathcal{F}_{\mathcal{P}}(L), L)$ is a $\mathbb{Q}$-compatible bounded continuous cozero lattice-valued map and φ is nonzero bounded Riesz map $\varphi\colon \mathcal{F}_{\mathcal{P}}(L) \to \mathbb{R}$. By Proposition 4.10, L is compact and, by Proposition 4.13, every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L) = \mathcal{F}^*_{\mathcal{P}}(L)$ is fixed. Since φ is an f-ring epimorphism, we infer that $\overline{\varphi}\colon \frac{\mathcal{F}_{\mathcal{P}}(L)}{ker(\varphi)} \to \mathbb{R}$ given by $f + ker(\varphi) \mapsto \varphi(f)$ is an isomorphism. Since $ker(\varphi)$ is a maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$, we conclude that there exists an element $p \in \Sigma L$ such that $\bigvee \text{coz}\,(\ker(\varphi)) \le p$, which from Proposition 2.3 implies that $\varphi = \widetilde{p_{\text{coz}}}$. □

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Ali Akbar Estaji
Faculty of Mathematics and Computer Sciences,
Hakim Sabzevari University,
Sabzevar, Iran.
aaestaji@hsu.ac.ir
Ahmad Mahmoudi Darghadam
Faculty of Mathematics and Computer Sciences,
Hakim Sabzevari University,
Sabzevar, Iran.
m.darghadam@yahoo.com