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A CLASS OF WELL-COVERED AND VERTEX DECOMPOSABLE GRAPHS ARISING FROM RINGS

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Abstract. Let $\mathbb{Z}_n$ be the ring of integers modulo n. The unitary Cayley graph of $\mathbb{Z}_n$ is defined as the graph $G(\mathbb{Z}_n)$ with the vertex set $\mathbb{Z}_n$ and two distinct vertices a, b are adjacent if and only if $a - b \in U(\mathbb{Z}_n)$, where $U(\mathbb{Z}_n)$ is the set of units of $\mathbb{Z}_n$. Let $\Gamma(\mathbb{Z}_n)$ be the complement of $G(\mathbb{Z}_n)$. In this paper, we determine the independence number of $\Gamma(\mathbb{Z}_n)$. Also it is proved that $\Gamma(\mathbb{Z}_n)$ is well-covered. Among other things, we provide condition under which $\Gamma(\mathbb{Z}_n)$ is vertex decomposable.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph theory language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [3], [6], [7] and [8]. Moreover, for the most recent study in this field see [1], [5], [12] and [17].

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Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. An *independent vertex set* of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set in G. For any $x \in V(G)$, $\deg_G(x)$ or $(\deg(x))$ represents the number of edges incident to x, called the *degree* of the vertex x in G. Let r be a non-negative integer. A graph G is called *r-regular* if $\deg_G(v) = r$, for each vertex v of G. Let G_1 and G_2 be two graphs. The *category product* of G_1 and G_2, $G_1 \otimes G_2$, is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, specified by putting (u, v) adjacent to $(u^{'}, v^{'})$ if and only if u is adjacent to $u^{'}$ in G_1 or v is adjacent to $v^{'}$ in G_2.

A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. For any positive integer n, there is a unique cycle on n vertices. This graph is denoted by C_n. A graph is called *chordal* if every cycle of length at least four has a chord. Let G and H be two graphs with disjoint vertex sets. For two graphs G and H, the graph on vertex set $V(G \cup H) = V(G) \cup V(H)$ with edge set $E(G \cup H) = E(G) \cup E(H)$ is denoted by $G \cup H$. If a graph G consists of $k(\geq 2)$ disjoint copies of a graph H, then we write $G = kH$. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$.

The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \mid u \in V(G),\ vu \in E(G)\}$. Similarly, for $A \subseteq V(G)$, we have $N_G(A) = \cup_{v \in A} N_G(v)$ and $N_G[A] = A \cup N_G(A)$. For any two non-adjacent vertices of $V(G)$, such as u, v, the graph with the vertex set $V(G)$ and the edge set $E(G) \cup \{u, v\}$ is denoted by $G \cup \{u, v\}$. Also the induced subgraph of G on the vertex set $V(G) \setminus A$ is denoted by $G \setminus A$. Moreover, the graph G is called *well-covered* if all its maximal independent sets are of the same size. Furthermore, if G has no isolated vertices and $|V(G)| = 2\alpha(G)$, then G is *very well-covered.* A vertex x is called *simplicial vertex* if $N_G[x]$ is a clique. A *simplicial complex* Δ, on a finite set V, is a set of subsets of V closed under inclusion. The elements of Δ is called *face* of Δ and the maximal faces with respect to inclusion are called *facet* of Δ. A simplicial complex that has only one faset, ia called *simplex.* Recall that a simplicial complex Δ is called *pure* if every facets has the same number of elements. Let $\sigma \in \Delta$, the *link* and the *deletion* of σ from Δ are given by $link_\Delta \sigma := \{\tau \in \Delta \mid \sigma \cap \tau = \varnothing, \sigma \cup \tau \in \Delta\}$ and $del_\Delta \sigma := \{\tau \in \Delta \mid \sigma \nsubseteq \tau\}$. The *independence complex* of a graph G, denoted by $Ind(G)$, is the simplicial complex whose faces are the independent sets of G. A simplicial complex Δ is recursively defined to be *vertex decomposable* if it is either a simplex or else has some vertex v so that (1) both $\Delta \setminus v$ and $\mathrm{link}_\Delta v$ are vertex decomposable, and (2) no face of $\mathrm{link}_\Delta v$ is a facet of $\Delta \setminus v$. During the last two decades, many researchers have been interested in finding well-covered or vertex decomposable graphs, see for instance [13] and [14].

Given a ring R, let $Z(R)$ denote the set of zero divisors of R, and $Z^*(R) = Z(R) \setminus \{0\}$. Let $\mathbb{Z}_n$ be the ring of integers modulo n. The unitary Cayley graph of $\mathbb{Z}_n$ is defined as the graph $G(\mathbb{Z}_n)$ with the vertex set $\mathbb{Z}_n$ and two distinct vertices a, b are adjacent if and only if $a - b \in U(\mathbb{Z}_n)$, where $U(\mathbb{Z}_n)$ is the set of units of $\mathbb{Z}_n$. Let $\Gamma(\mathbb{Z}_n)$ be the complement of $G(\mathbb{Z}_n)$. In this paper, new families of well-covered graphs and vertex decomposable graphs are given.

Now we are ready to start the first section.

2. When $\Gamma(R)$ is well-covered?

We start this section with the following propositions.

Proposition 2.1. [9, Proposition 1] *Let R be a finite local ring. Then $|R| = p^n$, for some prime p and some positive integer n.*

Lemma 2.2. *Suppose $R = \mathbb{Z}_{p^t}$, where p is a prime. For $Z(R) = \underline{m} = \{a_1, \cdots, a_{p^{t-1}}\}$, consider the cosets of $\underline{m}$ by $T_i(R) = \{a_1 + i1_R, \cdots, a_{p^{t-1}} + i1_R\}$, for $1 \leq i \leq p-1$. Then:*
(i) If x and y are two distinct elements in $Z(R) = \underline{m}$, then $x - y \in Z(R)$.
(ii) If x and y are two distinct elements in $T_i(R)$, then $x - y \in Z(R)$.
(iii) If x and y are two distinct elements in $Z(R) \cup (\bigcup_{i=1}^{p-1} T_i(R))$ that are not in the same set, then $x - y \in U(R)$.

Proof. It is clear that $\bigcup_{i=1}^{p-1} T_i(R) = U(R)$ and $\{Z(R), T_1(R), \cdots, T_{p-1}(R)\}$ is a partition of R into their subsets. Since $Z(R) = \underline{m}$ is a maximal ideal of R, (i) and (ii) are clear. Now suppose that $x \in Z(R)$ and $y \in T_i(R)$ where $i \in \{1, 2, \cdots, p-1\}$. If $x - y \in Z(R)$, then there exists $u \in \{1, 2, \cdots, p^{t-1}\}$ such that $x - y = a_u \in Z(R)$. Therefore $x - a_u = y \in Z(R) \cap U(R)$, a contradiction. Now we assume that $i, j \in \{1, 2, \cdots, p-1\}$ and $i \neq j$ such that $y \in T_i(R)$ and $x \in T_j(R)$. Without loss of generality, assume that $i \lneq j$. So there exist $1 \leq u, v \leq p^{t-1}$ such that $x = a_u + j1_R$ and $y = a_v + i1_R$. Therefore $x - y = (a_u + j1_R) - (a_v + i1_R) = (a_u - a_v) + (j-i)1_R \in T_{j-i}(R) \subseteq U(R)$. □

Proposition 2.3. *Let $R = \mathbb{Z}_n$. Then the following statements hold:*
(i) $\Gamma(R)$ is a $|Z^*(R)|$-regular graph.
(ii) If $R \simeq \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}$ where $p_i^{'}s$ are primes, then $\Gamma(R) = \otimes_{i=1}^{t} \Gamma(\mathbb{Z}_{p_i^{n}{}_i})$.

Proof. (i) Is clear, by [2, Proposition 2.2].

(ii) We know that each element of R is a vertex of $\otimes_{i=1}^{t} \Gamma(\mathbb{Z}_{p_i^{n_i}})$ and two vertices in $\otimes_{i=1}^{t} \Gamma(\mathbb{Z}_{p_i^{n_i}})$ are adjacent if and only if their difference is in at least one component, say $1 \leq i \leq t$, is a zero divisor of $\mathbb{Z}_{p_i^{n_i}}$. So the proof is complete. □

Proposition 2.4. *Let $(R, \underline{m})$ be a finite local ring. Then $\Gamma(R)$ is totally disconnected or $\Gamma(R) = pK_{|\underline{m}|}$, where p is a prime number.*

Proof. It is known that R is a field if and only if $Z(R) = \underline{m} = 0$ if and only if $Z^*(R) = \varnothing$. In this case, $\Gamma(R)$ is an empty graph. By part two of [2, Proposition 2.2] and this fact that $\Gamma(R)$ is complement of $G(R)$, we conclude that $\Gamma(R)$ is the union of (distinct) complete graphs of the form $K_{|\underline{m}|}$, where the number of such complete graphs is $|\dfrac{R}{\underline{m}}| = p$. □

Proposition 2.5. *Let $R = \mathbb{Z}_{p^n}$. Then the following statements hold:*

(i) If $n = 1$, then $\alpha(\Gamma(R)) = p$ and there exists exactly one maximal independent set of $\Gamma(R)$.
(ii) If $n \geqslant 2$, then $\alpha(\Gamma(R)) = p$ and the number of maximal independent sets of $\Gamma(R)$ is $p^{(n-1)p}$.

Proof. (i) If $n = 1$, then R is a field and $\underline{m} = 0$. So $\Gamma(R)$ is an empty graph and has one maximal independent set of the form $V(\Gamma(R))$. Then $\Gamma(R)$ is well-covered.

(ii) If $n \geq 2$, then $\Gamma(R)$ is of the form $pK_{|\underline{m}|}$. So by selecting one element of any $K_{|\underline{m}|}$, every maximal independent set has p elements and so the number of such maximal independent sets is $(p^{n-1})^p = p^{(n-1)p}$. Hence $\Gamma(R)$ is well-covered. □

Proposition 2.6. *Let $R = \mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}$ be a finite ring such that $p_1 \leq p_2 \leq \cdots \leq p_t$. Then:*

(i) $\Gamma(R)$ is a $|Z^*(R)|$-regular graph.
(ii) $\Gamma(R)$ is well-covered and $\alpha(\Gamma(R)) = p_1$.
(iii) For $t \geqslant 2$ the number of maximal independent sets of $\Gamma(R)$ is
$\left(\Pi_{i=1}^{t} p_i^{(n_i-1)p_1}\right)\left(\Pi_{i=1}^{t}\Pi_{j=1}^{p_1}(p_i - p_1 + j)\right)$.

Proof. (i) By part one of Proposition 2.3, the proof is clear.

(ii) Since $\Gamma(R)$ is p_1-partite whose parts are complete graphs and each vertex in every part is adjacent with the same number of vertices in other parts, we deduce $\alpha(\Gamma(R)) = p_1$.

(iii) Assume that $t \geqslant 2$. We need to find the number of all maximal independent sets of order p_1 in $\Gamma(R)$. The number of possible choices for the i-th component of j-th member, is equal with $(p_i - (j-1))p_i^{(n_i-1)}$ such that $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant p_1$. Multiplying all possible states for each of the components in each of the members, the ruling will result. □

3. When $\Gamma(R)$ is vertex decomposable?

A simplicial complex Δ is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex v so that (1) both $\Delta \setminus v$ and $\text{link}_\Delta v$ are vertex decomposable, and (2) no face of $\text{link}_\Delta v$ is a facet of $\Delta \setminus v$. Vertex decomposability is introduced by Provan and Billera in [16] in the pure case and extended to the non-pure case by Björner and Wachs in [10] and [11]. We call a graph G vertex decomposable if the independence complex $\text{Ind}(G)$ is vertex decomposable. In [19] Woodroofe translated the definition of vertex decomposability for graphs as follow.

Definition 3.1. A graph G is called vertex decomposable if either it is an edgeless graph or it has a vertex x such that:
(i) $G \setminus \{x\}$ and $G \setminus N_G[x]$ are both vertex decomposable.
(ii) For every independent set S of $G \setminus N_G[x]$, there is some vertex $y \in N_G(x)$ such that $S \cup \{y\}$ is an independent set of $G \setminus \{x\}$.

The vertex x of G that satisfies condition (ii) is called a *shedding* vertex of G.

Proposition 3.2. [19, Corollary 7] *A chordal graph is vertex decomposable.*

Example 3.3. Let C_4 be a cycle of order 4. Clearly C_4 is not a vertex decomposable graph.

Proposition 3.4. [15, Lemma 2.2] *Let G and H be two graphs that $V(G) \cap V(H) = \varnothing$, and set $W = G \cup H$. Then W is vertex decomposable if and only if G and H are vertex decomposable.*

The first main result of this section is the following theorem which provides a new method for constructing vertex decomposable graphs.

Theorem 3.5. *Let $R = \mathbb{Z}_{p^n}$. Then $\Gamma(R)$ is vertex decomposable.*

Proof. There are two cases, based on whether R is a field or not.

Case 1 We consider the case in which R is a field or equivalently $n = 1$. Then it is clear that $\Gamma(R)$ is totally disconnected and according to the definition, $\Gamma(R)$ is vertex decomposable.

Case 2 If R is not a field, then

$$\Gamma(R) = tK_{|Z(R)|} = K_{|Z(R)|} \cup ... \cup K_{|Z(R)|}$$

where t is the cardinality of $\dfrac{R}{Z(R)}$. Combining this observation with Propositions 3.2 and 3.4, we conclude that $\Gamma(R)$ is vertex decomposable.□

Theorem 3.6. *Let $R = \mathbb{Z}_n \simeq \mathbb{Z}_{2^r} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, where $2 \lneq p_2 \lneq p_3 \lneq \cdots \lneq p_n$. Then $\Gamma(R)$ is vertex decomposable.*

Proof. One can easily see that $\Gamma(R)$ contains two complete graphs of order

$$p_2^{r_2} \times p_3^{r_3} \times ... \times p_n^{r_n} \times 2^{r-1}$$

and every vertex in

$$K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}}|2^{r-1}}$$

is adjacent to exactly $2^{r-1}|Z(\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}})|$ vertices in other

$$K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}}|2^{r-1}} \, .$$

We call these two complete subgraphs S_9 and S_{10}. Assume that the first component of each of these parts belongs to cosets $T_1(\mathbb{Z}_{2^r})$ and $T_2(\mathbb{Z}_{2^r})$, respectively. Obviously, every vertex of each of these parts is a vector of length n, and t-th component of it belongs to $T_i(\mathbb{Z}_{p_t^{r_t}})$ such that $2 \leqslant t \leqslant n$ and $1 \leqslant i \leqslant p_t$. Consider S_9 and let $d \in T_1(\mathbb{Z}_{2^r})$. We suppose that $\beta = (\beta_1, ..., \beta_n) \in S_9$ such that $\beta_1 \neq d$. Without loss of generality, assume that $\beta_i \in T_1(\mathbb{Z}_{p_i^{r_i}}), 2 \leq i \leq n$. The subgraph $E = \Gamma(R) \setminus N[\beta]$ is complete and so is chordal. Hence, E is vertex decomposable. Every vertex in E, is a vector of length n and its first component belongs to $T_2(\mathbb{Z}_{2^r})$ and t-th component of it is belong to $T_i(\mathbb{Z}_{p_t^{r_t}})$ such that $2 \leqslant t \leqslant n$ and $2 \leqslant i \leqslant p_t$. Each independent set in E has at most one element. One may find a vertex in S_9 such that its firts component is d and its t-th component is in a coset different from the coset associated with the t-th component of the vertex in the given independent set. Thus we can make a larger independent set in $\Gamma(R) \setminus \{\beta\}$. Therefore every vertex in S_9 with its first component is different from d is a shedding vertex of $\Gamma(R)$ and we can remove it. The graphs obtain from $\Gamma(R)$ and S_9 by removing these vertices are denoted by $\Gamma(R)^{'}$ and S_{91}, respectively. All vertices in S_{91} are vectors of length n and its first component is d, and the t-th component of them is belong to $T_i(\mathbb{Z}_{p_t^{r_t}})$ that $2 \leqslant t \leqslant n$ and $1 \leqslant i \leqslant p_t$. Among all of the possible cosets for t-th component of vertices in S_{91}, $2 \leqslant t \leqslant n$, select two cosets and in every of these cosets consider an arbitrary element. We can remove other vertices in S_{91} as shedding vertices in $\Gamma(R)^{'}$. Hence S_{91} contains 2^{n-1} vertices. The graphs obtain from $\Gamma(R)^{'}$ and S_{91} by removing these vertices are denoted by $\Gamma(R)^{''}$ and S_{92}, respectively. The induced subgraph by S_{92} is $K_{2^{n-1}}$. No vertex of S_{92} in $\Gamma(R)^{''}$ is a shedding vertex. If a vector in S_{10} share a common component with a vector in S_{92}, then we can remove it from S_{10}. Repeating this procedure, S_{10} is reduced to a complete graph which is not adjacent to no vertex in S_{92}. So we get two complete separated graphs, that obviously it is vertex decomposable. Therefore $\Gamma(R)$ is vertex decomposable. □

The following example explains Theorem 3.6.

Example 3.7. Let $R = \mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$, and let $T_1(\mathbb{Z}_2) = \{\bar{0}\}$, $T_2(\mathbb{Z}_2) = \{\bar{1}\}$, $T_1(\mathbb{Z}_3) = \{\bar{0}\}$, $T_2(\mathbb{Z}_3) = \{\bar{1}\}$ and $T_3(\mathbb{Z}_3) = \{\bar{2}\}$. According to the previous argument, we can select $(\bar{0}, \bar{0})$, $(\bar{1}, \bar{1})$ and $(\bar{1}, \bar{2})$ as shedding vertices, respectively. It is obvious that $\Gamma(R)$ is vertex decomposable.

Theorem 3.8. *Let $R = \mathbb{Z}_n \simeq \mathbb{Z}_{3^r} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, where $3 \lneq p_2 \lneq p_3 \lneq \cdots \lneq p_n$. Then $\Gamma(R)$ is vertex decomposable.*

Proof. One can easily see that $\Gamma(R)$ contains three complete graphs of order

$$p_2^{r_2} \times p_3^{r_3} \times ... \times p_n^{r_n} \times 3^{r-1}$$

and every vertex in

$$K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}}|3^{r-1}}$$

is adjacent to exactly $3^{r-1}|Z(\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}})|$ vertices in other

$$K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times ... \times \mathbb{Z}_{p_n^{r_n}}|3^{r-1}}.$$

These induced subgraphs are named by S_6, S_7 and S_8. Assume that the first component of each of these parts, respectively belong to cosets $T_1(\mathbb{Z}_{3^r})$, $T_2(\mathbb{Z}_{3^r})$ and $T_3(\mathbb{Z}_{3^r})$. Focus on S_6. Consider $c \in T_1(\mathbb{Z}_{3^r})$, arbitrary and fixed. We arbitrarily select $z = (z_1, ..., z_n) \in S_6$ such that $z_1 \neq c$. Without loss of generality, assume that $z_i \in T_1(\mathbb{Z}_{p_i^{r_i}}), 2 \leq i \leq n$. Set $D = \Gamma(R) \setminus N[z]$. By Theorem 3.6, D is a vertex decomposable graph and We can expand every independent set of D, which has at most two members, to a larger independent set in $\Gamma(R) \setminus \{z\}$, by selecting a suitable vertex in S_6, in the way that, its first component be c, and its t-th component, $2 \leqslant t \leqslant n$, be in the cosets other than cosets of t-th component of vertices in that independent set. Therefore, we can eliminate all the vertices which are similar to z in S_6, as a shedding vertex. By doing so, S_6 is changing to S_{61} and $\Gamma(R)$ is changing to $\Gamma_1(R)$. Now by the process of Theorem 3.6, we select three arbitrary cosets, and from each one, just one element, among the cosets of $\mathbb{Z}_{p_t^{r_t}}$, that means the cosets $T_i(\mathbb{Z}_{p_t^{r_t}})$ that $2 \leqslant t \leqslant n$ and $1 \leqslant i \leqslant p_t$. These choices specify 3^{n-1} vertices in S_{61}. We can eliminate other vertices in S_{61}, as shedding vertices in $\Gamma_1(R)$. Thus S_{61} is changing to S_{62} and $\Gamma_1(R)$ is changing to $\Gamma_2(R)$. No vertex of S_{62} in $\Gamma_2(R)$ is a shedding vertex. Now, we focus on the vertices in S_7. Consider $e \in T_2(\mathbb{Z}_{3^r})$, arbitrary and fixed. We can eliminate all the vertices in S_7, whose first component is not e, as a shedding vertex in $\Gamma_2(R)$. So S_7 is changing to S_{71} and $\Gamma_2(R)$ is changing to $\Gamma_3(R)$. Now, we can eliminate all the vertices in S_{71}, whose t-th component is in the same coset of the t-th component of the vertices in S_{62}, as a shedding vertex in S_{71}, $2 \leqslant t \leqslant n$. Thus S_{71} is changing to S_{72} and $\Gamma_3(R)$ is changing to $\Gamma_4(R)$. In this case, there is no edge between S_{72} and S_{62} and edges are just between each of these sections and S_8. In the t-th component of vertices in S_{72}, we choose from each of the cosets only one element arbitrary, and eliminate others as a shedding vertex, $2 \leqslant t \leqslant n$. So S_{72} is changing to S_{73} and $\Gamma_4(R)$ is changing to $\Gamma_5(R)$. Consider $f \in T_3(\mathbb{Z}_{3^r})$, arbitrary and fixed. We can eliminate all the vertices in S_8, whose first component is not f, as a shedding vertex in $\Gamma_5(R)$. Thus S_8 is changing to S_{81}. Now we note that whether the number of choosen cosets for second component of vertices in S_{73} is greater or in S_{62}. From the greater one, we eliminate extra vertices as shedding vertices, as far as the number of cosets of their second components get equal. Now, we can eliminate all the vertices in S_{81}, whose second component is in the same coset of the second component of the vertices in S_{73} or S_{62}, as a shedding vertex. Thus S_{81} is changing to S_{82}. Among all of the remaining cosets for the second component of vertices in S_{82}, select an arbitrary element and eliminate the rest as a shedding vertex. Thus S_{82} is changing to S_{83}. By continuing and repeating this procedure on the t-th component of vertices in S_{83}, S_{73} and S_{62}, $3 \leqslant t \leqslant n$, we have a graph of three component, in which each of the component is a complete graph and there is no edges between them. As a result, $\Gamma(R)$ is vertex decomposable. □

Theorem 3.9. *Let* $R = \mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, *where* $5 \leq p_1 \lneq p_2 \lneq p_3 \lneq \cdots \lneq p_n$ *and* $p_2 \geq \lceil \frac{p_1}{2} \rceil + \lfloor \frac{p_1}{2} \rfloor (\lceil \frac{p_1}{2} \rceil + 1)$. *Then* $\Gamma(R)$ *is vertex decomposable.*

Proof. One can easily see that $\Gamma(R)$ contains p_1 complete graphs of order

$$p_2^{r_2} \times p_3^{r_3} \times \ldots \times p_n^{r_n} \times p_1^{r_1-1}$$

and every vertex in

$$K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times \ldots \times \mathbb{Z}_{p_n^{r_n}}| p_1^{r_1-1}}$$

is adjacent to exactly $p_1^{r_1-1}|Z(\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times \ldots \times \mathbb{Z}_{p_n^{r_n}})|$ vertices in every one of $(p_1 - 1)$ other $K_{|\mathbb{Z}_{p_2^{r_2}} \times \mathbb{Z}_{p_3^{r_3}} \times \ldots \times \mathbb{Z}_{p_n^{r_n}}| p_1^{r_1-1}}$ graphs. Select one of these p_1 parts arbitrary, and call it S_{11}. Vertices in it are vectors of length n whose first component belongs to a coset of $\mathbb{Z}_{p_1^{r_1}}$, such as $T_1(\mathbb{Z}_{p_1^{r_1}})$. Note that the t-th component of each of the vertices in R, $1 \leqslant t \leqslant n$, may belong to exactly one coset of type $T_j(\mathbb{Z}_{p_t^{r_t}})$, $1 \leqslant j \leqslant p_t$. Consider $a \in T_1(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We arbitrarily select $x = (x_1, ..., x_n) \in S_{11}$ such that $x_1 \neq a$. Without loss of generality, assume that $x_i \in T_1(\mathbb{Z}_{p_i^{r_i}}), 2 \leq i \leq n$. Now, the vertex set of $A = \Gamma(R) \setminus N[x]$ can be partitioned into $(p_1 - 1)$ parts: the subgraph induced on each part is a complete graph and A might have some edges between some of these parts. Observe that every vertex in each of these parts is a vector of length n with the property that its t-th component belongs to $T_i(\mathbb{Z}_{p_t^{r_t}})$, where $1 \leqslant t \leqslant n$ and $2 \leqslant i \leqslant p_t$. We now repeat this process with the graph $\Gamma(R)$ replaced by A and we do this up to $(p_1 - 5)$- times. The resulting graph, which we denote by B, consists of four parts: the subgraph induced on each part is a complete graph and B might have some edges between some of these parts. We name this four parts by S_2,S_3,S_4 and S_5. Assume that the first component of vertices in each of these parts, belong to cosets $T_{p_1-3}(\mathbb{Z}_{p_1^{r_1}})$, $T_{p_1-2}(\mathbb{Z}_{p_1^{r_1}})$, $T_{p_1-1}(\mathbb{Z}_{p_1^{r_1}})$ and $T_{p_1}(\mathbb{Z}_{p_1^{r_1}})$ respectively. According to the proof, every vertex in each of these parts is a vector of length n that t-th component of it, is belong to $T_i(\mathbb{Z}_{p_t^{r_t}})$ such that $2 \leqslant t \leqslant n$ and $p_1 - 3 \leqslant i \leqslant p_t$. Focus on S_2. Consider $b \in T_{p_1-3}(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We select $y = (y_1, ..., y_n) \in S_2$ such that $y_1 \neq b$ arbitrary. Without loss of generality, assume that $y_i \in T_{p_1-3}(\mathbb{Z}_{p_i^{r_i}}), 2 \leq i \leq n$. $C = B \setminus N[y]$ is a graph with three parts and according to the proof of the Theorem 3.8 is vertex decomposable. We can expand every independent set of C which has at most three members to a larger independent set in $B \setminus \{y\}$ by selecting a suitable vertex in S_2, in the way that, its first component be b, and its t-th component, $2 \leqslant t \leqslant n$, be in the coset other than cosets of t-th component of vertices in that independent set. Now, we can eliminate all the vertices in S_2 whose first component is not b, as a shedding vertex in B. So S_2 is changing to S_{21} and B is changing to B_1. In this step, we select four arbitrary cosets, and from each one, just one element, among the cosets of $\mathbb{Z}_{p_t^{r_t}}$, that means the cosets $T_i(\mathbb{Z}_{p_t^{r_t}})$, $2 \leqslant t \leqslant n$ and $p_1 - 3 \leqslant i \leqslant p_t$. These choices specify 4^{n-1} vertices in S_{21}. We can eliminate other vertices in S_{21} as shedding vertices in B_1. So S_{21} is changing to S_{22} and B_1 is changing to B_2. Now, we focus on the vertices in S_3. Consider $g \in T_{p_1-2}(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We can eliminate all the vertices in S_3, whose first component is not g, as a shedding vertex in B_2. So S_3 is changing to S_{31} and B_2 is changing to B_3. Now, we can eliminate all the vertices in S_{31}, whose t-th component is in the same coset with the coset which is related to the t-th component of the vertices in S_{22} as a shedding vertex in B_3, $2 \leqslant t \leqslant n$. Thus S_{31} is changing to S_{32} and B_3 is changing to B_4. In the t-th component of vertices in S_{32}, we choose from each of the cosets only one element arbitrary, and eliminate others as a shedding vertex, $2 \leqslant t \leqslant n$.Thus S_{32} is changing to S_{33} and B_4 is changing to B_5. In this case, there is no edge between S_{33} and S_{22}. The second component of vertices in S_{22}, is in the four cosets of the form $T_i(\mathbb{Z}_{p_2^{r_2}})$, $p_1 - 3 \leqslant i \leqslant p_2$, and the second component of vertices in S_{33} is in $p_2 - (p_1 - 4) - 4 = p_2 - p_1$ cosets. In this case, since other than S_{33} and S_{22} only S_4 and S_5 have remaind that we have not made any changes in their vertices, we can eliminate other vertices in $S_{33}(S_{22})$, $2 \leqslant t \leqslant n$. So, S_{22}, S_{33} and B_5 are changing to S_{23}, S_{34} and B_6, respectively. Now, we focus on the vertices in S_4. Consider $h \in T_{p_1-1}(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We can eliminate all the vertices in S_4, whose first component is not h, as a shedding vertex in B_6. So S_4 is changing to S_{41} and B_6 is changing to B_7. Now, we can eliminate all the vertices in S_{41} whose t-th component is in the same coset with the coset which is related to the t-th component of the vertices in S_{23} and S_{34}, as a shedding vertex in B_7, $2 \leqslant t \leqslant n$. In this case, for t-th component of vertices in S_{41}, we arbitrary choose from each of the cosets, only one element, and eliminate others as a shedding vertex, $2 \leqslant t \leqslant n$. So, S_{41} and B_7 are changing to

S_{42} and B_8, respectively. In this case, since only S_5 is remaining that we did not any change in its vertices, we can choose only two distinct cosets for each of t-th component of vertices in S_{42}, S_{34} and S_{23}, $2 \leqslant t \leqslant n$, and eliminate other vertices in S_{42}, S_{34} and S_{23} as a shedding vertex. So, S_{42}, S_{34}, S_{23} and B_8 are changing to S_{43}, S_{35}, S_{24} and B_9, respectively. Now, we focus on the vertices in S_5. Consider $i \in T_{p_1}(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We can eliminate all the vertices in S_5, whose first component is not i, as a shedding vertex in B_9. So S_5 is changing to S_{51} and B_9 is changing to B_{10}. Now, we can eliminate all the vertices in S_{51}, whose t-th component is in the same coset with the coset which is related to the t-th component of the vertices in S_{43}, S_{35} and S_{24}, as a shedding vertex in B_{10}. So S_{51} is changing to S_{52} and B_{10} is changing to B_{11}, and there is no edge between S_{52}, S_{43}, S_{35} and S_{24}. Clearly we get four complete vertex decomposable graphs. So B is vertex decomposable. If $p_1 = 5$, then $\Gamma(R)$ is vertex decomposable and proof is finished. In the following, we assume that $p_1 \gneqq 5$. Any independent set of B has at most four members. With proper implementation of the above mentioned process on graphs removed during the argument, which is obtained step by step from each other by removing the closed neighborhood a vertex of a graph, we go to the $\Gamma(R)$. We know that A is vertex decomposable and we can expand every independent set of A, which has at most $p_1 - 1$ members to a larger independent set in $\Gamma(R) \setminus \{x\}$, by selecting a suitable vertex in S_{11}, in the way that , its first component be a, and its t-th component, $2 \leqslant t \leqslant n$, be in the cosets other than cosets of t-th component of vertices in that independent set. Therefore, we can eliminate all the vertices which are similar to x in S_{11} as a shedding vertex. By doing so, S_{11} is changing to S_{111} and $\Gamma(R)$ is changing to $\Gamma_1(R)$. Other parts of the $\Gamma_1(R)$ are named S_{1j}. So that the first component of their vertices, is at $T_j(\mathbb{Z}_{p_1^{r_1}})$, respectively, $2 \leqslant j \leqslant p_1$. We select p_1 arbitrary cosets, and from each one, just one element, among the cosets of $\mathbb{Z}_{p_t^{r_t}}$, that means the cosets $T_j(\mathbb{Z}_{p_t^{r_t}})$, $1 \leqslant j \leqslant p_t$, $2 \leqslant t \leqslant n$. We can eliminate other vertices in S_{111} as a shedding vertex. Thus S_{111} is changing to S_{112} and $\Gamma_1(R)$ is changing to $\Gamma_2(R)$. These choices, specify p_1^{n-1} vertices in S_{112}. Now, we focus on the vertices in S_{12}. Consider $k \in T_2(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We can eliminate all the vertices in S_{12}, whose first component is not k, as a shedding vertex in $\Gamma_2(R)$. So S_{12} is changing to S_{121} and $\Gamma_2(R)$ is changing to $\Gamma_3(R)$. Now, we can eliminate all the vertices in S_{121}, whose t-th component are in the same coset with the coset which is related to the t-th component of the vertices in S_{112}, as a shedding vertex in $\Gamma_3(R)$, $2 \leqslant t \leqslant n$. In this case, for t-th component of vertices in S_{121}, we arbitrary choose from each of the cosets, only one element, and eliminate others as a shedding vertex, $2 \leqslant t \leqslant n$. So, S_{121} and $\Gamma_3(R)$ are changing to S_{122} and $\Gamma_4(R)$, respectively. The t-th component of vertices in S_{112} are in p_1 coset(one member of each coset)and t-th component of vertices in S_{122} are in $p_t - p_1$ coset(one member of each coset), $2 \leqslant t \leqslant n$. Except from these two parts in $\Gamma_4(R)$, $p_1 - 2$ other parts are untouched. $S_{122} = K_{(p_2-p_1)(p_3-p_1)\cdots(p_t-p_1)}$. So, we can select $p_1 - 1$

distinct cosets for t-th component of vertices in $S_{122}(S_{112})$and eliminate other vertices in S_{122} (S_{112}) as a shedding vertex in $\Gamma_4(R)$. So, S_{122}, S_{112} and $\Gamma_4(R)$ are changing to S_{123}, S_{113} and $\Gamma_5(R)$ respectively. Now, we focus on the S_{13}. Consider $l \in T_3(\mathbb{Z}_{p_1^{r_1}})$, arbitrary and fixed. We can eliminate all the vertices in S_{13}, whose first component is not l, as a shedding vertex in $\Gamma_5(R)$. So S_{13} is changing to S_{131} and $\Gamma_5(R)$ is changing to $\Gamma_6(R)$. Now, we can eliminate all the vertices in S_{131}, whose t-th component is in the same coset of the t-th component of the vertices in S_{123} or S_{113}, as a shedding vertex in $\Gamma_6(R)$, $2 \leqslant t \leqslant n$. In this case, for t-th component of vertices in S_{131}, we arbitrary choose from each of the cosets, only one element, and eliminate others as a shedding vertex, $2 \leqslant t \leqslant n$. So, S_{131} and $\Gamma_6(R)$ are changing to S_{132} and $\Gamma_7(R)$, respectively. Note that the number of remaining cosets for t-th component of vertices in S_{132} is equal to: $p_t - (p_1 - 1) - (p_1 - 1) = p_t - 2p_1 + 2 \geq 1$. Except from S_{132}, S_{113} and S_{123}, $p_1 - 3$ other parts in the $\Gamma_7(R)$ are still remaining untouched. So, we can select only $p_1 - 2$ distinct coset in t-th component for vertices in S_{132}, S_{113} and S_{123}, $2 \leq t \leq n$, and eliminate other vertices in S_{132}, S_{113} and S_{123} as a shedding vertex. Thus S_{132}, S_{113}, S_{123} and $\Gamma_7(R)$ are changing to S_{133}, S_{114}, S_{124} and $\Gamma_8(R)$, respectively. Therefore, the number of variety of cosets for t-th component of vertices in S_{133}, S_{114} and S_{124} are equal. There is no edge between this three part. By following this process for each of the $p_1 - 3$ remaining parts, respectively, we get to the p_1 distinct and different complete graph that each of them is vertex decomposable. So, their union is vertex decomposable. Thus $\Gamma(R)$ is vertex decomposable. □

We close this paper with the following example.

Example 3.10. Let $R = \mathbb{Z}_{25}$ and $T_1 = Z(\mathbb{Z}_{25}) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$, $T_2 = \{\overline{1}, \overline{6}, \overline{11}, \overline{16}, \overline{21}\}$, $T_3 = \{\overline{2}, \overline{7}, \overline{12}, \overline{17}, \overline{22}\}$, $T_4 = \{\overline{3}, \overline{8}, \overline{13}, \overline{18}, \overline{23}\}$, $T_5 = \{\overline{4}, \overline{9}, \overline{14}, \overline{19}, \overline{24}\}$. Clearly, all the elements in T_i, $2 \leq i \leq 5$, are unit. Moreover, difference between two arbitrary members in T_1 is an element of T_1, difference between two arbitrary elements in each T_i, $2 \leq i \leq 5$, is a zero divisor and difference between two arbitrary members in T_1 and every T_i, $2 \leq i \leq 5$, is unit. The difference between two arbitrary members in every pair T_i and T_j, $2 \leq i, j \leq 5$ is unit.

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