AN UPPER BOUND ON THE DISTINGUISHING INDEX OF GRAPHS WITH MINIMUM DEGREE AT LEAST TWO

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ABSTRACT. The distinguishing index of a simple graph \( G \), denoted by \( D'(G) \), is the least number of labels in an edge labeling of \( G \) not preserved by any non-trivial automorphism. We prove that for a connected graph \( G \) with maximum degree \( \Delta \), if the minimum degree is at least two, then \( D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1 \). We also present graphs \( G \) for which \( D'(G) \leq \lceil \sqrt{\Delta(G)} \rceil \).

1. INTRODUCTION

Let \( G = (V, E) \) be a simple connected graph. We use the standard graph notation \((\mathbb{R})\). In particular, \( \text{Aut}(G) \) denotes the automorphism group of \( G \) and is a form of symmetry in which the graph is mapped onto itself while preserving the edge-vertex connectivity. Formally, an automorphism of a graph \( G \) is a permutation \( \sigma \) of the vertex set \( V \), such that the pair of vertices \((u, v)\) form an edge if and only if the pair \((\sigma(u), \sigma(v))\) also form an edge. For simple connected graph \( G \), and \( v \in V \), the neighborhood of a vertex \( v \) is the set \( N_G(v) = \{u \in V(G) : uv \in E(G)\} \). The degree of a vertex \( v \) in a graph \( G \), denoted by \( \text{deg}_G(v) \), is the number of edges of \( G \) incident with \( v \). The number of neighbours of \( v \) in \( G \) is denoted by \( \text{deg}_G(v) \). We
denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of $G$. A graph $G$ is $k$-regular if $\deg_G(v) = k$ for all $v \in V$. The distance between two vertices $u$ and $v$ is denoted by $d(u, v)$ and is the number of edges in a shortest path connecting $u$ and $v$. The diameter of a graph $G$ is the largest distance between two vertices of $G$, and denoted by $\text{diam}(G)$. A connected graph $G$ is called 2-connected, if for every vertex $x \in V(G)$, $G - x$ is connected.

An edge-colouring of $G$ with colours in $C$ is a map $c : E \to C$. We say that $f \in \text{Aut}(G)$ preserves the edge-colouring $c$ if $c \circ f = c$. Call a colouring of $G$ distinguishing, if the identity is the only automorphism which preserves it. In other words, such a colouring $c$ of a graph $G$ breaks an automorphism $f \in \text{Aut}(G)$ if $f$ does not preserve colours of $c$. The distinguishing index $D'(G)$ of a graph $G$ is the least number $d$ such that $G$ has an edge labeling with $d$ labels that is preserved only by the identity automorphism of $G$. The distinguishing edge labeling was first defined by Kalinowski and Pilśniak [8] for graphs (inspired by the well-known distinguishing number $D(G)$ which has been defined for general vertex labelings by Albertson and Collins [1]). The distinguishing index of some examples of graphs was exhibited in [8]. For instance, $D'(P_n) = 2$ for every $n \geq 3$, and $D'(C_n) = 3$ for $n = 3, 4, 5$, $D'(C_n) = 2$ for $n \geq 6$. Also, for complete graphs $K_n$, we have $D'(K_n) = 3$ for $n = 3, 4, 5$, $D'(K_n) = 2$ for $n \geq 6$. It is known that for every graph $G$ of order at least three, we have $D'(G) \leq D(G) + 1$ ([8]). Recently Alikhani et.al in [2] characterized finite trees for which this inequality is sharp. They also showed that if $G$ is a connected unicyclic graph, then $D'(G) = D(G)$. Authors in [8] showed that if $G$ is a connected graph of order $n \geq 3$ and maximum degree $\Delta$, then $D'(G) \leq \Delta$, unless $G$ is $C_3$, $C_4$ or $C_5$. It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and $G$ is a cycle of length at most five. The equality $D'(G) = \Delta(G)$ holds for all paths, for cycles of length at least 6, for $K_4$, $K_{3,3}$ and for symmetric or bisymmetric trees. Also, Pilśniak showed that $D'(G) < \Delta(G)$ for all other connected graphs.

**Theorem 1.1.** [9] Let $G$ be a connected graph that is neither a symmetric nor an asymmetric tree. If the maximum degree of $G$ is at least 3, then $D'(G) \leq \Delta(G) - 1$ unless $G$ is $K_4$ or $K_{3,3}$.

Pilśniak put forward the following conjecture.

**Conjecture 1.2.** [1] If $G$ is a 2-connected graph, then $D'(G) \leq 1 + \lceil \sqrt{\Delta(G)} \rceil$.

In this paper, we prove the following theorem which proves the conjecture.

**Theorem 1.3.** Let $G$ be a connected graph of maximum degree $\Delta$. If the minimum degree is at least two, then $D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1$.

For our purposes, we consider graphs with specific construction that are from dutch-windmill graphs. Because of this, in Section 2, we compute the distinguishing index of the dutch windmill graphs. In Section 3, we use those results to prove the main result. In the last section we present graphs $G$ for which $D'(G) \leq \lceil \sqrt{\Delta(G)} \rceil$. 
2. Distinguishing index of Dutch windmill graphs

To obtain the upper bound for the distinguishing index of connected graphs with minimum degree at least two, we characterize such graphs with minimum number of edges. For this characterization we need the concept of Dutch windmill graphs. The Dutch windmill graph $D^k_n$ is the graph obtained by taking $n$, $(n \geq 2)$ copies of a cycle $C_k$, $(k \geq 3)$ with a vertex in common (see Figure 1). If $k = 3$, then we call $D^3_n$, a friendship graph. In the following theorem we compute the distinguishing number of the Dutch windmill graphs.

**Theorem 2.1.** For every $n \geq 2$ and $k \geq 3$, 
$$D(D^k_n) = \min\{r : \frac{r{k-1}}{2} - \frac{r}{2} \geq n\}.$$ 

**Proof.** We consider two cases:

**Case 1)** $k$ is odd. There is a natural number $m$ such that $k = 2m + 1$. We can consider a blade of $D^k_n$ as in Figure 2. Let $(x_1^{(i)}, x_1^{(i)'}, \ldots, x_m^{(i)}, x_m^{(i)'}$) be the label of vertices $(v_1, v_1', \ldots, v_m, v_m')$ of the $i$th blade where $1 \leq i \leq n$. Suppose that $L = \ldots$
\{(x_1^{(i)}, x_1^{(i)}, \ldots, x_m^{(i)}) \mid 1 \leq i \leq n; x_j^{(i)}, x_j^{(i)} \in \mathbb{N}, 1 \leq j \leq m\} \text{ is a labeling of the vertices of } \mathcal{D}_n^k \text{ except its central vertex. In an } r\text{-distinguishing labeling we must have:}

(i) There exists \( j \in \{1, \ldots, m\} \) such that \( x_j^{(i)} \neq x_j^{(i)} \) for all \( i \in \{1, \ldots, n\} \).

(ii) For \( i_1 \neq i_2 \) we must have \((x_1^{(i_1)}, x_1^{(i_1)}, \ldots, x_m^{(i_1)}, x_m^{(i_1)}) \neq (x_1^{(i_2)}, x_1^{(i_2)}, \ldots, x_m^{(i_2)}, x_m^{(i_2)})\)
and \((x_1^{(i_1)}, x_1^{(i_1)}, \ldots, x_m^{(i_1)}, x_m^{(i_1)}) \neq (x_1^{(i_2)}, x_1^{(i_2)}, \ldots, x_m^{(i_2)}, x_m^{(i_2)})\).

The number of \((2m)\)-arrays of labels which using \( r \) labels that satisfying the condition (i) is \( \frac{r^{2m} - r^m}{2} \). According to the condition (ii) and since there are \( n \) blades, we conclude that
\[
D(\mathcal{D}_n^k) = \min\{r : \frac{r^{2m} - r^m}{2} \geq n\}.
\]

**Case 2** \( k \) is even. There is a natural number \( m \) such that \( k = 2m \). We can consider a blade of \( \mathcal{D}_n^k \) as in Figure 2. Let \((x_0^{(i)}, x_1^{(i)}, \ldots, x_m^{(i)})\) be the label of vertices \((v_0, v_1, v_2, \ldots, v_{m-1}, v_{m-1}')\) of \( i \)th blade where \( 1 \leq i \leq n \). Suppose that \( L = \{(x_0^{(i)}, x_1^{(i)}, \ldots, x_m^{(i)}) \mid 1 \leq i \leq n; x_0^{(i)}, x_j^{(i)}, x_j^{(i)} \in \mathbb{N}, 1 \leq j \leq m - 1\} \) is a labeling of the vertices of \( \mathcal{D}_n^k \) except its central vertex. In an \( r\)-distinguishing labeling we must have:

(i) There exists \( j \in \{1, \ldots, m - 1\} \) such that \( x_j^{(i)} \neq x_j^{(i)} \) for all \( i \in \{1, \ldots, n\} \).

(ii) For \( i_1 \neq i_2 \) we must have
\[
(x_0^{(i_1)}, x_1^{(i_1)}, \ldots, x_m^{(i_1)}, x_m^{(i_1)}) \neq (x_0^{(i_2)}, x_1^{(i_2)}, \ldots, x_m^{(i_2)}, x_m^{(i_2)}),
\]
\[
(x_0^{(i_1)}, x_1^{(i_1)}, x_1^{(i_1)}, \ldots, x_m^{(i_1)}, x_m^{(i_1)}) \neq (x_0^{(i_2)}, x_1^{(i_2)}, x_1^{(i_2)}, \ldots, x_m^{(i_2)}, x_m^{(i_2)}).
\]

There are \( \frac{r^{2m-1} - r^m}{2} \) possible \((2m - 1)\)-arrays of labels using \( r \) labels satisfying condition (i)
and \( \frac{r^{2m-1} - r^m}{2} \) choices for \( x_0^{(i_1)}, x_1^{(i_1)}, \ldots, x_m^{(i_1)}, x_m^{(i_1)} \). According to the condition (ii) and since there are \( n \) blades, we conclude that
\[
D(\mathcal{D}_n^k) = \min\{r : \frac{r^{2m-1} - r^m}{2} \geq n\}.
\]

The following theorem gives the distinguishing index of \( \mathcal{D}_n^k \).

**Theorem 2.2.** For any \( n \geq 2 \) and \( k \geq 3 \), \( D'(\mathcal{D}_n^k) = \min\{r : \frac{r^{k-1} - r^m}{2} \geq n\} \).

**Proof.** There is a natural bijection between the edges of \( \mathcal{D}_n^k \) and the non-central vertices of \( \mathcal{D}_n^{k+1} \), so if we consider the non-central vertices of \( \mathcal{D}_n^{k+1} \) as the edges of \( \mathcal{D}_n^k \), then we have \( D'(\mathcal{D}_n^k) = D(\mathcal{D}_n^{k+1}) \). Therefore the result follows from Theorem 2.2. \( \square \)
3. Proof of conjecture

In this section, we shall prove Conjecture \[1.2\]. To do this, first we state some preliminaries.

By the result obtained by Fisher and Isaak \[5\] and independently by Imrich, Jerebic and Klavžar \[7\] the distinguishing index of complete bipartite graphs is as follows:

**Theorem 3.1.** \[3, 7\] Let \( p, q, d \) be integers such that \( d \geq 2 \) and \( (d - 1)p < q \leq d^p \). Then

\[
D'(K_{p,q}) = \begin{cases} 
    d & \text{if } q \leq d^p - \lceil \log_d p \rceil - 1, \\
    d + 1 & \text{if } q > d^p - \lceil \log_d p \rceil + 1.
\end{cases}
\]

If \( q = d^p - \lceil \log_d p \rceil \) then the distinguishing index \( D'(K_{p,q}) \) is either \( d \) or \( d + 1 \) and can be computed recursively in \( O(\log(q)) \) time.

**Corollary 3.2.** \[9\] If \( p \leq q \), then \( D'(K_{p,q}) \leq \lceil \sqrt[p]{q} \rceil + 1 \).

Let \( G \) be a connected graph of order \( n > 3 \) and let \( c : E(G) \to \{1, 2, \ldots, k\} \) be a colouring of the edges of \( G \) for some positive integer \( k \) (where adjacent edges may be coloured the same). The colour code of a vertex \( v \) of \( G \) (with respect to \( c \)) is the ordered \( k \)-tuple \( c(v) = (a_1, a_2, \ldots, a_k) \) (or simply, \( c(v) = a_1a_2\ldots a_k \)), where \( a_i \) is the number of edges incident with \( v \) that are coloured \( i \) for \( 1 \leq i \leq k \), see \[3\].

**Lemma 3.3.** \[3\] Let \( c \) be a \( k \)-colouring of the edges of a graph \( G \). The maximum number of different color codes of the vertices of degree \( r \) in \( G \) is \( \binom{r + k - 1}{r} \).

Now, we are ready to prove the main theorem of this paper.

**Proof of Theorem 1.3.** If \( \Delta \leq 5 \), then the result follows from Theorem \[1.1\]. So, we suppose that \( \Delta \geq 6 \). Let \( v \) be a vertex of \( G \) with the maximum degree \( \Delta \). If \( G \) does not contain a dutch windmill graph or a triangle attached to \( G \) at the vertex \( v \) (a proper subgraph \( H \) of \( G \) is called attached if it has only one vertex adjacent to vertices outside \( H \)), then we define \( G' \) to be \( G \). Otherwise, we delete the attached dutch windmill graph or attached triangle attached to \( G \) at the vertex \( v \), until we obtain a subgraph \( G' \). We first label the attached dutch windmill graph or attached triangle, and then we construct an edge labeling with \( 1 + \lceil \sqrt{\Delta} \rceil \) labels stabilizing all vertices of \( G' \) by every automorphism preserving the labeling. By Theorem \[2.2\], we can label the edges of a dutch windmill graph attached to \( G \) at the vertex \( v \) for which \( v \) is the central point of the dutch windmill graph, with at most \( 1 + \lceil \sqrt{\Delta} \rceil \) labels from label set \( \{0, 1, \ldots, \lceil \sqrt{\Delta} \rceil\} \), distinguishingly, such that the label 0 is used for at least one edge. If there exists a triangle attached to \( G \) at \( v \), then we label the two incident edges to \( v \) with 0 and 1, and the other edge of the triangle with label 2.

Let \( N^{(1)}(v) = \{v_1, \ldots, v_{|N^{(1)}(v)|}\} \) be the vertices of \( G' \) at distance one from \( v \), and continue the labeling by the following steps:
**Step 1)** In this step, we want to find a labeling of the edges of $G' \lfloor N^{(1)}(v) \rfloor$ such that each vertex of $N^{(1)}(v)$ is fixed by every automorphism preserving this labeling. Since $|N^{(1)}(v)| \leq \Delta$, for $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$ and $1 \leq j \leq \lceil \sqrt{\Delta} \rceil$, we label the edges $vv_i[v_i\sqrt{\Delta}+j]$ with label $i$, and we do not use the label 0 any more. With respect to the number of incident edges to $v$ with label 0, we conclude that the vertex $v$ is fixed under each automorphism preserving the labeling. Also, since the dutch windmill or the triangle graph attached to $G$ at $v$ has been labeled distinguishingly, so the vertices of attached graph are fixed under each automorphism preserving the labeling. Hence, every automorphism preserving the labeling must map the set of vertices of $G'$ at distance $i$ from $v$ to itself setwise, for any $1 \leq i \leq \text{diam}(G')$. We denote the set of vertices of $G'$ at distance $i$ from $v$ by $N^{(i)}(v)$ for $2 \leq i \leq \text{diam}(G')$.

If $N^{(2)}(v) = \emptyset$, then let $E_{k,j}$ be the set of unlabeled edges of $G'$ incident to the vertex $v_j[v_j\sqrt{\Delta}+k]$. For every $0 \leq j \leq \lceil \sqrt{\Delta} \rceil - 1$, we can label the elements of each $E_{k,j}$ with labels $\{1, \ldots, \lceil \sqrt{\Delta} \rceil\}$ such that for every pair of $(E_{k,j}, E_{k',j'})$, where $k \neq k'$, there exists a label $l$, $1 \leq l \leq \lceil \sqrt{\Delta} \rceil$, such that the number of label $l$ used for labeling of elements of $E_{k,j}$ and $E_{k',j'}$ are distinct (this is possible by Lemma 3.3). Therefore, all elements of $N^{(1)}(v)$ is fixed under each automorphism preserving the labeling.

Thus we suppose that $N^{(i)}(v) \neq \emptyset$, for some $i \geq 2$. Now, we partition the vertices of $N^{(1)}(v)$ into two sets $M^{(1)}_1$ and $M^{(1)}_2$ as follows:

$$M^{(1)}_1 = \{x \in N^{(1)}(v) : N(x) \setminus \{v\} \subseteq N(v)\},$$

$$M^{(1)}_2 = \{x \in N^{(1)}(v) : N(x) \setminus \{v\} \nsubseteq N(v)\}.$$

The sets $M^{(1)}_1$ and $M^{(1)}_2$ are mapped to $M^{(1)}_1$ and $M^{(1)}_2$, respectively, setwise, under each automorphism preserving the labeling. For $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$, we set $L_i = \{v_i[v_i\sqrt{\Delta}+j] : 1 \leq j \leq \lceil \sqrt{\Delta} \rceil\}$. By this notation, we get that for $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$, the set $L_i$ is mapped to $L_i$ setwise, under each automorphism preserving the labeling. Let the sets $M^{(1)}_{1i}$ and $M^{(1)}_{2i}$ for $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$ be as follows:

$$M^{(1)}_{1i} = M^{(1)}_1 \cap L_i, \quad M^{(1)}_{2i} = M^{(1)}_2 \cap L_i.$$

It is clear that the sets $M^{(1)}_{1i}$ and $M^{(1)}_{2i}$ are mapped to $M^{(1)}_{1i}$ and $M^{(1)}_{2i}$, respectively, setwise, under each automorphism preserving the labeling. Since for any $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$, we have $|M^{(1)}_{1i}| \leq \lceil \sqrt{\Delta} \rceil$, we can label all incident edges to each element of $M^{(1)}_{1i}$ with labels $\{1, 2, \ldots, \lceil \sqrt{\Delta} \rceil\}$, such that for any two vertices of $M^{(1)}_{1i}$, say $x$ and $y$, there exists a label $k$, $1 \leq k \leq \lceil \sqrt{\Delta} \rceil$, such that the number of label $k$ for the incident edges to vertex $x$ is different from the number of label $k$ for the incident edges to vertex $y$ (this is possible by Lemma 3.3). Hence, it can be deduced that each vertex of $M^{(1)}_{1i}$ is fixed under each automorphism preserving the labeling, where $0 \leq i \leq \lceil \sqrt{\Delta} \rceil - 1$. Thus every vertex of $M^{(1)}_1$ is fixed under each
automorphism preserving the labeling. In sequel, we want to label the edges incident to vertices of \( M_2^{(1)} \) such that \( M_2^{(1)} \) is fixed under each automorphism preserving the labeling, pointwise. For this purpose, we partition the vertices of \( M_2^{(1)} \) into the sets \( M_{2i}^{(1)} \), \( (1 \leq p \leq \Delta - 1) \) as follows:

\[
M_{2i}^{(1)} = \{ x \in M_2^{(1)} : |N(x) \cap N^{(2)}(v)| = p \}.
\]

Since the set \( N^{(i)}(v) \), for any \( i \), is mapped to itself, it can be concluded that \( M_{2i}^{(1)} \) is mapped to itself, setwise, under each automorphism preserving the labeling, for any \( i \) and \( p \). Let \( M_{2i}^{(1)} = \{ x_{i1}, \ldots, x_{ips} \} \). It is clear that \( |M_{2i}^{(1)}| \leq |M_2^{(1)}| \leq \lfloor \sqrt{\Delta} \rfloor \). Let \( x_{ipk} \in M_{2i}^{(1)} \), and \( N(x_{ipk}) \cap N^{(2)}(v) = \{ x'_{ipk1}, x'_{ipk2}, \ldots, x'_{ipk_p} \} \). We assign to the \( p \)-tuple \( (x_{ipk}, x'_{ipk1}, \ldots, x'_{ipk_p}) \) of edges, a \( p \)-tuple of labels such that for every \( x_{ipk} \) and \( x_{ipk'} \), \( 1 \leq k, k' \leq s_p \), there exists a label \( l \) in their corresponding \( p \)-tuples of labels for which the number of label \( l \) in the corresponding \( p \)-tuples of \( x_{ipk} \) and \( x_{ipk'} \), is distinct. For constructing \( |M_{2i}^{(1)}| \) numbers of such \( p \)-tuples we need, \( \min\{ r : \binom{p+r-1}{r-1} \geq |M_{2i}^{(1)}| \} \) distinct labels. Since for any \( 1 \leq p \leq \Delta - 1 \), we have

\[
\min\left\{ r : \binom{p+r-1}{r-1} \geq |M_{2i}^{(1)}| \right\} \leq \min\left\{ r : \binom{p+r-1}{r-1} \geq \lfloor \sqrt{\Delta} \rfloor \right\} \leq \lfloor \sqrt{\Delta} \rfloor,
\]

so we need at most \( \lfloor \sqrt{\Delta} \rfloor \) distinct labels from label set \( \{1, 2, \ldots, \lfloor \sqrt{\Delta} \rfloor \} \) for constructing such \( j \)-tuples. For instance, let \( p = 1 \), and \( M_{2i}^{(1)} = \{ x_{i1}, \ldots, x_{i,s_{i1}} \} \). By our method, we label the edge \( x_{i1}x'_{i1k} \) with label \( k \) for \( 1 \leq k \leq s_{i1} \) where \( s_{i1} \leq \lfloor \sqrt{\Delta} \rfloor \). Hence, the vertices of \( M_{2i}^{(1)} \), for any \( 1 \leq p \leq \Delta - 1 \), are fixed under each automorphism preserving the labeling. Therefore, the vertices of \( M_2^{(1)} \) for any \( 0 \leq i \leq \lfloor \sqrt{\Delta} \rfloor - 1 \), and so the vertices of \( M_2^{(1)} \) are fixed under each automorphism preserving the labeling. Now, we can get that all vertices of \( N^{(1)}(v) \) are fixed. If there exist unlabeled edges of \( G' \) with the two endpoints in \( N^{(1)}(v) \), then we assign them an arbitrary label, say 1.

**Step 2** Here we consider \( N^{(2)}(v) \). We partition this set such that the vertices of \( N^{(2)}(v) \) with the same neighbours in \( M_2^{(1)} \), lie in a set. In other words, we can write \( N^{(2)}(v) = \bigcup_i A_i \), such that \( A_i \) contains that elements of \( N^{(2)}(v) \) having the same neighbours in \( M_2^{(1)} \), for any \( i \). Since all vertices in \( M_2^{(1)} \) are fixed, so the set \( A_i \) is mapped to \( A_i \) setwise, under each automorphism of \( G' \) preserving the labeling. Let \( A_i = \{ w_{i1}, \ldots, w_{ipt_i} \} \), and we have

\[
N(w_{i1}) \cap M_2^{(1)} = \cdots = N(w_{ipt_i}) \cap M_2^{(1)} = \{ v_{i1}, \ldots, v_{ip} \}.
\]

We consider the following two cases:

Case 1) If for every \( w_{ij} \) and \( w_{ij'} \) in \( A_i \), where \( 1 \leq j, j' \leq t_i \), there exists a \( k \), \( 1 \leq k \leq p_i \), for which the label of edge \( w_{ij}v_{ik} \) is different from label of edge \( w_{ij'}v_{ik} \), then all vertices of \( G' \) in \( A_i \) are fixed under each automorphism preserving the labeling. Now, we can label the unlabeled edges of \( G' \) which are incident to the vertices in \( A_i \) and another their endpoint is \( N^{(3)}(v) \), arbitrarily.
Case 2) If there exist $w_{ij}$ and $w_{ij}'$ in $A_i$, where $1 \leq j, j' \leq t_i$, such that for every $k$, $1 \leq k \leq p_i$, the label of edge $w_{ij}v_{ik}$ and $w_{ij}'v_{ik}$ are the same, then we can make a labeling such that the vertices in $A_i$ have the same property as Case 1, and so are fixed under each automorphism preserving the labeling, by using at least one of the following actions:

- By permuting the coordinates of $j$-tuple of labels assigned to the incident edges to $v_{ik}$ with an end point in $N^{(2)}(v)$.
- By using a new $j$-tuple of labels, with labels $\{1, 2, \ldots, \lceil \sqrt{\Delta} \rceil\}$, for incident edges to $v_{ik}$ with an end point in $N^{(2)}(v)$, such that (by notations in Step 1) for every $x_{ij}^{k'}$ and $x_{ij}^{k''}$, $1 \leq k', k'' \leq s_j$, there exists a label $l$ in their corresponding $j$-tuples of labels with different number of label $l$ in their coordinates, where $1 \leq i' \leq \lceil \sqrt{\Delta} \rceil$.
- By labeling the unlabeled edges of $G'$ with the two end points in $N^{(2)}(v)$ which are incident to the vertices in $A_i$.
- By labeling the unlabeled edges of $G'$ which are incident to the vertices in $A_i$, and another their endpoint is $N^{(3)}(v)$.
- By labeling the unlabeled edges of $G'$ with the two end points in $N^{(3)}(v)$ for which the end points in $N^{(3)}(v)$ are adjacent to some of vertices in $A_i$.

Using at least one of above actions, it can be seen that every two vertices $w_{ij}$ and $w_{ij}'$ in $A_i$ have the property as Case 1. Thus we conclude that all vertices in $A_i$, for any $i$, and so all vertices in $N^{(2)}(v)$, are fixed under each automorphism preserving the labeling. Now, we can label the unlabeled edges of $G'$ which are incident to the vertices in $A_i$ and another their endpoint is $N^{(3)}(v)$, arbitrarily. If there exist unlabeled edges of $G$ with the two endpoints in $N^{(2)}(v)$, then we assign them an arbitrary label, say 1.

By continuing this method, in the next step we partition $N^{(3)}(v)$ exactly by the same method as partition of $N^{(2)}(v)$ to the sets $A_i$’s in Step 2, and so we can make a labeling such that the elements of $N^{(i)}(v)$ are fixed pointwise, under each automorphism preserving the labeling, for any $3 \leq i \leq \text{diam}(G')$.

For a 2-connected planar graph $G$, the distinguishing index may attain $1 + \lceil \sqrt{\Delta(G)} \rceil$. For example, consider the complete bipartite graph $K_{2,q}$ with $q = r^2$, where $r$ is a positive integer. By Theorem $\Box$, $D'(K_{2,q}) = r + 1$.

4. Graphs with $D'(G) \leq \lceil \sqrt{\Delta} \rceil$

In this section, we present graphs $G$ with specific construction such that $D'(G) \leq \lceil \sqrt{\Delta} \rceil$. To do this we state the following definition.
Definition 4.1. Let $G$ be a connected graph with $\delta(G) \geq 2$. The graph $G$ is called a $\delta$-minimally graph, if the minimum degree of each spanning subgraph of $G$, except $G$, is less than $\delta(G)$.

It can be concluded from Definition 4.1 that if $e$ is an edge of a connected $\delta$-minimally graph with end points $u$ and $w$, then without loss of generality we can assume that $\deg_G u = \delta$ and $\deg_G w \geq \delta$. In fact the distance between the two vertices of degree greater than $\delta$ is at least two.

The simplest connected 2-minimally graphs are cycles $C_n$ and complete bipartite graphs $K_{2,n}$. Now, we explain more on the structure of a 2-minimally graph. Let call a path of length at least two in the graph a simple path, if all its internal vertices have degree two. Let $G$ be a connected 2-minimally graph.

- If the degree of all vertices of $G$ is two, then $G$ is a cycle graph.
- If there exists a vertex $v$ of $G$ with degree at least three. We consider the two following cases:

Case 1) If $v$ is the only vertex of $G$ with degree greater than two, then $G$ is a graph which is made by identifying the central points of some dutch windmill graphs $D_{p_i}^n$ where $p_i \geq 3$, and hence $\Delta(G) = 2 \sum_{i \in I} n_i$ where $I$ is a set of indices. In this case we denote $G$ by Wind($v$) (for instance, see Figure 3).

Case 2) If $G$ has other vertex $w$ of degree greater than two, then either there exists at least a simple path between $v$ and $w$, or there exists a vertex of degree at least three on each path between $v$ and $w$. In the latter case, we can obtain a vertex $u$ of $G$ with degree greater than two such that there exists at least a simple path between $v$ and $u$ (see Figure 4).

Now we present graphs with $D'(G) \leq \lceil \sqrt{\Delta} \rceil$.

Theorem 4.2. Let $G$ be a connected 2-minimally graph with maximum degree $\Delta$. If $G$ is not a cycle $C_3, C_4, C_5$ or a complete bipartite graph $K_{2,r^2}$ for some integer $r$, then $D'(G) \leq \lceil \sqrt{\Delta} \rceil$.

Proof. If $\Delta = 2$, then $G$ is a cycle. It is known that the distinguishing index of cycle graph of order at least 6 is two. Hence, we suppose that $G$ is not a cycle, so $\Delta \geq 3$. Let $v$ be a vertex...
Figure 4. The state of vertices of degree greater than two in a connected 2-minimally graph.

of $G$ of maximum degree $\Delta$. Suppose that $V' = \{v_1, v_2, \ldots, v_k\}$ are all vertices of $G$ which are of degree at least three such that there exists at least a simple path between $v$ and $v_i$, for any $1 \leq i \leq k$ (it is possible that $V' = \emptyset$). Let there exist $n_{ij}$ disjoint simple paths of length $j$ between $v$ and $v_i$, for any $1 \leq i \leq k$ and $2 \leq j \leq \text{diam}(G)$ where $n_{ij}$ is a non-negative integer and $\sum_{j=2}^{\text{diam}(G)} n_{ij} > 0$. We can label these $n_{ij}$ simple paths of length $j$ with at most $\lceil \sqrt{\Delta} \rceil$ labels, by using $n_{ij}$ numbers of $j$-tuples such that the coordinates of each $j$-tuple are in the set $\{1, 2, \ldots, \lceil \sqrt{\Delta} \rceil \}$, for any $1 \leq i \leq k$ and $2 \leq j \leq \text{diam}(G)$, and for every two paths of length $j$, say $P_1$ and $P_2$ with labels $(a_1, \ldots, a_j)$ and $(b_1, \ldots, b_j)$ respectively, there exists $l$, $1 \leq l \leq j$, such that $a_l \neq b_l$. Let $P$ be a simple path between $v$ and $v_i$ for some $1 \leq i \leq k$, such that the label of edge of $P$ which is incident to $v$, is different from the label of edge of $P$ which is incident to $v_i$. We do not use labeling of the simple path $P$, for any other simple path (with the same length) between any two vertices of degree greater than two. Since $G$ is not a complete bipartite graph $K_{2,r}$ for some integer $r$, we can label these paths distinguishingly with at most $\lceil \sqrt{\Delta} \rceil$ labels. Now, we label the induced subgraph $\text{Wind}(x)$, for any vertex $x$ of degree greater than two, if there exists, with at most $\lceil \sqrt{\Delta} \rceil$ labels distinguishingly by Theorem 2.2, such that the distinguishing labeling of $\text{Wind}(v)$ is nonisomorphic to the remaining distinguishing labeling of $\text{Wind}(x)$, where $x \in V(G) - \{v\}$. Thus any automorphism of $G$ preserving this labeling should fix $v, v_1, \ldots, v_k$ and all vertices of degree two on the simple paths between $v$ and $v_i$ for any $1 \leq i \leq k$. Since for any $1 \leq i, j \leq k$ where $i \neq j$, the vertices $v_i$ and $v_j$ are fixed, so all the simple paths between $v_i$ and $v_j$, if there exist, are mapped to each other under each automorphism of $G$ preserving this labeling. Hence we can label all edges of these simple paths with at most $\sqrt{\Delta}$ labels, by assigning distinct ordered tuples of labels of length of the simple paths between $v_i$ and $v_j$ such that all vertices of these paths are fixed under each automorphism of $G$ preserving this labeling.

For any $1 \leq i \leq k$, we consider $v_i$, and suppose that $v_{i1}, \ldots, v_{ik_i}$ are all vertices of $V(G) \setminus \{v_1, \ldots, v_k\}$ with degree at least three such that there exists at least a simple path between $v_i$
and \(v_{ij}\) for any \(1 \leq j \leq k_i\). Now we do the same method as labeling of simple paths between \(v\) and \(\{v_1, \ldots, v_k\}\), for all simple paths between \(v_i\) and \(\{v_{i1}, \ldots, v_{ik_i}\}\) with at most \(\lceil \sqrt{\Delta} \rceil\) labels. Also, we do the same method as labeling of simple paths between \(v_i\) and \(v_j\), for all simple paths between \(v_{ip}\) and \(v_{iq}\) with at most \(\lceil \sqrt{\Delta} \rceil\) labels, where \(1 \leq p, q \leq k_i\). Note that we do not use labeling of \(P\) for any simple path with the same length as \(P\) between \(v_i\) and \(\{v_{i1}, \ldots, v_{ik_i}\}\). Thus the vertices \(\{v_i, v_{i1}, \ldots, v_{ik_i}\}\) and all vertices of the simple paths between them are fixed under each automorphism of \(G\) preserving this labeling. After the finite number of steps we can obtain a distinguishing edge labeling of \(G\) with at most \(\lceil \sqrt{\Delta} \rceil\) labels. □

5. Conclusion

We gave an upper bound for the distinguishing index of graphs \(G\) with minimum degree at least two. This result proves a conjecture by Pilśniak (2017). We also studied graphs \(G\) with \(D'(G) \leq \lceil \sqrt{\Delta} \rceil\). We think that the following conjecture is true, but until now all attempts to prove this failed. So, we end this paper by proposing the following conjecture.

**Conjecture 5.1.** Let \(G\) be a connected graph with maximum and minimum degree \(\Delta\) and \(\delta\), respectively.

(i) If \(G\) is a connected \(\delta\)-minimally graph with \(\delta(G) \geq 3\) such that \(G\) is not a complete bipartite or \(\delta\)-regular graph, then \(D'(G) \leq \lceil \delta(G) \sqrt{\Delta(G)} \rceil\).  
(ii) If \(G\) is a connected graph with \(\delta(G) \geq 3\), then \(D'(G) \leq 1 + \lceil \delta(G) \sqrt{\Delta(G)} \rceil\).

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