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CAYLEY GRAPH ASSOCIATED TO A SEMIHYPERGROUP

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Abstract. The purpose of this paper is the study of Cayley graph associated to a semihypergroup(or hypergroup). In this regards first we associate a Cayley graph to every semihypergroup and then we study the properties of this graph, such as Hamiltonian cycles in this graph. Also, by some of examples we will illustrate the properties and behavior of these Cayley graphs, in particulars we show that the properties of a Cayley graph associated to a semihypergroup is completely different with respect to the Cayley graph associated to a semigroup(group). Also, we briefly discuss on category of Cayley graphs associated to semihypergroups and construct a functor from this category to the category of digraphs. Finally, we give an application the Cayley graph of a hypergroupoid to a social network.

1. Introduction

Algebraic hyperstructure is one branch of algebra which deals with to structures endowed with multivalued operations. More precisely, a hyperoperation $\circ$ on a non-void set H is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of H. Hyperstructure came into being through the notion of the hypergroup. The hypergroup was introduced by

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F. Marty in 1934, during the 8^{th} congress of the Scandinavian Mathematicians [15] and it has been developed by many authors in viewpoint of theory and applications [4, 5]).

Let H be a non-empty set and let $P_*(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a map $\circ : H \times H \longrightarrow P^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid. For non-empty subsets A and B of H, $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$.

A hypergroupoid $(S, \circ)$ is called a semihypergroup if for x, y, z of S we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v$.

A semihypergroup $(H, \circ)$ is called a hypergroup if for all $x \in H$, we have $x \circ H = H \circ x = H$. In general a hyperalgebraic system is a set together with a family of a finitary hyperoperations(for more details see [2]).

Let H be a hypergroup(resp. semihypergroup) and the relation β^* is the smallest equivalence relation on H, such that the quotient space $H/\beta^* = \{\beta^*(a) | a \in H\}$ together with hyperoperation $\otimes$ defined by

$$\beta^*(a) \otimes \beta^*(b) = \beta^*(c), \quad \forall c \in a \circ b,$$

is a group(resp. semigroup). The group $(H/\beta^*, \otimes)$ is called *fundamental group* of $(H, \circ)$ and the relation β^* is called the *fundamental relation* on H. This relation plays an important role to the study of hypergroup theory. This relation was introduced by Koskas [13] and has been studied by Corsini [4]. Freni in [7] characterized β^* as follows:

If $\mathcal{U}$ denotes the set of all finite products of the elements of H, then the relation β is we denoted on H by $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$. Then $\beta^* = \beta$. Moreover, if we denote the equivalence class of $a \in H$ by $\beta(a)$, then $(H/\beta^*, \odot)$ is a group, where

$$\beta(a) \odot \beta(b) = \beta(c), \quad \forall \;\; c \in \beta(a).\beta(b).$$

Let H be a semihypergroup. For every integer $n > 0$, and $s \in H$, we get the powers of $s : s^1 = s, s^{n+1} = s^n \circ s \subset H$.

Using the original definition of cyclic semihypergroup as it can be seen in [18] as well, we give the following definitions.

A hypergroup H is called cyclic, if $H = h^1 \cup h^2 \cup ... \cup h^n \cup ...$, for some $h \in H$.

If there exists an integer $n > 0$, the minimum one with the following property:

$$H = h^1 \cup h^2 \cup ... \cup h^n,$$

then H is called cyclic with *finite period* and call h the *generator* of H with period n.

The definition of Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups being described by a set of generators. In the last 50 years, the theory of Cayley graphs has been grown into a substantial branch in algebraic graph theory [12].

The concept of generalized Cayley graphs of semigroups was first introduced in [20], where some fundamental properties of generalized Cayley graphs of semigroups were studied. Let G be a semigroup, and let S be a nonempty subset of G. The Cayley graph $Cay(G, S)$ of G relative to S is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y), such that $sx = y$ for some $s \in S$ [11](Also, for more details see [10, 21, 9]).

The purpose of this paper is the study Cayley graph of a semihypergroup(or a hypergroup) H and its application to social networks. This paper has been written in 6 sections. In Section 2, we introduce Cayley graph of a semigroup and investigate its basic properties. Also, by some examples of Cayley graphs of semihypergroups, we show that the behavior of a of Cayley graphs of a semigroup is different to Cayley graphs of a semigroup. In Section 3, we study Hamiltonian Cayley graphs of semihypergroups and obtains their properties. In Section 4, some kinds of Cayley graphs of a semihypergroup with the certain properties are considered and analogous algebraic properties of their semihypergroup are investigated.

In Section 5, we introduce **SHC**, category of semihypergroups with connection sets and defining a functor Cay_H from **SHC** to category of digraphs, which associate to each semihypergroup its associative Cayley graph

Finally, in Section 6, an application of Cayley graph of hypergroupoids in social networks are given. Also, an algorithm is designed to calculate the number spreading of advertising before seeing certain a person in a social network.

2. Cayley graph of a semihypergroup

Cayley graphs of a semihypergroup have been studied because they reflect the structure of a semihypergroup. Finite Cayley graphs of semigroups $Cay(S, A)$, where A is a one-element has been characterized by B. Zelinka in [20]. In this paper, we introduce Cayley graphs of semihypergroups, also we state and prove some related results of these graphs.

Let $(H, \circ)$ be a semihypergroup, and A be its subset. The graph $Cay_H(H, A)$ is a directed graph whose vertices are elements of H and in which there is a directed edge from a vertex u into a vertex v if and only if $v \in u \circ a$, where $a \in A$.

if $A = \{a\}$ is a one-element set, then instead of $Cay_H(H, A)$ we shall write simply $Cay_H(H, a)$.

The properties of Cayley graph of a semihypergroup is completely different to the Cayley graph of a semigroup.

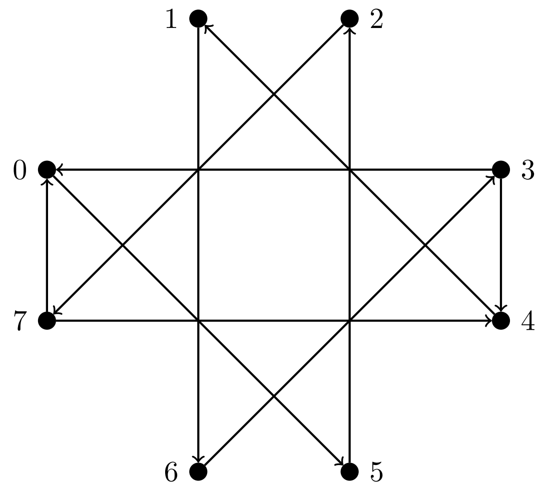

Figure 1. $Cay_H(H,5)$

For example in [20], every Cayley graph of a semigroup $Cay(G,a)$ has the property that the output degree of each of its vertices is 1, but example 2.1 shows that it does not hold for Cayley graph of a semihypergroup as $Cay_H(H,a)$.

Example 2.1. According to Example 3.6 of [8], $(H;\circ)$ is a hypergroup, where $\circ$ is defined as table 1. Cayley graph $Cay_H(H,5)$ shows in figure 1. It is clear that output degree vertices 3 and 7 of $Cay_H(H,5)$ is equal to 2.

$\circ$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0.4	5	6	7	0,4
2	2	3	0,4	1,5	6	7	0,4	1,5
3	3	0,4	1,5	2,6	7	0,4	1,5	2,6
4	4	5	6	7	0	1	2	3
5	5	6	7	0,4	1	2	3	0,4
6	6	7	0,4	1,5	2	3	0,4	1,5
7	7	0,4	1,5	2,6	3	0,4	1,5	2,6

Table 1. $(H,\circ)$ is a semihypergroup.

We have the following result for Cayley graphs of a semihypergroups with connection to singletons.

Lemma 2.2. *Let $(H,\circ)$ be a finite semihypergroup.*

(i) Let $a \in H$, and let $Cay_H(H,a)$ be Cayley graph of H with connection set $\{a\}$. if $u \in V(Cay_H(H,a))$, $uv, uv' \in E(Cay_H(H,a))$, then $v\beta v'$.

(ii) if $v\beta v'$, then there exists $a \in H$, such that v and v' are two output vertices of $Cay_H(H,a)$ such that has a common vertex.

Proof. (i) Suppose uv, uv' are edges in Cayley graph $Cay_H(H,a)$, then $v \in u \circ a$, $v' \in u \circ a$. So $\{v,v'\} \subseteq u \circ a$. It follows that $v\beta v'$.

(ii) Let $v\beta v'$. Then there exist $u, a \in H$, such that $\{v,v'\} \in u \circ a$. By definition, two edges (u,v) and (u,v') are members of the set of edges in $Cay_H(H,a)$. □

Every Cayley graph of $Cay(G,a)$ a semigroup G has the property that the cardinal numbers of the set of edges of the graph $Cay(G,a)$ is equal to the cardinal number of vertices, but this does not hold for e Cayley graph $Cay_H(S,a)$ of a semihypergroup.

Lemma 2.3. *Let $(H,\circ)$ be a finite semihypergroup and $a \in H$. Consider $Cay_H(H,a)$ as Cayley graph of a semihypergroup of H with the connection set $\{a\}$. Then we have*

$$|E(Cay_H(H,a))| = \sum_{x \in H} |x \circ a|,$$

where, $|X|$ denotes the cardinal number of set X.

Proof. Output vertices in the edge of the input vertex x are members of $x \circ a$. Then the number of output edges of any vertex x is equal to the number of $x \circ a$. It follows that the number of edges of the Cayley graph $Cay_H(H,a)$ is equal to the sum $|x \circ a|$, where $x \in H$. □

Lemma 2.4. *Let $(H,\circ)$ be a finite semihypergroup and $A \subseteq H$, and let $Cay_H(H,A)$ be the Cayley graph of H with connection set A. Then*

$$|E(Cay_H(H,A))| = \sum_{x \in H} |x \circ A|.$$

Proof. Output vertices in the edge of the input vertex x are the members of $x \circ A$. Then the number of output edges of any vertex x is equal to the number of $x \circ A$. It follows that the number of edges of the Cayley graph $Cay_H(H,A)$ is equal to the sum $|x \circ A|$, where $x \in H$. □

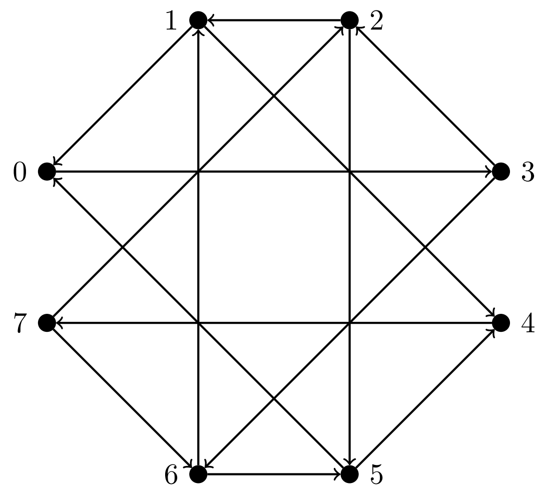

FIGURE 2. The graph $Cay_H(H,3)$

3. **Hamiltonian Cycles in Cayley graphs**

Since the 1984 survey of results on Hamiltonian cycles and paths in Cayley graphs by Witte and Gallian [19], many advances have been made.

Much of the focus of research has been directed towards proving special cases of the following conjecture made by many people.

Conjecture [6]: Every connected Cayley graph with more than 2 vertices is Hamiltonian.

In the seque, we study the properties of Cayley graphs of semi-hypergroups with Hamiltonian cycle.

Example 3.1. Let $(H;\circ)$ be the hypergroup in Example 2.5. Then the Cayley graph $Cay_H(H,3)$ is shown in figure 2. This graph has a Hamiltonian cycle which is as follows:

$$0 \to 3 \to 2 \to 1 \to 4 \to 7 \to 6 \to 5 \to 0.$$

But this Hamiltonian cycle is not unique, because there is another Hamiltonian cycle as follows:

$$0 \to 3 \to 2 \to 5 \to 4 \to 7 \to 6 \to 1 \to 0.$$

Note that it is not true that every Cayley graph of a semihypergroup H with a connected subset, necessarily contains a Hamiltonian cycle. Because: It is obvious that the Cayley graph $Cay_H(H,2)$ shown in figure 3 has not any Hamiltonian cycles, although H is a cyclic semihypergroup with 3 as its generator.

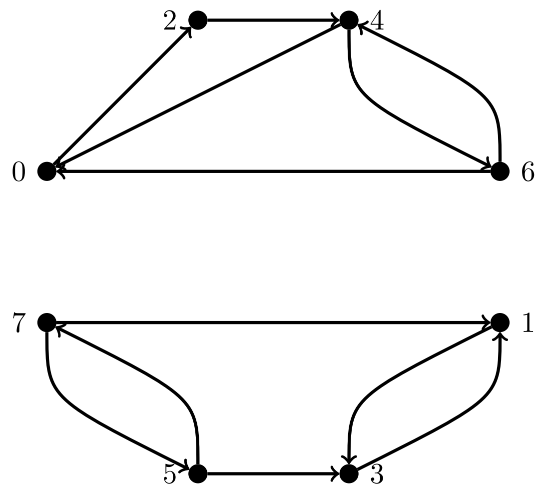

Figure 3. The Cayley graph $Cay_H(H, 2)$

A conjecture naturally gives rise: there is a relationship between Hamiltonian cycles and generators of cyclic semihypergroups. In the following we give some of this connections.

Theorem 3.2. *Let H be a finite semihypergroup and $\mid H \mid= n$, $A \subseteq H$. If $Cay_H(H, A)$ is a Hamiltonian graph, then we have*

$$H = A \cup A^2 \cup A^3 \cup \cdots \cup \ A^{n+1}.$$

Proof. By hypothesis $Cay_H(H, A)$ has a Hamiltonian cycle. Then there exists a cycle as follows:

$$a \to a_1 \to a_2 \to ... \to a_{n-2} \to a_{n-1} \to a,$$

then by definition it follows that:

$$\exists x_1 \in A, a_1 \in a \circ x_1 \subseteq A \circ A = A^2,$$

$$\exists x_2 \in A, a_2 \in a_1 \circ x_2 \subseteq A \circ A^2 = A^3,$$

$$\exists x_3 \in A, a_3 \in a_2 \circ x_3 \subseteq A \circ A^3 = A^4,$$

$\vdots$

$$\exists x_{n-1} \in A, a_{n-1} \in a_{n-2} \circ x_{n-1} \subseteq A^{n-1} \circ A = A^n,$$

TABLE 2. $(H;\circ)$ is a hypergroup.

$\circ$	a	b	c	d
a	b	a,c,d	b	b
b	a,c,d	b	a,c,d	a,c,d
c	b	a,c,d	b	b
d	b	a,c,d	b	b

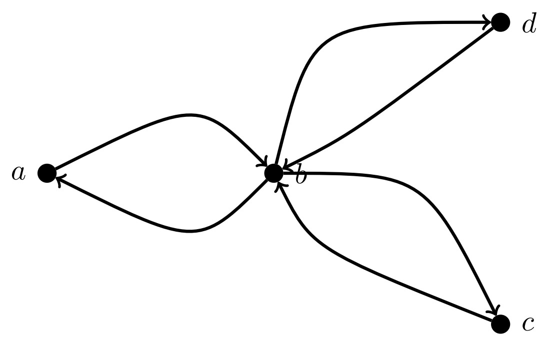

FIGURE 4. Cayley graph $Cay_H(H,a)$

$$\exists x_n \in A, a \in a_{n-1} \circ x_n \subseteq A^n \circ A = A^{n+1}.$$

Since for every $x \in H$, an i such that $x = a_i$, so $x \in A^{i+1}$. Therefore

$$H = A \cup A^2 \cup ... \cup A^n \cup A^{n+1}.$$

□

Corollary 3.3. *Let H be a semihypergroup and $a \in H$. If $Cay_H(H,a)$ is a Hamiltonian graph, then H is cyclic semihypergroup with a generator a.*

The converse of above theorem is not true, that is it is not true that every Cayley graph associated to a cyclic semihypergroup has not a Hamiltonian graph.

Example 3.4. According to an example V of [16], $(H;\circ)$ is a hypergroup, where $\circ$ is defined in table 3.4. Consider the Cayley graph $Cay_H(H,a)$ in figure 3.4.

Clearly, this graph has not any Hamiltonian cycle, although H is a cyclic semihypergroup with a generator a.

A non-empty subset A of a semihypergroup $(H, \circ)$ is called sub-semihypergroup of H if $A \circ A \subseteq A$ and it is called a complete part of H if for all $n \geq 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ the following implication holds:

$$\prod_{i=1}^{n} x_i \cap A \neq \varphi \Rightarrow \prod_{i=1} x_i \subseteq A.$$

The complete closure of A in H is the intersection of all complete parts which contains A and it is denoted by $\mathcal{C}(A)$. Recall that a semihypergroup $(H, \circ)$ is complete if for all $(x, y) \in H^2$, $\mathcal{C}(x \circ y) = x \circ y$.

Theorem 3.5. [16] *if H is cyclic and complete semihypergroup, then H is hypergroup .*

Corollary 3.6. *Let H be complete semihypergroup, $a \in H$. If $Cay_H(H, a)$ is a Hamiltonian graph, then H is cyclic hypergroup and a is a its generator.*

Proof. According to 3.3, H is a cyclic semihypergroup. So H is a complete semihypergroup. Furthermore, H is a hypergroup by 3.5. □

4. **Special Cayley graphs**

Connected Cayley graphs of semihypergroups:

The notion of paths and semi-paths is used to describe the connectedness of a digraph as follows:

Let D be a digraph and let u and v be two distinct vertices of D. The digraph D is strongly connected if the paths $u-v$ and $v-u$ exist. Connected Cayley graphs of a semigroup have been studied. Interesting results have been found. For instance, the paper [17] includes one section devoted to the study of strongly connected Cayley graphs of a semigroup. The condition given in the introduction determines whether a Cayley graph of a group is connected. But these results does not hold for semihypergroups instead of semigroups.

In this Section, we give a condition to determine whether a Cayley graph of a semihypergroup is strongly connected.

Let $(H, \circ)$ be a semihypergroup and $A = \{a_1, a_2, ..., a_n\}$. Then

$$A_2 = \bigcup_{1 \leq i \leq n, 1 \leq j \leq} a_i \circ a_j,$$

$$A_3 = \bigcup_{1 \leq r_j \leq n} \prod_{j=1}^{3} a_{r_j}$$

$$\vdots$$

$$A_n = \bigcup_{1\leq r_j \leq n} \prod_{j=1}^{n} a_{r_j}$$

then we agree:

$$[A] = \bigcup_{i=2}^{n} A_i$$

According to Corollary 3.2. in [17], A Cayley graph $Cay(S, A)$ of a semigroup S is strongly connected if and only if the linear equation $ux = v$ in the variable x has a solution in $\langle A \rangle$ for all $u, v \in S$.

Example 4.1. Let H be the semihypergroup in example 2.1, then for $A = \{2\}$, $[A] = \{0, 2, 4, 6\}$. According example 3.1, $Cay_H(H, 2)$ is not strongly connected and the path between vertices 1 and 2 does not exist. It is easy to see that the equation $1 \in 0 \circ x$ has not a solution in $[A]$. But if we consider the connected graph $Cay_H(H, a)$ in example 3.4, then for $A = \{a\}$, $[A] = H$ and for $v, u \in H$ the equation $v \in u \circ x$ in the variable x has a solution in $[A]$.

So the last example, induce the following conjecture:

An strongly connected Cayley graph $Cay_H(H, A)$ is related to the solution of the equation of the form $v \in ux$ in the variable x in $\langle A \rangle$ for all $u, v \in H$.

At the following we show that this conjecture is true in for semihypergroups.

Theorem 4.2. *Let $(H, \circ)$ be semihypergroup and u and v be distinct elements of H. A path from u to v in $Cay_H(H, A)$ exists if and only if the equation $v \in u \circ x$ in the variable x has a solution in $[A]$.*

Proof. Let $u = s_0, s_1, \ldots, s_n = v$ be a path. By definition, there are elements $a_1, a_2, \ldots, a_n$ of A, such that $s_i \in s_{i1} \circ a_i$ for $i = 1, 2, \ldots, n$. This proves $v \in u \circ (a_1 \circ a_2 \circ \ldots \circ a_n)$, and there exist $x \in a_1 \circ a_2 \circ \cdots \circ a_n \subseteq [A]$, such that x is a solution of equation $v \in u \circ x$.

Conversely, suppose there exists $x_0 \in [A]$ for which $v \in u \circ x_0$. Since $x_0 \in [A]$, we can write $x_0 \in a_1 \circ a_2 \circ \cdots \circ a_r$, where $a_i \in A$. Then

$$v \in u \circ x_0 \subseteq u \circ (a_1 \circ a_2 \circ \cdots \circ a_r) = (u \circ a_1 \circ a_2 \circ \cdots \circ a_{r-1}) \circ a_r.$$

Hence there exist $s_1 \in u \circ (a_1 \circ a_2 \circ \cdots \circ a_{r-1}$, such that $v \in s_1 \circ a_r$, so (s_1, v) is edges in $Cay_H(H, A)$. If we continue this process, we obtain the elements $s_1, \ldots, s_r \in H$, such that (s_i, s_{i-1}) are edges in $Cay_H(H, A)$. Therefore, there exists a path from u to v in $Cay_H(H, A)$, as desired. □

Corollary 4.3. *Let $(H, \circ)$ be a semihypergroup. A Cayley graph $Cay_H(H, A)$ is strongly connected if and only if the equation $v \in u \circ x$ in the variable x has a solution in $[A]$ for all $u, v \in H$.*

Proof. It is enough we show that only a solution for $u \in u \circ x$ there exists for $u \in H$. Since H is a nontrivial semihypergroup, there is an element $w \neq u$ of H, such that the path $u - w$ and the path $w - u$ exist. Hence, $w \in u \circ x_0$ and $u \in w \circ y_0$ for some $x_0, y_0 \in [A]$. So

$$u \in w \circ y_0 \subseteq (u \circ x_0) \circ y_0 = u \circ (x_0 \circ y_0)$$

Therefore, there is $h \in x_0 \circ y_0$, such that $u \in u \circ h$. This complete the proof. □

Undirected graph:

A digraph $D = (V, E)$ is said to be undirected if and only if, for every $(u, v) \in E$, the edge (v, u) belongs to E, too.

Example 4.4. The graph $Cay_H(H, a)$ in example 3.4 is an undirected graph. It is easy to see that $H = a^2 \circ H$.

Theorem 4.5. *Let $(H, \circ)$ be a semihypergroup and $a \in H$ and all vertices of $Cay_H(H, a)$ have a degree greater than 0. If $Cay_H(H, a)$ is an undirected graph, then $H = H \circ a^2$*

Proof. For $x \in H$, So there is $y \in H$ that $(x, y) \in E(Cay_H(H, a))$. By definition of undirected graph, one has

$$(y, x) \in E(Cay_H(H, a)),$$

thus

$$y \in x \circ a, x \in y \circ a.$$

So, we have

$$x \in y \circ a \subseteq (x \circ a) \circ a = x \circ (a \circ a) \subseteq H \circ a^2.$$

Therefore, $H \subseteq H \circ a^2$. This complete the proof. □

Theorem 4.6. *Let $(H, \circ)$ be a semihypergroup and $A \subseteq H$ and all vertices of $Cay_H(H, A)$ have a degree greater than 0. If $Cay_H(H, A)$ is an undirected graph, then $H = H \circ A^2$.*

Proof. For $x \in H$, there exists $y \in H$ such that $(x, y) \in E(Cay_H(H, A))$. By definition of an undirected graph we have

$$(y, x) \in E(Cay_H(H, A)).$$

Therefore,

$$\exists z \in A, x \in y \circ z \subseteq y \circ A,$$

$$\exists z' \in A, y \in x \circ z' \subseteq x \circ A.$$

So, it is concluded that

$$x \in y \circ A \subseteq (x \circ A) \circ A = x \circ (A \circ A) \subseteq H \circ A^2.$$

It follows that $H \subseteq H \circ A^2$.

This complete the proof. □

Eulerian graph:

A digraph D is said to be Eulerian if it contains a closed walk which traverses every arc of D exactly once.

Example 4.7. The graph $Cay_H(H, a)$ in example 3.4 is an Eulerian graph. It is easy to see that

$\sum_{u \in H} |\{z|u \in z \circ A\}| = 6$ and is equal to The number of the edges of the graph $Cay_H(H, a)$.

Theorem 4.8. *A digraph has a Euler cycle if and only if it is connected and indegree of each vertex is equal to its output degree*[3].

Theorem 4.9. *Let $Cay_H(H, A)$ be a Eulerian graph. Then*

(i) *for all $u \in H$, $|u \circ A| = |\{z|u \in z \circ A\}|$;*

(ii) $|E(Cay_H(H, A))| = \sum_{u \in H} |\{z|u \in z \circ A\}|$.

Proof. (i) Clearly, the output vertices in the edge of the input vertex u are members of $u \circ A$. Then the number of output edges of any vertex u is equal to the number of $u \circ A$. On the other hands, u is the edge of output vertex such that there exist $z \in H$, $u \in z \circ A$. According to the theorem4.8, for an Eulerian graph the indegree of each vertex is equal to its output degree. Therefore, $|u \circ A| = |\{z|u \in z \circ A\}|$.

(ii) is an immediate consequence of part $(i)a$ and proposition 2.4. □

Complete digraph:

Complete digraphs are digraphs in which every pair of vertices is connected by a bidirectional edge.

Theorem 4.10. *Let $(H, \circ)$ be a semihypergroup and $A \subseteq H$. If $Cay_H(H, A)$ is a complete digraph, then for all $x \in H$, $H = x \circ A$.*

Proof. Let $Cay_H(H, A)$ be a complete digraph, then every pair of vertices is connected by a bidirectional edge. So for $x, y \in H$, there exists $a \in A$, such that $y \in x \circ a$. Therefore, $y \in x \circ A$, then $H \subseteq x \circ A$. □

Cartesian Products:

Let $(H_1, *)$,$(H_2, \star)$ be semihypergroups. We define a binary hyperoperation on Cartesian products $H_1 \times H_2 = \{(x, y)|x \in H_1, y \in H_2\}$, as follows for all $(x, y), (x_1, y_1) \in H_1 \times H_2$:

$$(x, y) \bigcirc (x_1, x_2) = \{(r, s)|r \in x * x_1, s \in y \star y'\},$$

Theorem 4.11. *Let $(H_1, *)$ and $(H_2, \star)$ be semihypergroups, $A_1 \subseteq H_1$, $A_2 \subseteq H_2$ and $x, x' \in H_1, y, y' \in H_2$. There is an edge from (x, y) to (x', y') in Cayley graph $Cay_H(H_1 \times H_2, A_1 \times A_2)$ if and only if there are edges from x to x' on Cayley graph $Cay_H(H_1, A_1)$ and y to y' on Cayley graph $Cay_H(H_2, A_2)$.*

Proof. Suppose there is an edge from (x, y) to (x', y'), so there is $(s, t) \in A_1 \times A_2$, such that

$$(x', y') \in (x, y) \bigcirc (s, t).$$

Then there edges $s \in A_1$ that $x' \in x * s$ and $t \in A_1$ that $y' \in y \star t$. It followers that There are edge from x to x' on the Cayley graph $Cay_H(H_1, A_1)$ and y to y' on Cayley graph $Cay_H(H_2, A_2)$. □

5. **The $\mathcal{C}ay_H$ functor**

In this section we briefly present some basic results which describe the construction of Cayley graphs starting from semihypergroups with given connection sets via using the category toll (for more details of application of category theory in hyperstructures refer to [1].

Definition 5.1. Let $(H_1, \circ)$ and $(H_2, \bullet)$ be semihypergroups hypergroups). A function $f : H_1 \longrightarrow H_2$ is called an *inclusion homomorphism* if it satisfies the following conditions:

$$f(x \circ y) \subseteq f(x) \bullet f(y) \textit{ for all } x,\ y\ \in H,$$

and f is a strong(or good)homomorphism if

$$f(x \circ y) = f(x) \bullet f(y) \text{ for all } x,\ y \in H.$$

Consider the category **SHC** of semihypergroups with connection sets, that is its objects is as follows:

$$Ob\mathbf{SHC} = \{(S, C) | S \text{ is a semihypergroup} C \subseteq S\}.$$

For $(H, C), (H', D) \in Ob\mathbf{SHC}$,its morphisms set is as follows:

$$Morph\mathbf{SHC}((H, C), (H', D)) = \{f | f : H \to H' \text{ is a strong homomorphism with } f|_C : C \to D\}.$$

Then $Ob\mathbf{SHC}$ together with $Morph\mathbf{SHC}$ is a category, where $Morph\mathbf{SHC}$ denotes the class of all morphism sets in **SHC**.

Let $\mathcal{D}$ denotes the category of digraphs, which may have loops and multiple edges, with graph homomorphisms.

The Cayley graph of a semihypergroup $(H, \circ)$ with a connection set $C \subseteq H$, as $Cay_H(H, C) = (H, E)$, where

$$E = \{(s, z) | \ \exists c \in C, z \in s \circ c\}.$$

Theorem 5.2. *Let $(H, \circ)$ and $(H', \bullet)$ be semihypergroups, and let C and D be subsets of H and H', respectively. Then the rule $\mathcal{C}ay_H : \mathbf{SHC} \to \mathcal{D}$ given by*

$$\begin{array}{ccccc} (H, C) & \rightarrowtail & Cay_H(H, C) & & s \in H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Cay_H(f)} & & \downarrow \\ (H', D) & \rightarrowtail & Cay_H(H', D) & & f(s) \in H' \end{array}$$

for any $f \in \mathbf{SHC}((H, C), (H', D))$ and $s \in H$ is a covariant functor.

Proof. First, we show that Cay_H produces a homomorphism in **D**. Suppose that (s, z) is an edge in $Cay_H(H, C)$, where $s \in H$, $\exists c \in C$ such that $z \in s \circ c$. As $f(z) \in f(s \circ c) = f(s) \bullet f(c)$, then $(f(s), f(z))$ is edge in $Cay_H(H', D)$ for $f \in \mathbf{SHC}((H, C), (H'.D))$. It follows that $Cay_H(f)$ is a homomorphism from $Cay_H(H, C)$ into $Cay_H(H', D)$.

At this stage we verify two properties of a covariant functor.

(1) We have

$$Cay_H(id_{(H,C)} = id_{Cay_H(H,C)},$$

since $Cay_H(id_S)(s) = id(s) = s = id_{Cay(S,C)}(s)$.

(2) For $f \in \mathbf{SHC}((H,C),(H',D))$ and $g \in \mathbf{SHC}((H',D),(H'',E))$, we have

$$Cay_H(gf)(s) = gf(s) = g(f(s)) = Cay_H(g)Cay_H(f)(s),$$

for every $s \in H$. So $Cay_H(gf) = Cay_H(g)Cay_H(f)$. □

Definition 5.3. Consider the categories $\mathcal{C}$ and $\mathcal{D}$. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

(1) faithful if the mapping

$$Mor_{\mathcal{C}}(A, A') \longrightarrow Mor_{\mathcal{D}}(F(A), F(A')),$$

is injective for all $A, A' \in \mathcal{C}$;

(1) full if the mapping

$$Mor_{\mathcal{C}}(A, A') \longrightarrow Mor_{\mathcal{D}}(F(A), F(A')),$$

is surjective for all $A, A' \in \mathcal{C}$.

Corollary 5.4. *The functor* $Cay_H : \mathbf{SHC} \longrightarrow \mathbf{D}$ *is faithful.*

Proof. Let $f, g \in \mathbf{SHC}(H,C),(H',D))$ be such that $Cay_H(f) = Cay_H(g)$, hence for any $s \in H$ $f(s) = g(s)$, then $g = f$. which implies that that the functor $Cay_H : \mathbf{SHC} \longrightarrow \mathbf{D}$ is faithful. □

From the definition of the Cay_H functor and the fact that Cay_H is covariant and faithful, we get the following result.

Definition 5.5. A morphism $f \in \mathcal{C}(A, B)$ with $A, B \in \mathcal{C}$ is called an isomorphism if there exists a morphism $g \in \mathcal{C}(B, A)$ with the properties that $f \circ g = id_B$ and $g \circ f = id_A$.

A morphism $f \in \mathcal{C}(A, B)$ with $A, B \in \mathcal{C}$ is called an monomorphism if it is left cancelable, i.e. $g, h \in \mathcal{C}(C, A)$ with $f \circ g = f \circ h$, we get $g = h$.

A morphism $f \in \mathcal{C}(A, B)$ with $A, B \in \mathcal{C}$ is called an epimorphism if it is right cancelable, i.e. $g, h \in \mathcal{C}(B, C)$ with $g \circ f = h \circ f$ we get $g = h$.

Definition 5.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \to \mathcal{D}$ be an assignment of a unique object $F(A) \in \mathcal{D}$ to an object $A \in \mathcal{C}$ and a unique morphism $F(f)$ in $\mathcal{D}$ to a morphism $f : A \to A'$ in $\mathcal{C}$. We formulate the following two pairs of conditions, (1) and (2):

(1) $F(id_A) = id_{F(A)}$ for $A \in C$; we say F preserves identities.

(2) $F(f) : F(A) \longrightarrow F(A')$ and $F(f_2 f_1) = F(f_2)F(f_1)$ for $f_1 \in C(A_1, A_2)$ and $f_2 \in C(A_2, A_3)$, where $A_1, A_2, A_3 \in \mathcal{C}$, we say that F preserves the composition of morphisms.

If F satisfies (1) and (2), we call F a covariant functor.

Corollary 5.7. *The functor Cay_H preserves isomorphism.*

Proof. Suppose $(H, C), (H', D) \in \mathbf{SHG}$ and $f : (H, C) \longrightarrow (H', D)$ is isomorphism. Then there is $g : (H', D) \longrightarrow (H, C)$, such that

$$f \circ g = id_{H'}, g \circ f = id_H.$$

Also, we verify that

$$Cay_H(f) \circ Cay_H(g) = id_{Cay_H}, Cay_H(g) \circ Cay_H(f) = id_H,$$

for any $s \in H$:

$$Cay_H(g) \circ Cay_H(f) = Cay_H(g)(f(s)) = g \circ f(s) = id_H(s) = s.$$

Similarly, we obtain $Cay_H(f) \circ Cay_H(g) = id_{Cay_H}$. Therefore, the functor Cay_H preserves isomorphisms. □

Corollary 5.8. *The functor Cay_H preserves monomorphisms.*

Proof. Suppose $(H, C), (H', D) \in \mathbf{SHG}$ and $f : (H, C) \longrightarrow (H', D)$ is monomorphism. Then for every $g, h : (H', D) \longrightarrow (H, C)$, such that

$$f \circ g = f \circ h = id_H \Longrightarrow g = h,$$

Thus

$$Cay_H(f) \circ Cay_H(g) = Cay_H(f) \circ Cay_H(h).$$

By definition of functor, we have

$$Cay_H(f \circ g) = Cay_H(f \circ h).$$

Since the functor Cay_H is faithful, so $f \circ g = f \circ h$. Then $g = h$, because f is monomorphism. Therefore, $Cay_H(g) = Cay_H(h)$. It follows that the functor Cay_H preserves monomorphisms. □

Corollary 5.9. *The functor Cay_H preserves epimorphisms.*

Proof. Suppose $(H, C), (H', D) \in$ **SHG** and $f : (H, C) \longrightarrow (H', D)$ is epimorphism. Then for every $g, h : (H', D) \longrightarrow (H, C)$, such that

$$g \circ f = h \circ f = id_H \Longrightarrow g = h$$

Then we have

$$Cay_H(g) \circ Cay_H(f) = Cay_H(h) \circ Cay_H(f).$$

Then by definition of a functor, we obtain

$$Cay_H(g \circ f) = Cay_H(h \circ f).$$

As the functor Cay_H is faithful, then $g \circ f = h \circ f$. Then $g = h$, because f is epimorphism. Therefore, $Cay_H(g) = Cay_H(h)$. It follows that the functor Cay_H preserves epimorphisms. □

6. An application of Cayley graph to social networks

After expanding social networks and its influence on society, it can easily use social network for advertising. But one of the common challenges to analyzing these networks is to determine the least number of members starting to post an advertising. In this section we answer to the following questions:

A) Is the number of people selected for advertising enough at first?

B) How many times after republishing an advertisement everyone in the social networks are getting it?

To answer to these questions first we define a hyperoperation is defined on a social networks, and then the Cayley graph associated to this hypergroupoid related to the social network are being constructed. Also, we introduce two algorithms to finding the least number of members starting to post an advertising in a social network.

Let H be all members a social network set. We define $\odot$ on H, for $a, b \in H$

$$a \odot b = \{Common\ friends\ a\ and\ b\},$$

$(H, \odot)$ is a hypergroupoid, but not nor a hypergroup or a hypersemigroup.

The goal is to finding a number of members start to advertisers. Denote the first advertisers A. Suppose that the advertisers spread it to all friends and friends' friends also do this. With such a strong condition, if one does not receive this advertising, there is not enough advertisers in the first place.

Definition 6.1. Let $(H, \circ)$ be a hypergroupoid an A be a subset of H. The graph $Cay_P(H, A)$ is a directed graph whose vertices are elements of H and in which there is a directed edge from a vertex u into a vertex v if and only if $v \in u \circ a$, where $a \in A$.

Proposition 6.2. *Let $(H, \odot)$ be a social network groupoid and sub A be the set of advertisers. For a graph $Cay_P(H, A)$ we have the following properties, for $x, y \in H$:*

(a): *If $(x, y) \in E(Cay_P(H, A)$, then y is common friend's x and members of set A.*
(b): *Output degree of vertex x is equal to the common friends' x and member of set A.*
(c): *If output degree of a vertex x is zero, then x has not common friend with members of set A.*
(d): *If output degree of a vertex x is non-zero, then x has seen the advertising after the its second publication.*
(e): *Output degree of vertex x is equal to the maximum number of get advertising in second its publication.*
(f): *Indegree of vertex x is equal to the number of friends.*
(g): *If indegree of a vertex x is non-zero, then x is friend with a member of A.*
(h): *If indegree of a vertex x is non-zero, then x is the first person who has seen the ads.*

According to the proposition 6.2, vertices with non-zero degree inputs members of a social network that receive advertising from set A, and they can republish it. Set that vertices with non-zero input degree in $Cay_P(H, A)$ is represented by I. Also, Vertices with non-zero output degree Set in $Cay_P(H, A)$ is represented by O. Members of the set O are members of a social network that have seen an advertising after publishing it twice.

After replacing set A with the union of A and I, $Cay_P(H, A)$ is drawn again. This process can be continued, and adding members of the set A and drawing Cayley graph again, until members of the set A do not increase by replacing set A with the union of A and I. In this level, if $O = H$, this advertising has been seen by everyone on the network, otherwise members of the set are not enough for the advertisement. This general process can be seen in Algorithm 1. This algorithm checks that the members of the set in first is sufficient for universal advertising.

By making small changes to the Algorithm 1 and counting the number of repeated steps until to join a certain person in the set O, a new algorithm can be found. Algorithm 2 is

Algorithm 1 Universal advertising

step 1: Set H to all members of a social network set.

step 2: Set A to Set of first advertisers.

step 3: Draw set $Cay_P(H, A)$.

step 4: Set I to all vertices that has non-zero input degree. Set O to all vertices that non-zero output degree.

step 5: $B = A \cup I$

step 6: If $A \neq B$, then $A = B$ and go to step 3.

step 7: If $O \neq H$, then this advertising is not universal advertising.

designed to answer the second question. It determines the number of spreading of advertising before seeing certain person $x \notin A$.

Algorithm 2 Number spreading of advertising

step 1: Set H to all members of a social network set.

step 2: Set A to Set of first advertisers.

step 3: $i = 1$

step 4: Draw set $Cay_P(H, A)$.

step 5: Set I to all vertices that have non-zero input degree. Set O to all vertices that non-zero output degree.

step 6: If $x \in I$, then $s = i$ and go to step 10

step 7: If $x \in O$, then $s = i + 1$ and go to step 10

step 8: $B = A \cup I$

step 9: If $A \neq B$, then $A = B$ and $i = i + 1$ and go to step 4, else $s = -1$

step 10: If $s = -1$, then x does not see these advertising, otherwise x sees this advertising before s times of the advertising spread.

7. **Conclusion**

The Cayley graph of a semihypergroup is presented and the properties of this graph was studied. By some examples it was shown that the behavior of this Cayley graph is different from the Cayley graph of a semigroup. Also, Hamiltonian Cayley graphs was studied. Furthermore, some of kinds of Cayley graphs with certain properties has been studied and the properties of semihypergroups associated to these graphs was determined.

Also, a connection between category of semihypergroups and category of digraphs was established. Finally, two algorithms was introduced and used them to a social network. At the following in Table 3 the connection between algebraic properties of a semihypergroup and it's Cayley graph has been summarized.

Properties Cayley graphs $Cay_H(H, A)$	Properties semihypergroup H
Hamiltonian graph	$H = A \cup A^2 \cup \cdots \cup A^{n+1}$
Connected	equation $v \in u \circ x$ in the variable x has a solution in $[A]$
undirected	$H = H \circ A^2$
Complete digraph	$\forall x \in H : H = x \circ A$

Table 3. Properties Cayley graphs.

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