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ON GRADED HYPERRINGS AND GRADED HYPERMODULES

FARKHONDEH FARZALIPOUR AND PEYMAN GHIASVAND*

Abstract. Let G be a monoid with identity e. In this paper, first we introduce the notions of G-graded hyperrings, graded hyperideals and graded hyperfields in the sense of Krasner hyperring R. Also, we define the notion of a greded R-hypermodules and some examples are presented. Then we investigate graded maximal, graded prime and graded primary hyperideals of a graded hyperring R. Finally, we study graded maximal, graded prime and graded primary subhypermodules of a graded R-hypermodule M and some interesting results on these concepts are given.

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [13], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner [11]. Prime, primary, and maximal subhypermodules of a hypermodule were discussed by M. M.

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*Corresponding author

Zahedi and R. Ameri in [18]. Also, R. Ameri et al introduced Krasner (m, n)-hyperrings in [2] and in [3] studied prime and primary subhypermodules of (m, n)-hypermodules. The principal notions of algebraic hyperstructure theory can be found in [1, 5, 6, 7, 8, 15, 17]. Furthermore, the study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [16]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [4], [9], [10], [12] and [14]). Theory of greded hyperrings and greded hypermodules can be considered as an extension theory of hyperrings and hypermodules. In addition, graded hyperrings and graded hypermodules are extensions of graded rings and graded modules. In this paper, we define the notions of G-graded hyperrings, graded hyperideals and graded hyperfields in the sense of a Krasner hyperring R, and also introduce greded R-hypermodules. Then, some related results have been achieved and some respective examples have been provided in the following sections.

2. Preliminaries

Definition 2.1. [11] Let H be a nonempty set and $P^*(H)$ denotes the set of all nonempty subsets of H. If $+ : H \times H \longrightarrow P^*(H)$ is a map such that the following conditions hold, then we say that $(H, +)$ is a canonical hypergroup.

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;
(ii) for every $x, y \in H$, $x + y = y + x$;
(iii) there exists $0 \in H$ such that $0 + x = \{x\}$ for every $x \in H$;
(iv) for every $x \in H$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, it is denoted by $-x$;
(v) for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Definition 2.2. Let $A \subset H$. Then A is called a *subhypergroup* of H if $0 \in H$ and $(A, +)$ is itself a hypergroup.

Definition 2.3. [11] A *Krasner hyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$;

(3) the operation " $\cdot$" is distributive over the hyperoperation "+", which means that for all x, y, z of R we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

A Krasner hyperring $(R, +, \cdot)$ is called *commutative with identity* $1 \in R$; if we have

(a) $xy = yx$ for all $x, y \in R$,

(b) $1x = x1$ for all $x \in R$.

Example 2.4. [7] Let $(G, \cdot)$ be a finite group with m elements, $m > 3$, and define a hyperaddition and a multiplication on $H = G \cup \{0\}$, by:

$a + 0 = 0 + a = \{a\}$ for all $a \in H$,

$a + a = \{a, 0\}$ for all $a \in G$,

$a + b = b + a = H - \{a, b\}$ for all $a, b \in G, a \neq b$,

$a \otimes 0 = 0$ for all $a \in H$,

$a \otimes b = a \cdot b$ for all $a, b \in G$.

Then $(H, +, \otimes)$ is a hyperring.

Definition 2.5. Let $(R, +, \cdot)$ be a hyperring and $S \subset R$. Then S is said to be a *subhyperring* of R if $(S, +, \cdot)$ is itself a hyperring.

Definition 2.6. A subhyperring I of a hyperring R is a *left (right) hyperideal* of R if $rx \in I (xr \in I)$ for all $r \in R$, $x \in I$. I is called a *hyperideal* if I is both a left and a right hyperideal.

Definition 2.7. The direct sum of a family of subhypergroups $\{R_i | i \in I\}$ of R, denoted by $R = \bigoplus_{i \in I} R_i$, is the set

$\bigoplus_{i \in I} R_i = \{x \mid x \in \sum_{i \in I} x_i;\ x_i \neq 0 \text{ for at most finitely many } i \in I\}$

Definition 2.8. Let $(R, +, \cdot)$ be a hyperring. We say that R is the direct sum of subhypergroups $\{R_i\}_{i \in I}$ of R and we denote $R = \bigoplus_{i \in I} R_i$, if

(i) for every $i, j \in I$; $R_i \bigcap \sum_{i \neq j} R_j = \{0\}$,

(ii) for any $x \in R$, there exist unique elements $x_i \in R_i$ such that $x \in \sum_{i \in I} x_i$.

Definition 2.9. Let $(M, +)$ be a canonical hypergroup and $(R, +, \cdot)$ be a Krasner hyperring with identity. M is a left hypermodule over hyperring R if there exists a map

$$\cdot : R \times M \longrightarrow P^*(M); \quad (a, m) \mapsto a \cdot m$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2, m \in M$, the following are satisfied:

(1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_2 \cdot m_2$;

(2) $(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;

(3) $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$;

(3) $1m = m$ and $0m = 0$.

Definition 2.10. A nonempty subset N of an R-hypermodule M is called a *subhypermodule* if N is an R-hypermodule with the operations of M.

Definition 2.11. [17] Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows: $x\ \gamma\ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \cdots, k_n) \in \mathbb{N}^n$ and $\exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, (i = 1, \cdots, n)$ such that

$$x, y \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}).$$

Let γ^* be the transitive closure of γ and it is called the fundamental relation on R. The fundamental relation γ^* on R can be considered as the smallest equivalence relation such that the quotient R/γ^* be a ring. This quotient ring is called fundamental ring of R.

3. Graded hyperrings

In this section, we introduce and study graded hyperrings as a generalization of graded rings. In particular, we study graded (prime, primary, maximal) hyperideals and graded hyperfields and give some of their basic properties. In this section all hyperrings are Krasner commutative hyperring (with identity 1).

Definition 3.1. Let G be a monoid with identity e. A hyperring (R, G) is called a G-graded hyperring, if there exists a family $\{R_g\}_{g \in G}$ of canonical subhypergroups of R indexted by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For simplicity, we will denote the graded hyperring (R, G) by R. An element of a graded hyperring R is called homogeneous if it belongs to $\bigcup_{g \in G} R_g$ and this set of homogeneous elements is denoted by $h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g, and it is denoted by x_g.

Every $x \in R$ can be uniquely written as $x \in \sum_{g \in G} x_g$ with $x_g \in R_g$ such that all except finitely many x_g's are 0.

In fact, every hyperring is trivially a G-graded hyperring by letting $R_e = R$ and $R_g = 0$ for all $g \neq e$.

Lemma 3.2. *If $R = \bigoplus_{g \in G} R_g$ is a graded hyperring, then R_e is a subhyperring of R where e is the identity element of monoid G.*

Proof. We Know that $(R_e, +)$ is a canonical subhypergroup of R. Let $x_e, y_e \in R_e$. Then $x_e \cdot y_e \in R_e R_e \subseteq R_{e \cdot e} = R_e$, hence $x_e \cdot y_e \in R_e$. It is clear that for any $x_e, y_e, z_e \in R_e$, $x_e \cdot (y_e + z_e) = x_e \cdot y_e + x_e \cdot z_e$ and $(x_e + y_e) \cdot z_e = x_e \cdot z_e + y_e \cdot z_e$. Therefore, R_e is a subhyperring of R. □

Example 3.3. In Definition 3.1, let $G = (\mathbb{Z}_2, \cdot)$ be the monoid with identity $e = 1$ (by multiplication operation) and $R = \{0, a, b, c\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	{0, b}	{a, c}	{b}
b	{b}	{a, c}	{0, b}	{a}
c	{c}	{b}	{a}	{0}

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Letting $R_0 = \{0, b\}$ and $R_1 = \{0, c\}$. Then it is easy to verify that R_0 and R_1 are canonical hypergroups of $(R, +)$ and we can write $0 \in 0 + 0$, $a \in b + c$, $b \in b + 0$ and $c \in 0 + c$ uniquely, hence $R = R_0 \oplus R_1$. Also, $R_i R_j \subseteq R_{ij}$ for any $i, j \in \mathbb{Z}_2$ and so (R, G) is a graded hyperring.

Example 3.4. In Definition 3.1, let $G = (\mathbb{Z}_2, \cdot)$ be the monoid with identity $e = 1$ and $R = \{0, 1, 2\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{1}	{0, 1, 2}
2	{2}	{0, 1, 2}	{2}

.	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

It is easy to see that $H_0 = \{0, 1\}$ and $H_1 = \{0, 2\}$ are the only nontrivial subhypergroups of $(R, +)$. We have $0 \in 0 + 0$ and $0 \in 1 + 2$, hence 0 is not uniquely written as elements of H_0 and H_1. Therefore, $R \neq H_0 \oplus H_1$ and so (R, G) is not a graded hyperring.

Example 3.5. In Definition 2.1, let $R = \{0, a, b, c, d\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:

+	0	a	b	c	d
0	{0}	{a}	{b}	{c}	{d}
a	{a}	{0}	{c}	{b, d}	{c}
b	{b}	{c}	{0}	{a}	{0}
c	{c}	{b, d}	{a}	{0}	{a}
d	{d}	{c}	{0}	{a}	{0}

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	b	0	b	0
c	0	c	b	a	d
d	0	d	0	d	0

It is easy to see that $R_0 = \{0, a\}$ and $R_1 = \{0, c\}$ are subhypergroups of $(R, +)$. We have $0 \in 0 + 0$, $a \in a + 0$, $b \in b + c$, $c \in 0 + c$ and $d \in a + c$. Hence, $R = R_0 \oplus R_1$, but $R_1 R_1 \nsubseteq R_1$ since $c \cdot c = a \notin R_1$. Thus R is not a $\mathbb{Z}_2$-graded hyperring. In the other hand, we notice that $R_0 = \{0, a\}$, $R_1 = \{0, c\}$ and $R_2 = \{0, b\}$ are subhypergroups of $(R, +)$ but we have $a \in a + 0 + 0$ and $a \in 0 + c + b$, then $R \neq R_0 \oplus R_1 \oplus R_2$. So R is not a $\mathbb{Z}_3$-graded hyperring.

Example 3.6. In Definition 3.1, let $G = (\mathbb{Z}_2, \cdot)$ be the monoid with identity $e = 1$ and $R = \{0, 1, 2, 3\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0, 1}	{3}	{2, 3}
2	{2}	{3 }	{0 }	{1}
3	{3}	{2, 3}	{1}	{0, 1}

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

Clearly, $R_0 = \{0, 1\}$ and $R_1 = \{0, 2\}$ are subhypergroups of $(R, +)$. We have $0 \in 0 + 0$, $1 \in 1 + 0$, $2 \in 0 + 2$ and $3 \in 1 + 2$. Hence, $R = R_0 \oplus R_1$ and so (R, G) is a graded hyperring. Since $0\gamma 1$ and $2\gamma 3$ therefore $\gamma(0) = \gamma(1)$ and $\gamma(2) = \gamma(3)$. So The quotient R/γ^* is trivially a graded ring by letting $R_0 = R/\gamma^*$ and $R_1 = \gamma^*(0)$.

Open problem 3.7. For a given garded hyperring R, is its fundamenta ring R/γ^* a graded ring?

Definition 3.8. Let $R = \bigoplus_{g \in G} R_g$ be a graded hyperring. A subhyperring S of R is called a graded subhyperring of R, if $S = \bigoplus_{g \in G}(S \cap R_g)$. Equivalently, S is graded if for every element $f \in S$, all the homogeneous components of f (as an element of R) are in S.

Definition 3.9. Let I be a hyperideal of a graded hyperring R. Then I is a graded hyperideal, if $I = \bigoplus_{g \in G}(I \cap R_g)$. For any $a \in I$ and for some $r_g \in h(R)$ that $a \in \sum_{g \in G} r_g$, then $r_g \in I \cap R_g$ for all $g \in G$.

Lemma 3.10. *Let X be a nonempty subset of a commutative graded hyperring R. Let $\{A_i\}_{i \in I}$, be the family of all graded hyperideals in R which contains X. Then $\bigcap_{i \in I} A_i$ is also a graded hyperideal which contains X.*

Proof. By Lemma 2.9 in [18], $\bigcap_{i \in I} A_i$ is a hyperideal of R containing X. Now we show the grading. Let $r \in \cap_{i \in I} A_i$, hence $r \in \sum_{g \in G} r_g$ where $r_g \in R_g$. So we have $r \in A_i$ for all $i \in I$. Therefore, for any $g \in G$, $r_g \in A_i$ since A_i is a graded hyperideal. Hence for any $g \in G$, $r_g \in (\bigcap_{i \in I} A_i) \bigcap R_g$, and so $\bigcap_{i \in I} A_i$ is a graded hyperideal. □

The graded hyperideal $\bigcap_{i \in I} A_i$ is called graded hyperideal generated by X and is denoted by $< X >$. If $X = \{x_{g_1}, ..., x_{g_n}\}$, then $< X >$ is said to be finitely generated by $< x_{g_1}, ..., x_{g_n} >=$

$\{t \mid t \in \sum_{i=1}^{i=n} r_i x_{g_i}; r_i \in R\}$. A graded hyperideal generated by a single homogeneouse element $x_g \in h(R)$ is called principal and denoted by $< x_g >$.

Let $R = \bigoplus_{g\in G} R_g$ be a graded hyperring and I be a graded hyperideal of R. Then the quotient hyperring $(R/I, \oplus, \circ)$ where $(a+I)\circ(b+I) = ab+I$, for any $a, b \in R$ and $(a+I)\oplus(b+I) = \{t+I | t \in a+b\}$, for any $a, b \in R$ is also a graded hyperring with $R/I = \bigoplus_{g\in G}(R/I)_g$, where $(R/I)_g = (R_g + I)/I$.

Definition 3.11. If $P \neq R$ be a graded hyperideal of a graded hyperring R. Then P is called a graded prime hyperideal of R, if $a_g b_h \in P$, then $a_g \in P$ or $b_h \in P$ for $a_g, b_h \in h(R)$.

Definition 3.12. A graded hyperring $R = \bigoplus_{g\in G} R_g$ is a graded hyperintegral domain, if $a_g b_h = 0$, $a_g = 0$ or $b_h = 0$ for $a_g, b_h \in h(R)$.

Theorem 3.13. *Let $P \neq R$ be a graded hyperideal of a commutative graded hyperring R with identity 1. Then P is graded prime if and only if R/P is a graded hyperintegral domain.*

Proof. Let P be a graded prime hyperideal of R. Let $(a_g + P)\circ(b_h + P) = 0 + P = P$, so $a_g b_h + P = P$, then $a_g b_h \in P$ because $a_g b_h + P = \bigcup\{a_g b_h + x | x \in P\}$ and for any $t \in a_g b_h + P$, there exists $y \in P$ such that $t \in a_g b_h + y$, so $a_g b_h \in t - y \subseteq P$ since $t, y \in P$ and P is graded hyperideal. Hence $a_g \in P$ or $b_h \in P$ since P is a graded prime hyperideal, therefore $a_g + P = P$ or $a_h + P = P$, as needed. Conversely, let $a_g b_h \in P$ for some $a_g, b_h \in h(R)$. Hence $a_g b_h + P = (a_g + P)\circ(b_h + P) = P$, so $a_g + P = P$ or $b_h + P = P$ since R/P is a graded hyperintegral domain. Therefore, $a_g \in P$ or $b_h \in P$. □

Definition 3.14. Let R be a graded hyperring. The graded hyperideal M of R is said to be maximal, if for every graded hyperideal J of R; $M \subseteq J \subseteq R$, implies that $J = M$ or $J = R$.

Theorem 3.15. *Let R be a graded hyperring with identity 1. Then every graded maximal hyperideal is graded prime.*

Proof. Let M be a graded maximal hyperideal of R. Let $a_g b_h \in M$ and $a_g \notin M$ for $a_g, b_h \in h(R)$. So $M \subset < a_g > +M \subseteq R$, since M is a graded maximal hyperideal, then $R = < a_g > +M$. As $1 \in R$, we give $1 \in r a_g + x$ for some $r \in R$ and $x \in M$. Hence $b_h \in r b_h a_g + b_h x \subseteq M$ since M is a graded hyperideal, therefore $b_h \in M$, as needed. □

Definition 3.16. Let R be a commutative graded hyperring with identity 1. R is called a graded hyperfield, if whose non-zero homogeneous elements are invertible.

Theorem 3.17. *If R is a graded hyperring with identity, then R has a graded maximal hyperideal.*

Proof. The proof is similar to nongraded hyperrings. □

Theorem 3.18. *Let R be a commutative graded hyperring with identity and $M \neq R$ be a graded maximal hyperideal of R. Then M is graded maximal hyperideal if and only if R/M is a graded hyperfield.*

Proof. Let $M \neq a_g + M$ be a homogeneous elemet of R/M so $a_g \notin M$. Thus $M \subset < a_g > +M \subseteq R$, therefore $< a_g > +M = R$ since M is a graded maximal hyperideal. Hence $1 \in ra_g + x$ for some $r \in R$ and $x \in M$. So $1+M = ra_g+x+M$, hence $1+M = ra_g+M = (r+M)\circ(a_g+M)$, and so $a_g + M$ is unite. conversely, let $M \subset L \subseteq R$ where L is a graded hyperideal of R. So there exists $x \in L$ such that $x \notin M$. Hence we can write $x \in \sum_{g\in G} r_g$ where $r_g \in L \cap R_g$. Therefore there exists $g \in G$ such that $r_g \notin M$; because if for all $g \in G$, $r_g \in M$, then $\sum_{g\in G} r_g \subseteq M$ since M is a graded hyperideal, and so $x \in M$, which is a contradiction. Hence $r_g + M \neq 0_{R/M}$, then $(r_g + M) \circ (x + M) = r_g x + M = 1 + M$ for some $x + M \in R/M$. So $1 - r_g x \subseteq M \subseteq L$ and since $r_g x \in L$, we have $1 \in L$ since L is a graded hyperideal, then $L = R$, as needed. □

Definition 3.19. A nonempty subset S of $h(R)$ of a graded hyperring R is called multiplicative closed subset if $s_1 s_2 \in S$ for all $s_1, s_2 \in S$.

Let R be a graded hyperring and $S \subseteq h(R)$ be a multiplicative close subset of R. Then the hyperring of fractions $S^{-1}R$ is a graded hyperring which is called the graded hyperring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s | r \in R, s \in S; g = (degs)^{-1}(degr)\}$.

Theorem 3.20. *A graded hyperideal $P \neq R$ in a commutative graded hyperring R with identity is a graded prime hyperideal if and only if $h(R) - P$ is a multiplicative close subset in R.*

Proof. Let P be a graded prime hyperideal of R. Assume that $x_g, y_h \in h(R) - P$, so $x_g y_h \notin P$ since P is graded prime. Therefore, $x_g y_h \in h(R) - P$, and so $h(R) - P$ is a multiplicative close subset of R. Conversely, $x_g y_h \in P$ and $x_g \notin P$ where $x_g, y_h \in h(R)$. So $y_h \in P$, because if $y_h \notin P$, then $x_g y_h \in h(R) - P$, a contradiction. Hence P is graded prime. □

Definition 3.21. Let I be a graded hyperideal in a commutative graded hyperring R with identity. The graded radical of I (in abbreviation, $Grad(I)$) is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of R, then $r \in Grad(I)$ iff $r^n \in I$ for some positive integer n.

Definition 3.22. A graded hyperideal $Q \neq R$ in a commutative graded hyperring R is said to be graded primary, if $a_g b_h \in Q$, then $a_g \in Q$ or $b_h \in Grad(Q)$ for $a_g, b_h \in h(R)$.

Theorem 3.23. *If Q is a graded primary hyperideal in a commutative graded hyperring with identity, then $Grad(Q)$ is graded prime.*

Proof. Let $a_a b_h \in Grad(Q)$ and $a_g \notin Grad(Q)$. So there exists a positive integer n such that $(a_g b_h)^n \in Q$ and $a_g^n \notin Q$. Hence $b_h^n \in Q$, and so $b_h \in Grad(Q)$, as required. □

Definition 3.24. Let $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ be graded hyperrings. A mapping φ from R into S is said to be a graded good homomorphism, if for all $a, b \in R$;

(1) $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(0) = 0$
(2) $\varphi(ab) = \varphi(a)\varphi(b)$, and
(3) For any $g \in G$; $\varphi(R_g) \subseteq S_g$.

Definition 3.25. A graded good homomorphism $\varphi : R \to S$ is a graded isomorphism, if φ is one to one and onto and we write $R \cong S$.

4. Graded hypermodules

In this section, we introduce and study graded hypermodules over a commutative graded Krasner hyperring R with identity 1. Also, state and investigate graded prime subhypermodules of a graded hypermodule.

Definition 4.1. Let M be an R-hypermodule. We say that M is a graded R- hypermodule (or has an grading) if there exists a family of canonical subhypergroups $\{M_g\}_{g \in G}$ of M such that

(1) $M = \bigoplus_{g \in G} M_g$;
(2) $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$.

The set of all homogeneous elements of M is denoted by $h(M)$, and so $h(M) = \bigcup_{g \in G} M_g$.

Definition 4.2. A nonempty subset N of a graded R-hypermodule M is called a graded subhypermodule, if N is a graded R-hypermodule with the operations of M restricted to N.

Lemma 4.3. *Let N be a graded subhypermodule of graded hypermodule M. Then the quotient hypermodule M/N is a graded R- hypermodue with $M/N = \bigoplus_{g \in G} (M/N)_{g \in G}$ such that $(M/N)_{g \in G} = (M_g + N)/N$, which is called the quotient graded hypermodule of M by N.*

Example 4.4. Let $(R, +, \cdot)$ be the graded hyperring in Example 3.3. Set $M = R$ and $\oplus = +$, then $(M, \oplus)$ is an R-hypermodule with the following hyperoperation

$$\forall (r, m) \in R \times M;\ r \circ m = \{rm\}$$

We know that $M_0 = \{0, b\}$ and $M_1 = \{0, c\}$ are subhypergroups of $(M, \oplus)$ and $M = M_0 \bigoplus M_1$. Moreover, $R_0 M_0 \subseteq M_0$, $R_0 M_1 \subseteq M_0$, $R_1 M_0 \subseteq M_0$, $R_1 M_1 \subseteq M_1$. Hence M is a graded R-hypermodule.

Example 4.5. Let $(R, +, \cdot)$ be the graded hyperring in Example 3.6. Set $M = R$ and $\oplus = +$, then $(M, \oplus)$ is an R-hypermodule with the following hyperoperation

$$\forall (r, m) \in R \times M;\ r \circ m = r \oplus m.$$

We know that $M_0 = \{0, 1\}$ and $M_1 = \{0, 2\}$ are subhypergroups of $(M, \oplus)$ and $M = M_0 \bigoplus M_1$ because $0 \in 0 \oplus 0$, $1 \in 1 \oplus 0$, $2 \in 0 \oplus 2$ and $3 \in 1 \oplus 2$. But $R_0 M_1 \nsubseteq M_0$, since $1 \in R_0$ and $2 \in M_1$, but $1 \circ 2 = 1 \oplus 2 = \{3\} \nsubseteq M_0$. Hence, the condition (2) in Definition 4.1 is not hold. So, M is not a graded R-hypermodule.

From now on, consider the graded R-hypermodule M with operation $. : R \times M \to M$, denoted by $(r, m) \mapsto rm \in M$.

Definition 4.6. Let $M = \bigoplus_{g \in G} M_g$ and $N = \bigoplus_{g \in G} N_g$ be two graded R- hypermodules. A mapping φ from M into N is said to be a graded good homomorphism, if for all $m, n \in M$;

(1) $\varphi(m + n) = \varphi(m) + \varphi(n)$, $\varphi(0) = 0$
(2) $\varphi(rm) = r\varphi(m)$, for any $r \in R$ and $m \in M$.
(3) For any $g \in G$; $\varphi(M_g) \subseteq N_g$.

Definition 4.7. A graded R-hypermodule M is called graded finitely generated, if $M = Rm_{g_1} + ... + Rm_{g_n}$ where, for $i = 1, 2, ..., n$, $m_{g_i} \in h(M)$.

Definition 4.8. Let M be a graded R-hypermodule. A proper graded subhypermodule N of M is said to be graded maximal, provided that for graded subhypermodule K of M with $N \subseteq K \subseteq M$, then $N = K$ or $K = M$.

Theorem 4.9. *Let M be a graded finitely generated R-hypermodule. Then every proper graded subhypermodule of M is contained in a graded maximal subhypermodule.*

Proof. Let $M = Rm_{g_1} + ... + Rm_{g_n}$ for some $m_{g_i} \in h(M)$ and N a proper graded subhypermodule of M. Let S denote the set of all proper graded subhypermodules C of M such that $N \subseteq C \subseteq M$. Partially order S by the set theoric inclusion. Let $\{C_i\}_{i \in I}$ be a chain in S and $C = \cup_{i \in I} C_i$. Then clearly, $N \subseteq C$. Now we show that $C \subset M$. To this end it suffices to show that for some k; $1 \leq k \leq n$, we have $m_{g_k} \notin C$. On the contrary, let $m_{g_j} \in C$ for any $1 \leq j \leq n$. Then there exists $i \in I$ such that $m_{g_j} \in C_i$ for all $j = 1, 2, ..., n$. Thus $M = Rm_{g_1} + ... + Rm_{g_n} \subseteq C_i$, which is a contradiction, since $C_i \in S$. Hence $m_{g_k} \notin C$ for some $1 \leq k \leq n$, and so $C \in S$. Clearly, C is an upper bound for the chain $\{C_i\}_{i \in I}$ by Zorn Lemma, which clearly is a graded maximal subhypermodule that contains N. □

Definition 4.10. Let M be a graded R-hypermodule. The graded Jacobson radical of M, denoted by $GJ(M)$, is the intersection of all graded maximal subhypermodules of M. If no graded maximal subhypermodule exists, then we set $GJ(M) = M$.

Theorem 4.11. *Let M be a graded finitely generated R-hypermodule. Then $GJ(M) = M$ if and only if $M = \{0\}$.*

Proof. The proof is trivial by the fact that every graded subhypermodule of M is contained in a graded maximal subhypermodule. □

Definition 4.12. A homogeneous element u_g of a graded R-hypermodule M is said to be unit if u_g is not contained in any graded maximal subhypermodule of M.

Theorem 4.13. *Let M be a graded finitely generated R-hypermodule. Then $u_g \in h(M)$ is unit if and only if $Ru_g = M$.*

Proof. Let $Ru_g \neq M$. Then there exists a graded maximal subhypermodule N such that $Ru_g \subseteq N$, so $u_g \in N$, a contradiction. Conversely, let $u_g \in N$ for some graded maximal subhypermodule N, then $Ru_g \subseteq N$, and hence $N = M$, a contradiction. □

Theorem 4.14. *Let M and N be two graded R-hypermodules, where M is graded finitely generated and $\varphi : M \to N$ is a graded good epimorphism. If $u_g \in h(M)$ is unit, then $\varphi(n_g)$ is also a unit in N.*

Proof. Let u_g be a unit in M. Then $Ru_g = M$. Let $n \in N$, and so $n = \varphi(m)$ for some $m \in M$. Therefore, $m = ru_g$ for some $r \in R$. Hence $n = \varphi(m) = r\varphi(u_g)$, so $N =< \varphi(u_g) >$. Thus $\varphi(u_g)$ is unit in N. □

Definition 4.15. Let N be a graded subhypermodule of a graded R-hypermodule M. The subset $\{r \in R \mid rM \subseteq N\}$ is denoted by $(N : M)$. It is clear that $(N : M)$ is a graded hyperideal of R.

Definition 4.16. A proper graded subhypermodule P of a graded R-hypermodule M is called graded prime (primary) whenever $r_g m_h \in P$ with $r_g \in h(R)$ and $m_h \in h(M)$, implies that $m_h \in N$ or $r_g M \subseteq N$ ($m_g \in N$ or $r_g^n M \subseteq N$ for some positive integer n).

Theorem 4.17. *every graded maximal subhypermodule is graded prime.*

Proof. Let P be a graded maximal subhypermodule of a graded R-hypermodule M. Let $r_g m_h \in P$ for some $r_g \in h(R)$ and $m_h \in h(M) - P$. Hence $P \subseteq Rm_h + P \subset M$, and so $Rm_h + P = M$ since P is graded maximal. Thus for any $m \in M$, there exist $s \in R$ and $p \in P$ such that $m \in sm_h + p$. So $r_g m \in r_g s m_h + r_g p \subset P$. Therefore $r_g M \subseteq P$, and so P is graded prime. □

It is clear that every graded prime is graded primary.

Theorem 4.18. *Let M and N be two graded R-hypermodules and P, Q be graded (prime, primary) subhypermodules of M, N respectively. If $f : M \to N$ is a graded R-hypermodule homomorphism, then*

(i) if f is onto and $Ker(f) \subseteq P$, then $f(P)$ is a graded (prime, primary) subhypermodule of N.

(ii) $f^{-1}(Q)$ is a graded (prime, primary) subhypermodule of M.

(iii) if f is onto, then there is a bijective between, the set of all graded (prime, primary) subhypermodules of M of the set of all graded (prime, primary) subhypermodules of N.

Proof. (i) Let P be a graded prime subhypermodule of M. Assume that $r_g n_h \in f(P)$ and $n_h \notin f(P)$ for some $r_g \in h(R)$ and $n_h \in h(N)$. Hence $n_h = f(m_h)$ for some $m_h \in h(M)$, because f is onto. Thus $r_g f(m_h) = f(r_g m_h) \in f(P)$ and so $f(r_g m_h) = f(x)$ for some $x \in P$. Hence $f(r_g m_h - x) = 0$, so $r_g m_h - x \subseteq Kerf \subseteq P$ and $x \in P$. Therefore $r_g m_h \in P$ since P is graded subhypermodule, hence $r_g M \subseteq P$ because P is graded prime. Hence $r_g N = r_g f(M) = f(r_g M) \subseteq f(P)$, and so $f(P)$ is a graded prime subhypermodule of N. Now it is clear that, if P is a graded primary subhypermodule of M, then $f(P)$ is a graded primary subhypermodule of N.

(ii) Let Q be a graded prime subhypermodule of N. Suppose that $r_g m_h \in f^{-1}(Q)$ and $m_h \notin f^{-1}(Q)$ where $r_g \in h(R)$ and $m_h \in h(M)$. Hence $f(r_g m_h) = r_g f(m_h) \in Q$ and $f(m_h) \notin Q$. Thus $r_g N \subseteq Q$ since Q is a graded prime subhypermodule of N. Therefore $r_g M \subseteq r_g f^{-1}(N) \subseteq f^{-1}(Q)$, so $f^{-1}(Q)$ is a graded prime subhypermodule of M. Also, it is clear that, if Q is a graded primary subhypermodule of N, then $f^{-1}(Q)$ is a graded primary subhypermodule of M.

(iii) It is hold by (i) and (ii). □

Theorem 4.19. *Let M be a graded R-hypermodule and N be a graded subhypermodule of M.*
(i) if N is a graded prime subhypermodule of M, then $(N : M)$ is a graded prime hyperideal of R.
(ii) if N is a graded primary subhypermodule of M, then $(N : M)$ is a graded primary hyperideal of R.

Proof. (i) Let N be graded prime and $r_g s_h \in (N : M)$ and $s_h \notin (N : M)$ for some $r_g, s_h \in h(R)$. Thus $s_h m \notin N$ for some $m \in M$. Since $m \in \sum_{g\in G} m_g$ with $m_g \in h(M)$, so there exists $m_{g'} \in h(M)$ such that $s_h m_{g'} \notin N$ because if for any $g \in G$, $s_h m_g \in N$, then $s_h m \in s_h(\sum_{g\in G} m_g) \subseteq N$, a contradiction. As $r_g(s_h m_{g'}) \in N$, then $r_g \in (N : M)$ since N is a graded prime subhypermodule of M. Consequently, $(N : M)$ is graded prime.
(ii) It is similar to (i). □

Definition 4.20. Let M be a graded R-hypermodule. Then the set $\{r \in R \,|\, rM = 0\}$ is said to be annihilator of M and denoted by $ann(M)$. It is clear that $ann(M)$ is a graded hyperideal of R.

Proposition 4.21. *Let $\{N_i\}_{i\in I}$ be a collection of graded subhypermodules of a graded R-hypermodule M. Then $(\cap_{i\in I} N : M) = \bigcap_{i\in I}(N_i : M)$.*

Proof. It is strightforward. □

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Farkhondeh Farzalipour

Department of mathematics

Payame Noor University (PNU)

P.O.BOX 19395-3697 Tehran, Iran.

f_farzalipour@pnu.ac.ir

Peyman Ghiasvand

Department of mathematics

Payame Noor University (PNU)

P.O.BOX 19395-3697 Tehran, Iran.

p_ghiasvand@pnu.ac.ir