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Research Paper

THE SECONDARY RADICALS OF SUBMODULES

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Abstract. Let R be a commutative ring with identity and let M be an R-module. In this paper, we will introduce the *secondary radical* of a submodule N of M as the sum of all secondary submodules of M contained in N, denoted by $sec^*(N)$, and explore the related properties. We will show that this class of modules contains the family of second radicals properly and can be regarded as a dual of primary radicals of submodules of M.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$, see [13]. Let N be a submodule of M. The intersection of all prime submodules of M containing N is said to be the (*prime*) *radical* of N and denote by $rad_M N$ (or simply by $rad(N)$). In case N does not contained in

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any prime submodule, the radical of N is defined to be M, see [20]. A proper submodule Q of M is said to be *primary*, if $rm \in Q$, where $r \in R$, $m \in M$, then $m \in Q$ or $r^n M \subseteq Q$ for some $n \in \mathbb{N}$, see [21]. The *primary radical* of a submodule N of M, denoted by $prad_M(N)$ is defined as the intersection of all primary submodules of M which contain N. If there exists no primary submodule of M containing N, then $prad_M(N) = M$, see [14].

A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism of M given by multiplication by a is either surjective or zero, see [23]. For a submodule N of M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0). $N \neq 0$ is said to be a *second radical submodule of M* if $sec(N) = N$, see [4] and [12]. In [18], I.G. Macdonald introduced the notion of secondary modules. A non-zero R-module M is said to be *secondary* if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent, see [18].

The main purpose of this paper is to introduce and study the notion of secondary radicals of submodules and provide some useful information concerning this notion. We show that this class of modules contains the family of second radicals properly and can be regarded as a dual of primary radicals of submodules of M.

In the following, we recall some definitions which are needed in the sequel.

Definition 1.1. (a) An R-module M is said to be *finitely cogenerated* if for every set $\{M_\lambda\}_{\lambda \in \Lambda}$ of submodules of M, $\cap_{\lambda \in \Lambda} M_\lambda = 0$ implies $\cap_{i=1}^{n} M_{\lambda_i} = 0$ for some positive integer n, see [1].

(b) Let P be a prime ideal of R and let N be a submodule of an R-module M. *The P-interior of N relative to M* is as the set, see [2],

$$I_P^M(N) = \bigcap \{L \mid L \; is \; a \; completely \; irreducible \; submodule \; of \; M \; and$$

$$rN \subseteq L \; for \; some \; r \in R - P\}.$$

(c) Let R be an integral domain. An R-module M is said to be *cotorsion-free* if $I_0^M(M) = M$ and is *cotorsion* if $I_0^M(M) = 0$, see [3].

(d) An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$, see [5].

(e) Let M be an R-module. A submodule N of M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R, see [7].

(f) M is said to be *fully copure* if every submodule of M is copure, see [8].

(g) A submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M. Thus the intersection of all completely irreducible submodules of M is zero, see [15].

(h) An R-module M is said to be *atomic* if every nonzero submodule of M contains a minimal submodule, see [16].

(i) A family $\{N_i\}_{i\in I}$ of submodules of M is said to be an *inverse family of submodules of M* if the intersection of two of its submodules again contains a module in $\{N_i\}_{i\in I}$. Also M *satisfies the property $AB5^*$* if for every submodule K of M and every inverse family $\{N_i\}_{i\in I}$ of submodules of M, $K + \cap_{i\in I} N_i = \cap_{i\in I}(K + N_i)$, see [22].

We refer the reader to [19] and [1] for all concepts and basic properties of modules not defined here.

2. The secondary radicals of submodules

Definition 2.1. Let N be a submodule of an R-module M. We define the *secondary radical* of N as the sum of all secondary submodules of M contained in N and it is denoted by $sec^*(N)$. In case N does not contain any secondary submodule, the secondary radical of N is defined to be (0). Also, we say that $N \neq 0$ is a *secondary radical submodule of M* if $sec^*(N) = N$.

Example 2.2. Clearly every second radical submodule of an R-module M is a secondary radical submodule of M. But the converse is not true in general. Take the submodules $N_i =< 1/p^i + \mathbb{Z} >$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ for $i = 2, 3, ...$ Then each N_i is a secondary radical submodules of $\mathbb{Z}_{p^\infty}$ which it is not a second radical submodule.

Example 2.3. By [4, Proposition 2.1], for each submodule N of an R-module M, $sec(N) \subseteq (0 :_M \sqrt{Ann_R(N)})$. But if we consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ and set $N_2 =< 1/p^2 + \mathbb{Z} >$, then $sec^*(N_2) \not\subseteq (0 :_M \sqrt{Ann_R(N_2)})$.

Proposition 2.4. Let N and K be two submodules of an R-module M. Then we have the following.

(a) $sec(N) \subseteq sec^*(N) \subseteq N$.

(b) If $N \subseteq K$, then $sec^*(N) \subseteq sec^*(K)$.

(c) $sec^*(sec^*(N)) = sec^*(N)$.

(d) $sec^*(N) + sec^*(K) \subseteq sec^*(N + K)$.

(e) $sec^*(N \cap K) = sec^*(sec^*(N) \cap sec^*(K))$.

(f) If S is a P-secondary submodule of M such that $S \subseteq N + K$ and $Ann_R(N) \nsubseteq P$, then $S \subseteq K$.

(g) If $N + K = sec^*(N) + sec^*(K)$, then $sec^*(N + K) = N + K$.

Proof. The proofs are straightforward. □

Theorem 2.5. *Let M be a comultiplication R-module. Then we have the following.*

(a) *$sec^*(M) = 0$ if and only if $M = 0$.*

(b) *If N and K are two submodules of M, then $sec^*(N) \cap sec^*(K) = 0$ if and only if $N \cap K = 0$.*

(c) *If m a maximal ideal of R and Q is an m-secondary submodule of M, then $sec^*(Q)$ is an m-secondary submodule of M.*

(d) *If R is an integral domain with $dimR = 1$ and M is a secondary R-module, then $sec^*(M)$ is a secondary submodule of M.*

Proof. (a) By [6, Theorem 3.2], M contains a minimal submodule. Now the result follows from the fact that every minimal submodule is secondary.

(b) This follows from part (a) and Proposition 2.4 (e).

(c) We have $m = \sqrt{Ann_R(Q)} \subseteq \sqrt{Ann_R(sec^*(Q))}$ so that $\sqrt{Ann_R(sec^*(Q))} = R$ or $\sqrt{Ann_R(sec^*(Q))} = m$. If $\sqrt{Ann_R(sec^*(Q))} = R$, then $sec^*(Q) = 0$. Thus by part (a), $Q = 0$, a contradiction. Hence $\sqrt{Ann_R(sec^*(Q))} = m$. This implies that $Ann_R(sec^*(Q))$ is a primary ideal of R. Therefore, $sec^*(Q)$ is a m-secondary submodule of M by [9, Lemma 2.25].

(d) If $Ann_R(M) = 0$, then clearly M is a second R-module and hence $sec^*(M) = M$ is a secondary submodule of M. So assume that $Ann_R(M) \neq 0$. Then $\sqrt{Ann_R(M)}$ is a non-zero prime ideal of R. Since $dimR = 1$ and R is a domain, it follows that $\sqrt{Ann_R(M)}$ is a maximal ideal of R. Thus the result follows from part (c). □

Proposition 2.6. If V is a vector space, then $sec^*(N_1 + N_2) = sec^*(N_1) + sec^*(N_2)$ for every pair of subspaces N_1, N_2 of V.

Proof. This follows from the fact that every non-zero subspace of V is a (0)-secondary submodule. □

Proposition 2.7. Let N and K be two submodules of an R-module M such that whenever $S \subseteq N + K$, we have $S \subseteq N$ or $S \subseteq K$ for every secondary submodule S of M. Then

$$sec^*(N + K) = sec^*(N) + sec^*(K).$$

Proof. If $sec^*(N+K)=0$, then clearly, $sec^*(N)=sec^*(K)=0$ and so $sec^*(N+K)=sec^*(N)+sec^*(K)$. If $sec^*(N+K)\neq 0$, then there exists a secondary submodule S of M such that $S\subseteq N+K$. By hypothesis, $S\subseteq N$ or $S\subseteq K$. Hence $S\subseteq sec^*(N)$ or $S\subseteq sec^*(K)$. Since this is true for all secondary submodules S contained in $N+K$, $sec^*(N+K)\subseteq sec^*(N)+sec^*(K)$. The reverse inclusion is clear. □

We use the following basic fact without further comment.

Remark 2.8. Let N and K be two submodules of an R-module M. To prove $N\subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K\subseteq L$, then $N\subseteq L$.

Theorem 2.9. *Let M be an R-module. If N and K are two submodules of M such that $\sqrt{Ann_R(N)}$ and $\sqrt{Ann_R(K)}$ are comaximal, then $sec^*(N+K)=sec^*(N)+sec^*(K)$.*

Proof. Clearly, $sec^*(N)+sec^*(K)\subseteq sec^*(N+K)$. If S is a secondary submodule contained in $N+K$, then $Ann_R(N)\cap Ann_R(K)\subseteq\sqrt{Ann_R(S)}$. Thus $Ann_R(N)\subseteq\sqrt{Ann_R(S)}$ or $Ann_R(K)\subseteq\sqrt{Ann_R(S)}$ because $\sqrt{Ann_R(S)}$ is a prime ideal of R. We can assume that $Ann_R(N)\subseteq\sqrt{Ann_R(S)}$. Then $Ann_R(K)\not\subseteq\sqrt{Ann_R(S)}$, otherwise it contradicts comaximality. We show that $S\subseteq N$. Suppose that L is a completely irreducible submodule of M such that $N\subseteq L$ and let $r\in Ann_R(K)-\sqrt{Ann_R(S)}$. Since $K\subseteq(L:_M r)$ and $N\subseteq L\subseteq(L:_M r)$, $N+K\subseteq(L:_M r)$. Hence $rS\subseteq L$. As S is secondary, it follows that $S\subseteq L$. This implies that $S\subseteq N$. Thus $S\subseteq sec^*(N)\subseteq sec^*(N)+sec^*(K)$. Therefore, $sec^*(N+K)\subseteq sec^*(N)+sec^*(K)$. □

Corollary 2.10. Let $K_1,...,K_n$ be submodules of an R-module M such that $\sqrt{Ann_R(K_i)}$ are pairwise comaximal. Then $sec^*(K_1+...+K_n)=sec^*(K_1)+...+sec^*(K_n)$.

Definition 2.11. Let M be an R-module. We call the set of all secondary submodules of M the *secondary spectrum* of M and denote by $Spec^{s*}(M)$. The map $\varphi: Spec^{s*}(M)\to Spec(R/Ann_R(M))$ defined by $\varphi(S)=\sqrt{Ann_R(S)}/Ann_R(M)$ for every $S\in Spec^{s*}(M)$, is called the *natural map* of $Spec^{s*}(M)$.

Proposition 2.12. Let M be an R-module and let N be a submodule of M such that the natural map φ of $Spec^{s*}(N)$ is surjective. Then $\sqrt{Ann_R(sec^*(N))}=\sqrt{Ann_R(N)}$.

Proof. When $N=0$, the proposition is trivially true. So suppose that $N\neq 0$. Clearly, we have $\sqrt{Ann_R(N)}\subseteq\sqrt{Ann_R(sec^*(N))}$. Now let $\sqrt{Ann_R(N)}=\cap_i P_i$, where P_i runs through $V(\sqrt{Ann_R(N)})=\{P\in Spec(R)\mid \sqrt{Ann_R(N)}\subseteq P\}$. Since φ is surjective, for each $P_i\in$

$V(\sqrt{Ann_R(N)})$ there exists $S_i \in Spec^{s*}(N)$ such that $\sqrt{Ann_R(S_i)} = P_i$. Hence $\sum_i S_i \subseteq sec^*(N)$. Therefore,

$$\sqrt{Ann_R(sec^*(N))} \subseteq \sqrt{Ann_R(\sum_i S_i)} = \cap_i P_i = \sqrt{Ann_R(N)},$$

as required. □

Definition 2.13. The *secondary submodule dimension* of an R-module M, denoted by $dim^{s*}M$, is defined to be the supremum of the length of chains of secondary submodules of M if $Spec^{s*}(M) \neq \emptyset$ and -1 otherwise.

Theorem 2.14. *Let K be a field and let M be a K-vector space with $dim_K M = n$. Then $dim^{s*}M = n - 1$.*

Proof. Note that every non-zero submodule of M is secondary. Suppose that $dim^{s*}M = k$. Then there exists a chain

$$S_k \supset S_{k-1} \supset ... \supset S_1 \supset S_0$$

of secondary submodules. By using the fact that $dim_K S_i > dim_K S_{i-1}$ $(0 \leq i \leq k)$, we have $k < n$. Now let $\{e_1, e_2, ..., e_n\}$ be a basis for M. Thus we have the following chain of secondary submodules of M

$$< e_1, e_2, ..., e_n > \supset < e_1, e_2, ..., e_{n-1} > \supset ... \supset < e_1, e_2 > \supset < e_1 > .$$

Hence $k \geq n - 1$. Therefore, $dim^{s*}M = n - 1$. □

Theorem 2.15. *If R is a one dimensional domain and M is a finitely cogenerated cotorsion R-module with $dim^{s*}M = 1$, then the following are equivalent.*

(a) *M is a secondary module.*
(b) *$S_1 + S_2 = M$ for any distinct secondary submodules S_1 and S_2.*
(c) *Every proper submodule contains exactly one secondary submodule.*
(d) *Every proper secondary submodule is minimal.*

Proof. $(a) \Rightarrow (b)$. Since M is finitely cogenerated cotorsion, $Ann_R(M) \neq 0$. As M is a secondary module, $\sqrt{Ann_R(M)}$ is a prime ideal of R. Thus if S is any secondary submodule of M, then we have $\sqrt{Ann_R(M)} = \sqrt{Ann_R(S)}$ because $dimR = 1$. Therefore, for any distinct secondary submodules S_1 and S_2 of M, $\sqrt{Ann_R(S_1)} = \sqrt{Ann_R(S_2)} = \sqrt{Ann_R(M)}$ so that $S_1 + S_2$ is secondary. This implies that $S_1 + S_2 = M$ because $dim^{s*}M = 1$.

$(b) \Rightarrow (c)$. Since M is finitely cogenerated, by [1], every submodule of M contains a minimal submodule which is secondary. Now if N is a proper submodule of M and S_1 and S_2 are distinct secondary submodules of M with $S_1 \subseteq N$ and $S_2 \subseteq N$, then we have $M \subseteq N$ by part (b) which is a contradiction.

$(c) \Rightarrow (d)$. Let S be a proper secondary submodule of M. By [1], S contains a minimal submodule K because S is finitely cogenerated. But every minimal submodule is secondary. Therefore, $S = K$ by part (c).

$(d) \Rightarrow (a)$. Since $dim^{s*}M = 1$, there must exist a chain of secondary submodules $S_1 \supset S_2$. Now part (d), implies that S_1 must be M. □

Corollary 2.16. If R is a one dimensional domain and M is a secondary finitely cogenerated cotorsion R-module with $dim^{s*}M = 1$, then $sec^*(N)$ is secondary for every non-zero submodule N of M.

Proof. This follows from part $(a) \Rightarrow (c)$ of Theorem 2.15. □

Theorem 2.17. *Let R be an integral domain and let M be an R-module with $dim^{s*}M = 1$. If M is Artinian cotorsion-free, then $sec^*(N) = Soc_R(N)$ for any proper submodule N of M, where $Soc_R(N)$ is the sum of all minimal submodules of N.*

Proof. Since M is cotorsion-free, $I_0^M(M) = M \neq 0$ is a second submodule by [4] and so secondary submodule this together with the assumption that M is Artinian (and hence every non-zero submodule of M contains a minimal submodule) and $dim^{s*}M = 1$ implies that every proper secondary submodule of M is minimal. Hence $sec^*(N) \subseteq Soc_R(N)$. The reverse inclusion is clear. □

Definition 2.18. Let M be an R-module. We say that a non-zero submodule N of M is *secondary cocompactly packed* if for each family $\{S_\lambda\}_{\lambda \in \Lambda}$ of secondary submodules of M with $\cap_{\lambda \in \Lambda} S_\lambda \subseteq N$, we have $S_\lambda \subseteq N$ for some $\lambda \in \Lambda$. A module M is called *secondary cocompactly packed* if every submodule of M is secondary cocompactly packed.

Proposition 2.19. Let $f : M \to \acute{M}$ be an R-module monomorphism. Then we have the following.

(a) A submodule N of M is a secondary submodule of M if and only if $f(N)$ is a secondary submodule of $\acute{M}$.
(b) If $\acute{N}$ is a secondary submodule of $\acute{M}$ with $\acute{N} \subseteq Im(f)$, then $f^{-1}(\acute{N})$ is a secondary submodule of M.
(c) If $\acute{M}$ is secondary cocompactly packed, then M is so. The converse is true if $sec^*(\acute{M}) \subseteq Im(f)$.

Proof. (a) and (b) are straightforward.

(c) The proof is straightforward by using parts (a) and (b). □

Lemma 2.20. Let M be a secondary cocompactly packed R-module. Then for each non-zero submodule N of M there exists a completely irreducible submodule L of M such that $sec^*(N) = sec^*(L)$.

Proof. Let N be a non-zero submodule of M. Then there exists a family $\{L_\lambda\}_{\lambda\in\Lambda}$ of completely irreducible submodules of M such that $N = \cap_{\lambda\in\Lambda} L_\lambda$. Clearly, $sec^*(N) \subseteq sec^*(L_\lambda)$ for each $\lambda \in \Lambda$. Suppose that $sec^*(L_\lambda) \nsubseteq sec^*(N)$ for each $\lambda \in \Lambda$. Then far each $\lambda \in \Lambda$ there exists a secondary submodule S_λ such that $S_\lambda \nsubseteq N$ and $S_\lambda \subseteq L$. Now $\cap_{\lambda\in\Lambda} S_\lambda \subseteq N$. Thus M is not a secondary cocompactly packed module, which is a contradiction. □

Proposition 2.21. Let M be a fully copure R-module which satisfies the $AB5^*$ property. If M is a secondary cocompactly packed module, then each non-zero submodule of M is completely irreducible.

Proof. Let N be a proper submodule of M. Then by Lemma 2.20, there exists a completely irreducible submodule L of M with $N \subseteq L$ such that $sec^*(N) = sec^*(L)$. But $K = sec^*(K)$ for each non-zero submodule K of M by [2, Theorem 3.6]. Therefore $N = L$ as required. □

Example 2.22. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_6$. Then M is a fully copure $\mathbb{Z}$-module which satisfies the $AB5^*$ property. Also M is a secondary cocompactly packed module. Thus by Proposition 2.21, each non-zero submodule of M is completely irreducible.

Theorem 2.23. *Let M be an R-module. Then the following statements are equivalent.*

(a) *M is a secondary cocompactly packed module.*

(b) *For each non-zero submodule N of M, if $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of submodules of M and $\cap_{\lambda\in\Lambda} N_\lambda \subseteq N$, then $sec^*(N_\mu) \subseteq N$ for some $\mu \in \Lambda$.*

(c) *For each non-zero submodule N of M, if $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of secondary radicals submodules of M and $\cap_{\lambda\in\Lambda} N_\lambda \subseteq N$, then $N_\mu \subseteq N$ for some $\mu \in \Lambda$.*

Proof. $(a) \Rightarrow (b)$. Let N be a non-zero submodule of M and $sec^*(N_\mu) \nsubseteq N$ for each $\mu \in \Lambda$. Then for each $\mu \in \Lambda$ there exists a secondary submodule S_μ such that $S_\mu \subseteq N_\mu$ and $S_\mu \nsubseteq N$. Therefore, $\cap_{\mu\in\Lambda} S_\mu \subseteq N$. This implies that M is not a secondary cocompactly packed module, which is a contradiction.

$(b) \Rightarrow (c)$. This is clear.

$(c) \Rightarrow (a)$. This follows from the fact that each secondary submodule is a secondary radical submodule. $\square$

3. Descending chain condition on secondary radical submodules

Theorem 3.1. *If M is a secondary cocompactly packed module which has one minimal submodule, then M satisfies the DCC on secondary radicals submodules.*

Proof. Let $N_1 \supseteq N_2 \supseteq ...$ be a descending chain of secondary radical submodules of M and let $T = \cap_i N_i$. If $T = 0$ and K is a minimal submodule of M, then $T \subset K$. As M is a secondary cocompactly packed module, $N_j \subseteq K$ for some j by Theorem 2.23. Therefore, $N_j = K$. Since $N_j \supseteq N_{j+n} \supseteq T = 0$ for $n = 1, 2, 3, ...$ and N_j is a minimal submodule of M, we have $N_j = 0$ or $N_{j+n} = 0$ for all $n = 1, 2, 3, ...$ These contradictions show that $T \neq o$. Now by Theorem 2.23, $N_j \subseteq T$ for some j as desired. $\square$

Corollary 3.2. Let M be a secondary cocompactly packed R-module. If M is an atomic R-module (in particular, if M is a comultiplication or finitely cogenerated R-module), then M satisfies the DCC on secondary radical submodules.

Theorem 3.3. *Let M be an R-module which satisfies the DCC for secondary radical submodule. Then every secondary radical submodule is a secondary radical of a submodule K of M such that M/K is finitely cogenerated.*

Proof. Assume that there exists a secondary radical submodule N which is not secondary radical of a submodule K of M such that M/K is a finitely cogenerated module. Let L_1 be a completely irreducible submodule of M such that $N \subseteq L_1$ and $N_1 = sec^*(L_1)$. Then $N \subset N_1$ so that there exists a completely irreducible submodule L_2 of M such that $N \subseteq L_2$ but $N_2 \not\subseteq L_2$. Set $N_2 = sec^*(L_1 \cap L_2)$. Then $N \subset N_2 \subset N_1$. Thus there exists a completely irreducible submodule L_3 of M such that $N \subseteq L_3$ but $N_2 \not\subseteq L_3$ etc. This gives a descending chain of secondary radical submodules $N_1 \supset N_2 \supset N_3 \supset ...$, which is a contradiction. $\square$

Proposition 3.4. Let R be a Noetherian ring, P be a prime ideal of R, and M be an R-module which has a finitely generated P-secondary submodule. Then $I^M_P(0 :_M P^n) = (0 :_M P^n)$ for some positive integer n.

Proof. Let N be a finitely generated P-secondary submodule of M. Then there exists a positive integer n such that $P^n N = 0$ since R is Noetherian. Let L be a completely irreducible submodule of M such that $(L :_R (0 :_M P^n)) \cap (R \setminus P) \neq \emptyset$. Since N is P-secondary, $(L :_R$

$(0:_M P^n))N = N$. Hence, $Ann_R(N) + (L:_R (0:_M P^n)) = R$ by using [17, Theorem 76]. Thus, $(0:_M P^n) \subseteq L$. It follows that $I_P^M(0:_M P^n) = (0:_M P^n)$. □

Theorem 3.5. *Let R be a Noetherian ring and M be an Artinian R-module. Then*

$$sec^*(M) = \sum\{I_P^M((0:_M P^n)) : n \in \mathbb{N} \textit{ and } P \textit{ is a prime ideal of } R\}.$$

Proof. Let $sec^*(M) = \sum S$, where S is a secondary submodules of M. Set $P = \sqrt{Ann_R(S)}$. Then there exists a positive integer t such that $P^t S = 0$. Hence $S \subseteq (0:_M Ann_R(S)) \subseteq (0:_M P^t)$. By [10, Lemma 2.3], there exists $r \in R \setminus P$ such that $r(0:_M P^t) \subseteq I_P^M((0:_M P^t))$. Hence $rS \subseteq I_P^M((0:_M P^t))$. Since S is P-secondary, $rS = S$. Therefore, $S \subseteq I_P^M((0:_M P^t))$. Hence, $sec^*(M) \subseteq \sum I_P^M((0:_M P^t))$. Now the claim follows from the fact that $I_P^M((0:_M P^t))$ is a secondary submodule of M by [11, Corollary 2.4]. □

Theorem 3.6. *Let M be an R-module which satisfies the descending chain condition for secondary radical of its submodules. Then every secondary radical submodule of M is a sum of a finite number of secondary submodules.*

Proof. Let N be a secondary radical submodule of M and $N = \sum_{i\in I} S_i$, where S_i is a secondary submodule of M for each $i \in I$ and the expression is reduced. Assume that I is an infinite index set. Without loss of generality we may assume that I is countable, then

$$N = \sum_{i=1}^{\infty} S_i \supseteq \sum_{i=2}^{\infty} S_i \supseteq \sum_{i=3}^{\infty} S_i \supseteq \cdots$$

is a descending chain of secondary radical submodules of M. Since by Proposition 2.4,

$$\sum_i S_i \supseteq sec^*(\sum_i S_i) \supseteq (sec^*(\sum_i S_i)) = \sum_i S_i,$$

by hypothesis this descending chain must terminate, so there exists $j \in I$ such that $\sum_{i=j}^{\infty} S_i = \sum_{i=j+1}^{\infty} S_i$, whence $S_j \subseteq \sum_{i=j}^{\infty} S_i$ which contradicts that the expression $N = \sum_{i\in I} S_i$ is reduced. Thus I must be finite. □

Corollary 3.7. Let M be an Artinian R-module. Then there exists a positive integer m such that

$$sec^*(M) = \sum_{j=1}^{m}\{I_{P_j}^M((0:_M P_j^n)) : n \in \mathbb{N} \textit{ and } P_j \textit{ is a prime ideal of } R\}.$$

Proof. This follows from Theorems 3.5 and 3.6. □

Lemma 3.8. Let M be an R-module which satisfies the property $AB5^*$ and $\{S_i\}_{i\in I}$ be a chain of secondary submodules of M such that $\cap_{i\in I}S_i \neq 0$. Then $\cap_{i\in I}S_i$ is a secondary submodule of M.

Proof. Let $r \in R$ and L be a completely irreducible submodule of M such that $r(\cap_{i\in I}S_i) \subseteq L$ and $\cap_{i\in I}S_i \not\subseteq L$. Then $\cap_{i\in I}S_i \subseteq (L :_M r)$. By [10, Lemma 2.1], $(L :_M r)$ is a completely irreducible submodule of M. Therefore, $(L :_M r) = (L :_M r) + \cap_{i\in I}S_i = \cap_{i\in I}(S_i + (L :_M r))$ implies that there exists $j \in I$ such that $S_j \subseteq (L :_M r)$. Thus as $rS_j \subseteq L$ and S_j is a secondary submodule of M by [11, Theorem 2.8], there is a positive integer n such that $r^nS_j = 0$. Hence $r^n(\cap_{i\in I}S_i) = 0$ and the result follows from [11, Theorem 2.8]. □

Theorem 3.9. *Let M be a finitely cogenerated R-module which satisfies the property $AB5^*$. If every secondary submodule S of M is a secondary radical of a submodule K of M such that M/K is a finitely cogenerated R-module. Then M satisfies the descending chain condition for secondary submodules.*

Proof. Let $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ be a descending chain of secondary submodules of M. Since M is finitely cogenerated, $S = \cap_{i\in I}S_i \neq 0$. So by Lemma 3.8, $S = \cap_{i\in I}S_i$ is a secondary submodule of M. Thus by hypothesis, there exists a submodule K of M such that $S = sec^*(K)$ and M/K is a finitely cogenerated R-module. Let $\{L_j\}_{j\in J}$ be a family of completely irreducible submodules of M such that $K = \cap_{j\in J}L_j$. Since M/K is a finitely cogenerated R-module, this intersection must be finite. So suppose that $K = \cap_{j=1}^{n}L_j$. Hence $L_j \supseteq \cap_{j=1}^{n}L_j = K \supseteq sec^*(K) = S = \cap_{i\in I}S_i$. Now since M satisfies the property $AB5^*$,

$$L_j = \cap_{i\in I}S_i + L_j = \cap_{i\in I}(S_i + L_j)$$

for $j = 1, 2, ..., n$. Since L_j is a completely irreducible submodule of M, there exists $t_j \in I$ such that $S_{t_j} \subseteq L_j$. Thus $\cap_{j=1}^{n}S_{t_j} \subseteq \cap_{j=1}^{n}L_j = K$. Therefore, there exists a positive integer h, where $t_1 \leq h \leq t_n$ such that $S_h \subseteq K$. Hence $\cap_{i\in I}S_i = S = sec^*(K) \supseteq sec^*(S_h) = S_h$. It follows that $\cap_{i\in I}S_i = S_h$ and the chain of secondary submodules S_i terminates. □

A secondary submodule N of an R-module M is said to be a *maximal secondary submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a secondary submodule L of M such that $N \subset L \subset K$, see [11].

Theorem 3.10. *Let M be an R-module. If M satisfies the descending chain condition on secondary radicals submodules, then every non-zero submodule of M has only a finite number of maximal secondary submodules or has no maximal secondary submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that N has an infinite number of maximal secondary submodules and look for a contradiction. Then $sec^*(N)$ is a secondary radical submodule of M and it has an infinite number of maximal secondary submodules. Let S be a secondary radical submodule of M chosen minimal such that S has an infinite number of maximal secondary submodules. Then S is not secondary. Thus there exists a submodule L of M and an element r of R such that $L \subset S$ and $S \not\subseteq (0 :_M r^t)$ for each $t \in \mathbb{N}$. Let V be a maximal secondary submodule of M contained in S. Then $V \subseteq (0 :_S r^n)$ for some $n \in \mathbb{N}$ or $V \subseteq L$. By the choice of S, both the modules $(0 :_S r^n)$ and L have only finitely many maximal secondary submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. □

Corollary 3.11. Every Artinian R-module contains only a finite number of maximal secondary submodules or contains no maximal secondary submodules.

Proof. This follows from Theorem 3.10 □

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