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Research Paper

ON MEDIAL FILTERS OF BE-ALGEBRAS

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Abstract. In this paper, the notion of a medial filter in a BE-algebra is defined, and the theory of filters in BE-algebras is developed. These filters are very important for the study of congruence relations in BE-algebras. Moreover, the relationships between implicative filters, medial filters and normal filters are investigated.

1. Introduction

H.S. Kim et al. introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras [3]. A. Walendziak investigated the relationship between BE-algebras, implicative algebras and J-algebras [9]. B.L. Meng introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCK/BCI/BCH-algebras, then discussed it′s important

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properties [4]. The filter theory of BE-algebras was established by B.L. Meng [5]. A. Borumand saeid et al. introduced some types of filters in BE-algebras and stated some relationship between implicative, positive implicative, fantastic, normal, obstinate and maximal filters in BE-algebras [2]. The notion of a normal filter in BE-algebras also defined by A. Walendziak [10]. Some properties of θ-filters of BE-algebras were studied by M.S. Rao [6].

It is known that the filters theory is one of the important concepts in algebraic structure and have drawn the attention of many researchers in the last decades. We give the construction of quotient algebra $\frac{\mathfrak{X}}{F}$ of $\mathfrak{X}$ via a medial filter F of $\mathfrak{X}$. Moreover, we apply the notion of a medial filter on a BE/CI-algebra, to get some of their useful basic properties, and explore the characteristics of a medial filter in a BE/CI-algebra and investigate relationship between some of filters. It is observed that every implicative filter of a BE-algebra is a medial filter, but not the converse in general. Also, we show that if $\{1\}$ is a medial filter, then BE-algebra $\mathfrak{X}$ is transitive, and so every equivalence relation induced by any filter is a congruence relation.

2. Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for introducing the new filters in BE-algebras.

Definition 2.1. [4] An algebra $\mathfrak{X} = (X; \to, 1)$ of type $(2,0)$ is called a *CI-algebra,* if it satisfies the following axioms: for all $x, y, z \in X$;

(CI_1) $x \to x = 1$,

(CI_2) $1 \to x = x$,

(CI_3) $x \to (y \to z) = y \to (x \to z)$.

Definition 2.2. [3] A CI-algebra $\mathfrak{X}$ is said to be a *BE-algebra*, if for all $x \in X$;

(BE) $x \to 1 = 1$.

From now on, $\mathfrak{X}$ is a CI-algebra unless it is stated. We define a relation $\leq$ on X by:

$$x \leq y \quad \textit{if and only if} \quad x \to y = 1.$$

We note that $\leq$ is *reflexive* by (CI_1), but $(X; \leq)$ is not a *partially ordered* set, in general. For this, consider the CI-algebra $([0,1]; \to, 1)$, where $\to$ is defined as follows:

$$x \to y = \begin{cases} 1 & \text{if } x \neq 1 \\ y & \text{if } x = 1 \end{cases}$$

Then $\leq$ is not an *antisymmetric* relation, since

$$\frac{1}{2} \to \frac{1}{3} = 1 \text{ and } \frac{1}{3} \to \frac{1}{2} = 1, \text{ but } \frac{1}{2} \neq \frac{1}{3}.$$

BE-algebra $\mathfrak{X}$ is said to be *self distributive* [3] if for all $x, y, z \in X$;

$$x \to (y \to z) = (x \to y) \to (x \to z).$$

Any self distributive CI-algebra $\mathfrak{X}$ is a BE-algebra [4]. BE-algebra $\mathfrak{X}$ is said to be *transitive* [4] if for all $x, y, z \in X$;

$$(y \to z) \to [(x \to y) \to (x \to z)] = 1.$$

Note that every self distributive BE-algebra is transitive, but the converse is not valid in general (see [1]).

BE-algebra $\mathfrak{X}$ is said to be *commutative* [9] if for all $x, y \in X$;

$$(x \to y) \to y = (y \to x) \to x.$$

Remark 2.3. A. Rezaei et al. showed that if X is a self distributive and commutative BE-algebra, then $(X; \leq)$ is a partially ordered set (see [8]).

Example 2.4. Let $X := \mathrm{R} - \{-n\}$, $0 \neq n \in \mathbb{Z}^+$, where R be the set of all real numbers and $\mathbb{Z}^+$ be the set of all positive integers. If we define the binary operation $\to$ as $x \to y = \frac{n(y-x)}{n+x}$, then $\mathfrak{X} = (X; \to, 0)$ is a CI-algebra, but it is not a BE-algebra. Since $x := 3$ and $n := 5$, we get $3 \to 0 = \frac{5(0-3)}{5+3} = \frac{-15}{8} \neq 0$.

Let $\mathfrak{X}$ be a CI-algebra. If $\to$ is an associative operation on X, then $x \to 1 = x$, since

$$x = 1 \to x = (x \to x) \to x = x \to (x \to x) = x \to 1.$$

Also, applying (BE), if $\mathfrak{X}$ is a BE-algebra and $\to$ is an associative operation, then it is a trivial BE-algebra (i.e., $X = \{1\}$).

Conversely, if $x \to 1 = x$, for all $x \in X$, then the operation $\to$ is an associative operation, and so $\mathfrak{X} = (X; \to, 1)$ is an abelian group. Since, let $x, y, z \in X$, using (CI_3) we have

$$\begin{aligned}
(x \to y) \to z &= (x \to y) \to (z \to 1) \\
&= z \to [(x \to y) \to 1] \\
&= z \to (x \to y) \\
&= z \to [x \to (y \to 1)] \\
&= x \to [z \to (y \to 1)] \\
&= x \to [y \to (z \to 1)] \\
&= x \to (y \to z).
\end{aligned}$$

Also, we have

$$x \to y = x \to (y \to 1) = y \to (x \to 1) = y \to x.$$

Thus, $\mathfrak{X} = (X; \to, 1)$ is an abelian group.

Proposition 2.5. [3, 7] In a BE-algebra $\mathfrak{X}$, the following properties hold: for all $x, y, z \in X$;

(p_1) $x \to (y \to x) = 1$,

(p_2) $x \to ((x \to y) \to y) = 1$,

(p_3) if $x \leq y \to z$, then $y \leq x \to z$,

(p_4) $1 \leq x$, implies $x = 1$,

(p_5) if $x \leq y$, then $x \leq z \to y$.

Theorem 2.6. [7] *Let $\mathfrak{X}$ be a self distributive BE-algebra. Then*

(i) *if $x \leq y$, then $z \to x \leq z \to y$,*

(ii) *$x \to y \leq (y \to z) \to (x \to z)$.*

Definition 2.7. [3] A non-empty subset S of X is called a *subalgebra* of $\mathfrak{X}$ if $x \to y \in S$ for every $x, y \in S$. A non-empty subset F of X is called a *filter* of $\mathfrak{X}$ if it satisfies the following axioms:

(F_1) $1 \in F$,

(F_2) $x \in F$ and $x \to y \in F$ implies $y \in F$.

S.S. Ahn et al. introduced the notion of an ideal in a BE-algebra (see [1]).

Definition 2.8. [1] A non-empty subset I of X is called an *ideal* of $\mathfrak{X}$ if it satisfies the following axioms:

(I_1) $\forall x \in X$ and $\forall a \in I$ implies $x \to a \in I$, i.e., $X \to I \subseteq I$,

(I_2) $\forall x \in X$ and $\forall a, b \in I$ implies $[a \to (b \to x)] \to x \in I$.

The following shows that in BE-algebras ideals and filters coincide. For this,
Let F be a filter of BE-algebra $\mathfrak{X}$ and $a \in F$. Using (CI_3), (CI_2) and (BE) we have

$$a \to (x \to a) = x \to (a \to a) = x \to 1 = 1 \in F.$$

Since F is a filter and $a \in F$, we get $x \to a \in F$, and so (I_1) holds.

Now, let $a, b \in F$. Using (CI_3), (CI_2) and (BE) we have

$$\begin{aligned} a \to [b \to ([a \to (b \to x)] \to x)] &= a \to [(a \to (b \to x)) \to (b \to x)] \\ &= [a \to (b \to x)] \to [a \to (b \to x)] \\ &= 1 \in F. \end{aligned}$$

Hence $a \to [b \to ([a \to (b \to x)] \to x)] \in F$. Since F is a filter and $a \in F$, $b \to [(a \to (b \to x)) \to x] \in F$. Also, since $b \in F$, $[a \to (b \to x)] \to x \in F$, and so ($\mathrm{I}_2$) holds.

Conversely, let I be an ideal of BE-algebra $\mathfrak{X}$. Since I is a non-empty set, there exists $a \in I$. By (CI_1) and (I_1) we have $1 = a \to a \in I$ (i.e., (F_1) holds). Let $x \in F$ and $x \to y \in F$. Take $a = x \to y$, $b = x$ and $x = y$. Thus, $[(x \to y) \to (x \to y)] \to y = 1 \to y = y \in F$.

Definition 2.9. [2] A non-empty subset F of $\mathfrak{X}$ satisfies (F_1) is called a/an

(PIF) *positive implicative filter if* $z \to [(x \to y) \to x] \in F$ *and* $z \in F$ *implies* $x \in F$,

(OF) *obstinate filter if* $x, y \notin F$ *implies* $x \to y \in F$ *and* $y \to x \in F$,

(FF) *fantastic filter if* $z \to (x \to y) \in F$ *and* $z \in F$ *implies* $[(y \to x) \to x] \to y \in F$,

(IF) *implicative filter if* $x \to (y \to z) \in F$ *and* $x \to y \in F$ *implies* $x \to z \in F$, *for all* $x, y, z \in X$.

3. Medial filters in BE/CI-algebras

The aim of this section is to introduce the notion of medial filter in a BE/CI-algebra, to generalize a filter to a medial filter, and to give a number of it′s useful properties.

Definition 3.1. A non-empty subset F of $\mathfrak{X}$ is called a *medial filter* of $\mathfrak{X}$ if it satisfies (F_1) *and* (MF), where for all $x, y, z \in X$;

(MF) $x \to z \in F$ and $z \to y \in F$ implies $x \to y \in F$.

From (MF) by setting $x := 1$ and using (CI_2) it holds that every medial filter is a filter, but the following example shows that the converse is not valid in general.

Example 3.2. (i). Let $X = \{1, a, b\}$ and the binary operation $\to$ is defined as follows:

$\to$	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

In this case, $\mathfrak{X} = (X; \to, 1)$ is a BE-algebra (and so a CI-algebra). Then $F = \{1, b\}$ is not a filter, since $b \in F$, $b \to a = 1 \in F$, but $a \notin F$. Also, it is not a medial filter, since:

$$a \to 1 = 1 \in F \text{ and } 1 \to \mathrm{b} = \mathrm{b} \in \mathrm{F},\ \text{ but } \mathrm{a} \to \mathrm{b} = \mathrm{a} \notin \mathrm{F}.$$

(ii). Let $X = \{1, a, b, c\}$ and the binary operation $\to$ is defined as follows:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

In this case, $\mathfrak{X} = (X; \to, 1)$ is a BE-algebra (and so a CI-algebra). Then $F = \{1, a\}$ is a medial filter.

(iii). Let $X = \{1, a, b, c, d\}$ and the binary operation $\to$ is defined as follows:

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	d	1	c	d
c	1	d	1	1	d
d	1	b	b	c	1

In this case, $\mathfrak{X} = (X; \rightarrow, 1)$ is a BE-algebra (and so a CI-algebra). Then $F = \{1, d\}$ is a filter of $\mathfrak{X}$, but it is not a medial filter of $\mathfrak{X}$, since $b \rightarrow a = d \in F$ and $a \rightarrow c = 1 \in F$, but $b \rightarrow c = c \notin F$.

(iv). Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the binary operation $\rightarrow$ is defined as follows:

$$x \rightarrow y = \begin{cases} 0 & \text{if } y \leq x \\ y - x & \text{otherwise} \end{cases}$$

In this case, $\mathfrak{N} = (\mathbb{N}_0; \rightarrow, 0)$ is a BE-algebra (and so CI-algebra). Then $F = \{0\}$ is a medial filter.

(v). Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Define $x \rightarrow y = y - x$. Then $\mathfrak{Z} = (\mathbb{Z}; \rightarrow, 0)$ is a CI-algebra, but it is not a BE-algebra, since $x \rightarrow 0 = 0 - x \neq 0$. Also, $\mathbb{N}_0$ is a medial filter of $\mathfrak{Z}$. Let $x \rightarrow z \in \mathbb{N}_0$ and $z \rightarrow y \in \mathbb{N}_0$. Hence $z - x = r_1 \geq 0$ and $y - z = r_2 \geq 0$. Thus,

$$x \rightarrow y = y - x = z + r_2 - z + r_1 = r_2 + r_1 := r_3 \geq 0.$$

So, $x \rightarrow y \in \mathbb{N}_0$. Also, it is not a subalgebra of $\mathfrak{Z}$ since if $x := 4$ and $y := 2$, then $x \rightarrow y = y - x = 2 - 4 = -2 \notin \mathbb{N}_0$.

We note that X and $\{1\}$ are trivial filters of BE-algebra $\mathfrak{X}$. The following example shows that it is not a medial filter, in general.

Example 3.3. Let $X = \{1, a, b, c\}$ and the binary operation $\rightarrow$ is defined as follows:

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	1
c	1	1	1	1

Then $\mathfrak{X} = (X; \rightarrow, 1)$ is a BE-algebra (and so a CI-algebra) and $F = \{1\}$ is not a medial filter. Since

$$b \rightarrow c = 1 \in F \text{ and } c \rightarrow a = 1 \in F \text{ but } b \rightarrow a = a \notin F.$$

A. Borumand Saeid et al. was showed that if F_1 and F_2 are filters of (self distributive) BE-algebra $\mathfrak{X}$, $F_1 \subseteq F_2$ and F_1 is an (positive implicative, fantastic) implicative filter, so is F_2 (extension property, see [2, Theorems 2.4, 2.17, 3.4]). Consequently, if $\{1\}$ is a (positive implicative, fantastic) implicative filter of (self distributive) BE-algebra $\mathfrak{X}$, then all of filters are so. The following example shows that it is not valid for medial filters.

Example 3.4. Let $X = \{1, a, b, c, d\}$ and the binary operation $\to$ is defined as follows:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	b	1	c	c
c	1	a	b	1	b
d	1	1	1	1	1

In this case, $\mathfrak{X} = (X; \to, 1)$ is a BE-algebra. Then $\{\{1\}, \{1, c\}, X\}$ is the set of all filters and $\{\{1\}, X\}$ is the set of all medial filters. We can see that $\{1\} \subseteq \{1, c\}$ and $\{1\}$ is a medial filter, but $\{1, c\}$ is not a medial filter of $\mathfrak{X}$, since

$$b \to d = c \in \{1, c\} \text{ and } d \to a = 1 \in \{1, c\}, \text{ but } b \to a = b \notin \{1, c\}.$$

Theorem 3.5. *The filter $\{1\}$ is a medial filter if and only if $\mathfrak{X}$ is a transitive BE-algebra.*

Proof. Assume that $F = \{1\}$ is a medial filter of $\mathfrak{X}$. Let $x \leq y$ and $y \leq z$. Hence $x \to y \in \{1\}$ and $y \to z \in \{1\}$. Thus, $x \to z \in \{1\}$, and so $x \leq z$. Therefore, $\mathfrak{X}$ is a transitive BE-algebra.

Conversely, assume that $\mathfrak{X}$ is a transitive BE-algebra, $x \to y \in \{1\}$ and $y \to z \in \{1\}$. Hence $x \leq y$ and $y \leq z$. Thus, $x \leq z$, and so $x \to z \in \{1\}$. Consequently, $\{1\}$ is a normal filter. □

Theorem 3.6. *Let $\mathfrak{X}$ be a transitive BE-algebra. Then filters and medial filters coincide.*

Proof. It is sufficient to show that every filter is a medial filter. Assume that F is a filter of $\mathfrak{X}$, $x \to z \in F$ and $z \to y \in F$. Since $\mathfrak{X}$ is a transitive, we get $(z \to y) \to [(x \to z) \to (x \to y)] = 1$. Applying twice ($F_2$), we get $x \to y \in F$. □

Meng in [5] gave a procedure by which one could generate a filter by a subset in a transitive BE-algebra and gave some characterizations of Noetherian and Artinian BE-algebras. He defined a congruence relation related to any filter as follows: for all $x, y \in X$;

$$x \sim_1 y \text{ if and only if x} \to \text{y} \in \text{F and y} \to \text{x} \in \text{F}.$$

Then he constructed a quotient algebra $\frac{X}{F}$ of a *transitive* BE-algebra $\mathfrak{X}$ via a filter F of $\mathfrak{X}$. Now, since every medial filter F is a filter of $\mathfrak{X}$, we can generalized this results.

Let F be a filter of $\mathfrak{X}$. A binary relation $\sim_2$ on X can be defined as follows: for all $x, y \in X$;

$$x \sim_2 y \text{ if and only if } z \to (x \to y) \in F \text{ and } z \to (y \to x) \in F, \text{ for all } z \in X.$$

Remark 3.7. We note that $\sim_1$ and $\sim_2$ are the same. For this, let F be a filter of BE-algebra $\mathfrak{X}$. Indeed, if $x \sim_1 y$, then $x \to y$, $y \to x \in F$. Now, for any $z \in X$,

$$(x \to y) \to [z \to (x \to y)] = z \to [(x \to y) \to (x \to y)] = z \to 1 = 1 \in F.$$

Since F is a filter, we have $z \to (x \to y) \in F$. Similarly, $z \to (y \to x) \in F$. Thus, $x \sim_2 y$. Obviously, $\sim_2 \subseteq \sim_1$. Consequently, $\sim_1 = \sim_2$.

Remark 3.8. Let $\mathfrak{X}$ be a transitive BE-algebra and F is a filter of $\mathfrak{X}$. By [5, Lemma 5.1], $\sim_1$ is an equivalence relation on $\mathfrak{X}$. In the following we show that if F is a medial filter, then $\sim_2$ (resp., $\sim_1$) is an equivalence relation for every BE-algebra $\mathfrak{X}$.

Proposition 3.9. *Let F be a medial filter of BE-algebra $\mathfrak{X}$. Then $\sim_2$ is an equivalence relation on $\mathfrak{X}$.*

Proof. Assume that F be a medial filter of BE-algebra $\mathfrak{X}$ and $x, y, z \in X$. Since $z \to (x \to x) = z \to 1 = 1 \in F$, we have $x \sim_2 x$. Hence $\sim_2$ is reflexive. From definition $\sim_2$ is symmetric. For transitivity, let $x \sim_2 y$ and $y \sim_2 z$. Then $t \to (x \to y) \in F$, $t \to (y \to x) \in F$, $t \to (y \to z) \in F$ and $t \to (z \to y) \in F$, for all $t \in X$. Also, take $t = 1$ and using (CI_2), then $x \to y \in F$, $y \to x \in F$, $y \to z \in F$ and $z \to y \in F$. Since F is a medial filter, we get $x \to z \in F$ and $z \to x \in F$. Thus, for all $t \in X$, we have

$$(z \to x) \to [t \to (z \to x)] = t \to [(z \to x) \to (z \to x)] = t \to 1 = 1 \in F.$$

Now, since F is a filter and $z \to x \in F$, we have $t \to (z \to x) \in F$. By a similar argument we can prove $t \to (x \to z) \in F$. Thus, $x \sim_2 z$. $\square$

The following example shows that if $\mathfrak{X}$ is a CI-algebra, then $\sim_2$ is not an equivalence relation on $\mathfrak{X}$.

Example 3.10. Let $X = \{1, a, b, c\}$ and the binary operation $\to$ is defined as follows:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	c	c	1	1
c	c	c	a	1

In this case, $\mathfrak{X} = (X; \rightarrow, 1)$ is a CI-algebra (which is not a BE-algebra since $c \rightarrow 1 = c \neq 1$). Then $F = \{1, a\}$ is a medial filter. The induced relation $\sim_2$ is not an equivalence relation, since $b \rightarrow (1 \rightarrow 1) = b \rightarrow 1 = c \notin F$, we get $1 \not\sim_2 1$.

The following example shows that if F is not a medial filter, then $\sim_2$ is not an equivalence relation on $\mathfrak{X}$.

Example 3.11. Consider the BE-algebra given in Example 3.2(iii), $F = \{1, d\}$ is a filter, but it is not a medial filter. We have $b \sim_2 a$ and $a \sim_2 c$, but $b \not\sim_2 c$, since

$$1 \rightarrow (b \rightarrow c) = 1 \rightarrow c = c \notin F.$$

Then $\sim_2$ is not an equivalence relation on $\mathfrak{X}$.

Remark 3.12. Let F be a medial filter of $\mathfrak{X}$. Then $\sim_1$ $(= \sim_2)$ is a right congruence relation on $\mathfrak{X}$. Indeed, assume that $x \sim_1 y$ and $u \in X$. Then $x \rightarrow y \in F$ and $y \rightarrow x \in F$. Since $x \rightarrow [(x \rightarrow u) \rightarrow u] = 1 \in F$ and $y \rightarrow x \in F$, we get

$$y \rightarrow [(x \rightarrow u) \rightarrow u] = (x \rightarrow u) \rightarrow (y \rightarrow u) \in F.$$

Similarly, from $x \rightarrow y \in F$ and $y \rightarrow [(y \rightarrow u) \rightarrow u] = 1 \in F$, we get

$$x \rightarrow [(y \rightarrow u) \rightarrow u] = (y \rightarrow u) \rightarrow (x \rightarrow u) \in F.$$

Thus, $(x \rightarrow u) \sim_1 (y \rightarrow u)$.

The following example shows that for every medial filter F of $\mathfrak{X}$, $\sim_2$ $(= \sim_1)$ may be not a left congruence relation, and so it is not a congruence relation in general.

Example 3.13. Let $X = \{1, a, b, c, d\}$ and the binary operation $\rightarrow$ is defined as follows:

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	1	1	1

We can see that $\mathfrak{X} = (X; \rightarrow, 1)$ is a BE-algebra. Then $F = \{1, c\}$ is a medial filter. The induced relation $\sim_2$ is not a left congruence relation, (and so it is not a congruence relation) since $1 \sim_2 c$, but $a \rightarrow 1 = 1 \not\sim_2 a \rightarrow c = b$ (since $1 \rightarrow (1 \rightarrow b) = 1 \rightarrow b = b \notin \{1, c\}$). Also, $\mathfrak{X}$ is not a self distributive BE-algebra, since

$$a \rightarrow (b \rightarrow c) = a \rightarrow a = 1 \neq (a \rightarrow b) \rightarrow (a \rightarrow c) = a \rightarrow b = a.$$

Remark 3.14. Let F be a medial filter of a self distributive BE-algebra $\mathfrak{X}$. By Remark 3.7 and [5, Lemma 5.2], $\sim_2$ is a congruence relation on $\mathfrak{X}$ and by [5, Lemma 5.3], $[1]_F = F$, and so by [5, Proposition 5.4], $(\frac{X}{F}; \rightsquigarrow, F)$ is a BE-algebra.

Remark 3.15. We note that if $\{1\}$ is a medial filter, then $\mathfrak{X}$ is a transitive BE-algebra, and so $\sim_2$ $(= \sim_1)$ is a congruence relation on $\mathfrak{X}$.

The following example shows that there is a self distributive BE-algebra $\mathfrak{X}$ and a congruence relation $\sim$ on it in which $\sim$ does not coincide with any induced relation of any medial filter of $\mathfrak{X}$.

Example 3.16. Let $X = \{1, a, b, c, d\}$ and the binary operation $\to$ is defined as follows:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	1

Then $\mathfrak{X} = (X; \to, 1)$ is a BE-algebra (and so a CI-algebra). Then $\mathfrak{P} = \{\{1\}, \{a, b\}, \{c, d\}\}$ is a partition on X. We can see that the equivalence relation $\sim$ derived from $\mathfrak{P}$ is a congruence relation on $\mathfrak{X}$. Also, $\{\{1\}, \{X\}\}$ is the set of all medial filters of $\mathfrak{X}$, but $\sim$ does not coincide with $\{1\}$ nor X.

A. Borumand saeid et al. introduced some types of filters in BE-algebras and show the relationship between them (see [2]). Now, we discuss on relationship between medial filters and implicative filters. From [2, Proposition 2.2], every implicative filter is a filter, but the converse is not valid in general. Also, it was shown that in self distributive BE-algebras implicative filters and filters coincide (see [2, Proposition 2.7]).

Theorem 3.17. *Every implicative filter of BE-algebra $\mathfrak{X}$ is a medial filter of $\mathfrak{X}$.*

Proof. Let F be an implicative filter and $x \to z$, $z \to y \in F$. Using Proposition 2.5(p_1), we have

$$(z \to y) \to [x \to (z \to y)] = x \to [(z \to y) \to (z \to y)] = x \to 1 = 1 \in F.$$

Since F is a filter and $z \to y \in F$, we get $x \to (z \to y) \in F$. Now, since F is an implicative filter, applying (IF), we have $x \to y \in F$. □

The following example shows that the converse of Theorem 3.17, may be not valid in general.

Example 3.18. Let $X = \{1, a, b, c, d\}$ and the binary operation $\rightarrow$ is defined as follows:

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	d	1	1	1
c	1	d	d	1	1
d	1	c	c	c	1

In this case, $\mathfrak{X} = (X; \rightarrow, 1)$ is a transitive BE-algebra (and so a CI-algebra). Then $F = \{1\}$ is a medial filter of $\mathfrak{X}$, but it is not implicative, since

$$b \rightarrow (b \rightarrow a) = 1 \in F \text{ and } b \rightarrow b = 1 \in F, \text{ but } b \rightarrow a = d \notin F.$$

Theorem 3.19. *Let F be a medial filter of BE-algebra $\mathfrak{X}$. Then, for all $x, z \in X$, $y \in F$,*

(D) *$x \rightarrow (y \rightarrow z) \in F$ implies $x \rightarrow y \in F$ and $x \rightarrow z \in F$.*

Proof. Assume that F is a medial filter of $\mathfrak{X}$, $x \rightarrow (y \rightarrow z) \in F$ and $x, z \in X$, $y \in F$. Using (CI_3), (CI_1) and (BE), we have

$$\begin{aligned} y \rightarrow [(y \rightarrow z) \rightarrow y] &= (y \rightarrow z) \rightarrow (y \rightarrow y) \\ &= (y \rightarrow z) \rightarrow 1 \\ &= 1 \in F. \end{aligned}$$

Since F is a medial filter, so is a filter, $y \in F$, we get $(y \rightarrow z) \rightarrow y \in F$, and so $x \rightarrow y \in F$. Also, by using (CI_3) and (CI_1), we have

$$\begin{aligned} y \rightarrow [(y \rightarrow z) \rightarrow z] &= (y \rightarrow z) \rightarrow (y \rightarrow z) \\ &= 1 \in F. \end{aligned}$$

Since F is a medial filter, so is a filter, $y \in F$, we get $(y \rightarrow z) \rightarrow z \in F$, and so $x \rightarrow z \in F$. □

The following example shows that the converse of Theorem 3.19, is not valid in general.

Example 3.20. Consider the BE-algebra given in Example 3.2(iii), $\{1, b\}$ satisfies (D), but it is not a medial filter.

4. Some remarks on normal filters in BE-algebras

In this section, we generalize the notion of normal filter in a BE-algebra was introduced by A. Walendziak (see [10, Definition 3.1]) and give a number of it′s useful properties.

Definition 4.1. [10] A filter F of $\mathfrak{X}$ is said to be *normal* if it satisfies (NF), where for all $x, y, z \in X$;

(NF) $x \to y \in F$ implies $(z \to x) \to (z \to y) \in F$ and $(y \to z) \to (x \to z) \in F$.

Example 4.2. Consider the BE-algebra given in Example 3.2(iii), $F = \{1, b, c\}$ is a normal filter. Further, $\{1\}$ is a medial filter, but it is not a normal filter, since

$$a \to c = 1 \in F, \text{ but } (\mathrm{b} \to \mathrm{a}) \to (\mathrm{b} \to \mathrm{c}) = \mathrm{a} \to \mathrm{c} = \mathrm{c} \notin \{1\}.$$

Proposition 4.3. [10] *If $\mathfrak{X}$ is a transitive BE-algebra, then every filter of $\mathfrak{X}$ is normal.*

Proposition 4.4. *Every normal filter is a medial filter.*

Proof. Assume that F is a normal filter of $\mathfrak{X}$ and $x \to z$, $z \to y \in F$. Since F is a normal filter and $z \to y \in F$, we have $(x \to z) \to (x \to y) \in F$, and so $x \to y \in F$. Thus, F is a medial filter. □

Proposition 4.5. *If $\mathfrak{X}$ is a transitive BE-algebra, then every filter of $\mathfrak{X}$ is medial.*

Proof. Using Propositions 4.3 and 4.4 the proof is obvious. □

Theorem 4.6. *The filter F of $\mathfrak{X}$ is a normal filter if and only if it satisfies* (NF)$'$, *where for all $x, y, z \in X$;*

(NF)$'$ $x \to y \in F$ *implies* $(z \to x) \to (z \to y) \in F$.

Proof. It is obvious that every normal filer F satisfies (NF)$'$.

Conversely, first we prove that F is a medial filter. Assume that $x \to z$, $z \to y \in F$. Hence $(x \to z) \to (x \to y) \in F$. Since F is a filter and $x \to z \in F$, we get $x \to y \in F$. Thus, F is a medial filter. Using (CI_3) and (CI_1) we have

$$y \to [(y \to z) \to z] = (y \to z) \to (y \to z) = 1 \in F.$$

Since F is a medial filter, $x \to y \in F$ and $y \to [(y \to z) \to z] \in F$, we get:
$x \to [(y \to z) \to z] = (y \to z) \to (x \to z) \in F$. Thus, (NF) holds. □

The following example shows that the converse of Proposition 4.4, may be not valid in general.

Example 4.7. Consider the BE-algebra given in Example 3.2(iii), $F = \{1\}$ is a medial filter, but it is not a normal filter (see Example 4.2).

Remark 4.8. We note that, the filter F satisfies (NF) if and only if it satisfies (NF)$'$. For this, let filter F satisfies (NF), it is obvious that (NF)$'$ holds. Conversely, since every normal filter is a medial filter, we can see that if $x \to y \in F$ imply $(y \to z) \to (x \to z) \in F$ by applying Theorem 2.6(ii).

Theorem 4.9. *(Extension property) Let F_1 and F_2 be filters of BE-algebra $\mathfrak{X}$ such that $F_1 \subseteq F_2$. If F_1 is a normal filter, so is F_2.*

Proof. Suppose that $y \to z \in F_2$. Then $y \to [(y \to z) \to z] = (y \to z) \to (y \to z) = 1 \in F_1$. Since F_1 is normal, applying (NF$'$), we get

$$(x \to y) \to [x \to ((y \to z) \to z)] \in F_1 \subseteq F_2.$$

Also, since

$$\begin{aligned}(y \to z) \to [(x \to y) \to (x \to z)] &= (x \to y) \to [(y \to z) \to (x \to z)] \\ &= (x \to y) \to [x \to ((y \to z) \to z)] \\ &\in F_2.\end{aligned}$$

Hence $(y \to z) \to [(x \to y) \to (x \to z)] \in F_2$, and so $(x \to y) \to (x \to z) \in F_2$. Thus, F_2 is a normal filter of $\mathfrak{X}$. □

Corollary 4.10. *The filter $\{1\}$ is a normal filter if and only if all filters of $\mathfrak{X}$ are normal filter.*

Theorem 4.11. *The filter $\{1\}$ is a normal filter if and only if $\mathfrak{X}$ is a transitive BE-algebra.*

Proof. Assume that $\{1\}$ is a normal filter of $\mathfrak{X}$. Using Proposition 4.4, the filter $\{1\}$ is a medial filter, and so from Theorem 3.5, we conclude that $\mathfrak{X}$ is a transitive BE-algebra.

Conversely, assume that $\mathfrak{X}$ is a transitive BE-algebra and $x \to y \in \{1\}$. Since $(y \to z) \to [(x \to y) \to (x \to z)] = (y \to z) \to [1 \to (x \to z)] = (y \to z) \to (x \to z) = 1$, we get $(y \to z) \to (x \to z) \in \{1\}$. Also, since $(z \to x) \to (z \to y) = (x \to y) \to [(z \to x) \to (z \to y)] = 1$, we get $(z \to x) \to (z \to y) \in \{1\}$. Thus, $\{1\}$ is a normal filter. □

Let F be a filter of $\mathfrak{X}$. A binary relation $\sim_3$ on X can be defined as follows: for all $x, y \in X$;

$$x \sim_3 y \text{ if and only if } \mathrm{x} \to \mathrm{y} \in \mathrm{F} \text{ and } \mathrm{y} \to \mathrm{x} \in \mathrm{F}.$$

Theorem 4.12. [10] *If F is a normal filter of a BE-algebra $\mathfrak{X}$, then $\sim_3$ is a congruence relation on X.*

Remark 4.13. For the relation $\sim_1$, BE-algebra $\mathfrak{X}$ must be transitive, while for $\sim_3$ it is sufficient that, the filter F be a normal filter.

Remark 4.14. We note that $\sim_1 = \sim_3$. Consequently, by Remark 3.7, $\sim_1 = \sim_2 = \sim_3$.

Remark 4.15. If $\{1\}$ is a normal filter on $\mathfrak{X}$, then $\sim_1$ $(=\sim_2, \sim_3)$ is a congruence relation on $\mathfrak{X}$.

Conclusions

Now, in the following diagram we summarize the results of this paper and the previous results in this filed. The notion "$A \longrightarrow B$ (respectively, $A \xrightarrow{\text{sd}} B$)", means A conclude B (respectively, A conclude B with condition "*self distributive*" briefly "sd" and we show that the set of all "*filters*" briefly "**F**", the set of all "*implicative filters*" briefly "**IF**", the set of all *positive implicative filters*" briefly "**PIF**", the set of all "*fantastic filters*" briefly "**FF**", the set of all "*normal filters*" briefly "**NF**" and the set of all "*medial filters*" briefly "**MF**".

$$\begin{tikzcd}
 & \mathbf{NF} \arrow[d, shift left] & \\
 & \mathbf{MF} \arrow[u, shift left, "sd"] \arrow[d, shift left] & \\
\mathbf{IF} \arrow[ur] \arrow[r, shift left] & \mathbf{F} \arrow[l, shift left, "sd"] \arrow[u, shift left, "sd"] & \mathbf{FF} \arrow[l] \\
 & \mathbf{PIF} \arrow[ul, "sd"] \arrow[u] \arrow[ur, "sd"'] &
\end{tikzcd}$$

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