



Research Paper

**CONVEX, BALANCED AND ABSORBING SUBSETS OF HYPERVECTOR SPACES**

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**ABSTRACT.** In this paper, we define convex, balanced and absorbing subsets of a hypervector space  $V$  over a field  $K$ , where  $K$  is considered  $\mathbb{R}$  or  $\mathbb{C}$  and give some examples of them. We prove that every subspace of a hypervector space is a convex and balanced subset. Also, for every regular equivalence relation  $\rho$  on a hypervector space  $V$ , we show that if  $A$  is a convex, balanced or an absorbing subset of  $V$ , then  $A/\rho$  is respectively a convex, balanced or an absorbing subset of a hypervector space  $V/\rho$ .

**1. Introduction**

In 1934 Marty [7] introduced a new mathematical structure as a generalization of groups and called it hypergroup. Also he was motivated to introduce this structure to study several problems of the non-commutative algebra and has constructed some other structures such as hyperrings, hypermodules, hyperfields and hypervector spaces. Then many researchers

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have studied on hyperalgebraic structures and developed this theory (ref. [2], [3] and [6]). These structures have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, and etc (ref. [4], [5], [14] and [15]). The notion of the hypervector spaces was introduced by M. Scafati Tallini [13] in 1988. Authors, in [8]-[12] considered hypervector spaces in viewpoint of analysis. In mentioned papers, authors, introduced concepts such as dimension of hypervector spaces, normed hypervector spaces, operator on these spaces and other important concepts.

In this paper, we introduce the concepts of convex, balanced and absorbing subsets of a hypervector space. Then we study the properties of these subsets and give some examples of them.

Before to state our results we describe some fundamental concepts.

**Definition 1.1.** A hypervector space over a field  $K$  is a quadruplet  $(V, +, \circ, K)$  such that  $(V, +)$  is an abelian group and

$$\circ : K \times V \rightarrow P^*(V)$$

is a mapping of  $K \times V$  into the set of all non-empty subsets of  $V$ , such that

$$(a + b) \circ x \subseteq (a \circ x) + (b \circ x), \forall a, b \in K, \forall x \in V,$$

$$a \circ (x + y) \subseteq (a \circ x) + (a \circ y), \forall a \in K, \forall x, y \in V,$$

$$a \circ (b \circ x) = (ab) \circ x, \forall a, b \in K, \forall x \in V,$$

$$x \in 1 \circ x, \forall x \in V,$$

$$a \circ (-x) = -a \circ x, \forall a \in K, \forall x \in V.$$

**Definition 1.2.** Let  $(V, +, \circ)$  be a hypervector space over the field  $K$ . Then a non-empty subset  $H$  of  $V$  is a subspace of  $V$ , if

$$\text{i) } H - H \subseteq H,,$$

$$\text{ii) } r \circ H \subseteq H \text{ for every } r \in K,$$

where  $H - H = \{x - y : x, y \in H\}$ .

**Definition 1.3.** Let  $(V, +, \circ)$  and  $(W, \oplus, *)$  be two hypervector spaces over the same field  $K$ . A mapping

$$f : V \rightarrow W$$

is called a

1) homomorphism, if  $\forall r \in K, \forall x, y \in V$ :

$$f(x + y) = f(x) \oplus f(y),$$

$$f(r \circ x) \subseteq r * f(x).$$

2) strong homomorphism, if  $\forall r \in K, \forall x, y \in V$ :

$$\begin{aligned} f(x + y) &= f(x) \oplus f(y), \\ f(r \circ x) &= r * f(x). \end{aligned}$$

Throughout the paper, the field  $K$  is considered  $\mathbb{R}$  or  $\mathbb{C}$ .

## 2. Convex, balanced and absorbing subsets of a hypervector space

In this section we define a convex, balanced and an absorbing subset of a hypervector space and study algebraic properties of these subsets. Also we give some concrete examples of them.

**Definition 2.1.** Let  $C$  be a subset of a hypervector space  $(V, +, \circ, K)$ . The set  $C$  is called a convex subset of  $V$ , if for all  $x, y \in C$  and  $t \in [0, 1]$ , we have

$$t \circ x + (1 - t) \circ y \subseteq C.$$

**Definition 2.2.** Let  $(V, +, \circ, K)$ , be a hypervector space, a subset  $B$  of  $V$  is balanced, if  $\mathcal{D} \circ B \subseteq B$ , where

$$\mathcal{D} = \{r \in K : |r| \leq 1\}.$$

**Definition 2.3.** A subset  $A \subseteq V$  is absorbing if for any  $x \in V$ , there exists a non-negative real number  $\varepsilon_x$  such that  $t \circ x \subseteq A$ , for all  $t \in K$  with  $0 \leq t < \varepsilon_x$ .

**Proposition 2.4.** *If  $A$  is an absorbing subset of  $V$ , then every set containing  $A$  is absorbing.*

*Proof.* It is trivial.  $\square$

**Example 2.5.** In  $(\mathbb{R}^2, +)$  we define

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \rightarrow P^*(\mathbb{R}^2) \\ a \circ (x, y) = \{(ax, ay)\}. \end{cases}$$

Then  $(\mathbb{R}^2, +, \circ, \mathbb{R})$  is a hypervector space. For every  $r > 0$ , let

$$A_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}.$$

We show that  $A_r$  is an absorbing subset of  $(\mathbb{R}^2, +, \circ, \mathbb{R})$ . For every  $X = (a, b) \in \mathbb{R}^2$ , suppose that  $\varepsilon_X = \frac{r}{a^2 + b^2}$ . So, for all  $0 \leq t < \varepsilon_X$ ,

$$(ta)^2 + (tb)^2 < \varepsilon_X^2 (a^2 + b^2) = r^2.$$

It follows that  $t \circ (a, b) \subseteq A_r$ , for all  $0 \leq t < \varepsilon_X$ . Therefore  $A_r$  is absorbing. By Proposition 2.4, any set containing an open ball centered at 0 is absorbing. This implies that an absorbing

set does not have to be convex.

As an example of a balanced set that is neither absorbing nor convex, consider the union of  $x$  and  $y$  axes.

**Example 2.6.** Let  $V = \mathbb{R}^2$ . Define

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \rightarrow P^*(\mathbb{R}^2) \\ a \circ (x, y) = ax \times \mathbb{R}. \end{cases}$$

Then  $(\mathbb{R}^2, +, \circ, \mathbb{R})$  is a hypervector space. Suppose that  $C = I \times \mathbb{R}$  such that  $I$  is an interval of  $\mathbb{R}$ . Then,  $C$  is a convex set.

For  $I = (1, 2)$ ,  $C$  is a convex set that is neither absorbing nor balanced.

**Theorem 2.7.** Let  $(V, +, \circ, K)$  be a hypervector space. The following statements are valid:

- i)  $\emptyset$  and  $V$  are convex,
- ii) Arbitrary intersection of convex subsets of  $V$  is convex,
- iii) The summation of two convex subset of  $V$  is convex.

*Proof.* The proof is straightforward.  $\square$

**Remark 2.8.** The union of convex sets is not generally convex.

**Lemma 2.9.** Let  $\{B_\alpha : \alpha \in I\}$  be a family of subsets of  $V$ . Then

$$\text{i) } \bigcap \{a \circ B_\alpha : \alpha \in I\} = a \circ \bigcap \{B_\alpha : \alpha \in I\},$$

$$\text{ii) } \bigcup \{a \circ B_\alpha : \alpha \in I\} = a \circ \bigcup \{B_\alpha : \alpha \in I\},$$

for all  $a \in K$ .

*Proof.* i) Suppose that  $x \in \bigcap \{a \circ B_\alpha : \alpha \in I\}$ , for  $a \in K$ . Thus,  $x \in a \circ B_\alpha$ , for all  $\alpha \in I$ . Consequently  $x \in \bigcup \{a \circ y : y \in B_\alpha\}$ , for all  $\alpha \in I$ , it follows that

$$x \in \bigcup \{a \circ y : y \in \bigcap B_\alpha, \alpha \in I\}.$$

Hence,  $x \in a \circ \bigcap \{B_\alpha : \alpha \in I\}$ . Therefore,  $\bigcap \{a \circ B_\alpha : \alpha \in I\} \subseteq a \circ \bigcap \{B_\alpha : \alpha \in I\}$ . Similarly, we obtain the converse inclusion.

ii) Let  $x \in \bigcup \{a \circ B_\alpha : \alpha \in I\}$ . Then, there exist  $\alpha \in I$  and  $y \in B_\alpha$  such that  $x \in a \circ y$  and so  $x \in a \circ \bigcup \{B_\alpha : \alpha \in I\}$ . Therefore,  $\bigcup \{a \circ B_\alpha : \alpha \in I\} \subseteq a \circ \bigcup \{B_\alpha : \alpha \in I\}$ . Similarly, we can prove  $a \circ \bigcup \{B_\alpha : \alpha \in I\} \subseteq \bigcup \{a \circ B_\alpha : \alpha \in I\}$ .  $\square$

**Theorem 2.10.** *Let  $(V, +, \circ, K)$  be a hypervector space. The following properties hold:*

- i)  $\emptyset$  and  $V$  are balanced,
- ii) Arbitrary union of balanced subsets of  $V$  is balanced,
- iii) Arbitrary intersection of balanced subsets of  $V$  is balanced,
- v) The summation of two balanced subsets of  $V$  is balanced.

*Proof.* i) It is trivial.

ii) By Lemma 2.9 we have

$$\mathcal{D} \circ \bigcup \{B_\alpha : \alpha \in I\} = \bigcup \{\mathcal{D} \circ B_\alpha : \alpha \in I\}.$$

Since for all  $\alpha \in I$ ,  $B_\alpha$  is balanced, it follows that

$$\mathcal{D} \circ \bigcup \{B_\alpha : \alpha \in I\} \subseteq \bigcup \{B_\alpha : \alpha \in I\}.$$

iii) Let  $\{B_\alpha : \alpha \in I\}$  be a family of balanced subsets of  $V$ . According to the Lemma 2.9 we obtain that

$$\mathcal{D} \circ \bigcap \{B_\alpha : \alpha \in I\} = \bigcap \{\mathcal{D} \circ B_\alpha : \alpha \in I\}.$$

Since  $B_\alpha$  is balanced, for all  $\alpha \in I$ , it follows that

$$\mathcal{D} \circ \bigcap \{B_\alpha : \alpha \in I\} \subseteq \bigcap \{B_\alpha : \alpha \in I\}.$$

v) Let  $x \in \mathcal{D} \circ (A + B)$ , where  $A$  and  $B$  are balanced. Then, there exist  $r \in \mathcal{D}$ ,  $a \in A$  and  $b \in B$  such that  $x \in r \circ (a + b)$ . So  $x \in r \circ a + r \circ b \subseteq \mathcal{D} \circ A + \mathcal{D} \circ B$ . Therefore,

$$\mathcal{D} \circ (A + B) \subseteq \mathcal{D} \circ A + \mathcal{D} \circ B.$$

Since  $A$  and  $B$  are balanced, we have  $\mathcal{D} \circ (A + B) \subseteq A + B$ .  $\square$

**Theorem 2.11.** *If  $V$  is a hypervector space. Then*

- i)  $V$  is absorbing ,
- ii) Arbitrary union of absorbing subsets of  $V$  is absorbing,
- iii) Arbitrary intersection of absorbing subsets of  $V$  is absorbing,
- v) The summation of two absorbing subsets of  $V$  is absorbing.

*Proof.* i) It is trivial.

ii) It follows from Lemma 2.4.

iii) Let  $\{A_\alpha : \alpha \in I\}$  be a family of absorbing subsets of  $V$ . Thus, for every  $\alpha \in I$  and  $x \in V$  there exists  $\varepsilon_x^\alpha > 0$ , such that  $t \circ x \subseteq A_\alpha$ , for all  $0 \leq t < \varepsilon_x^\alpha$ .

So  $t \circ x \subseteq \bigcap \{A_\alpha : \alpha \in I\}$ , for all  $0 \leq t < \varepsilon$ , where  $\varepsilon = \min\{\varepsilon_x^\alpha : \alpha \in I\}$ . Therefore,  $\bigcap \{A_\alpha : \alpha \in I\}$  is an absorbing subset of  $V$ .

v) If  $A$  and  $B$  are absorbing subsets of  $V$ . then, for all  $x \in V$ , there exist  $\varepsilon_x^\alpha > 0$  and  $\varepsilon_x^\beta > 0$  such that  $t \circ x \subseteq A$  for all  $0 \leq t < \varepsilon_x^\alpha$  and  $t \circ x \subseteq B$ , for every  $0 \leq t < \varepsilon_x^\beta$ . Also for all  $0 \leq t < \varepsilon = \min\{\varepsilon_x^\alpha, \varepsilon_x^\beta\}$ , we have  $t \circ x \subseteq A$  and  $t \circ x \subseteq B$ . Thus  $t_0 \circ x \subseteq A + B$ , for all  $0 \leq t_0 \leq 2\varepsilon$ . Hence,  $A + B$  is an absorbing subset of  $V$ .  $\square$

### 3. Main results

We start with the following lemma:

**Lemma 3.1.** [1] *A nonempty subset  $W$  of  $V$  is subspace if and only if*

$$a \circ u + b \circ v \subseteq W,$$

for all  $a, b \in K$  and  $u, v \in W$ .

**Theorem 3.2.** *Let  $W$  be a subspace of a hypervector space  $V$ . Then  $W$  is a convex and balanced set.*

*Proof.* By Lemma 3.1, it is trivial.  $\square$

**Definition 3.3.** Let  $A$  be a subset of  $V$ . The balanced hull for a set  $A$  is the smallest balanced set containing  $A$  defined and denoted as follows:

$$\text{bal}(A) = \bigcap \{B \subseteq V : A \subseteq B, B \text{ is a balanced subset of } V\}.$$

**Theorem 3.4.** *Let  $A$  be a nonempty subset of a hypervector space  $(V, +, \circ, K)$ . Then,*

- i)  $\text{bal}(A) \neq \emptyset$ ,
- ii)  $A \subseteq \mathcal{D} \circ A$ ,
- iii)  $\mathcal{D} \circ A = \text{bal}(A)$ .

*Proof.* i) Since  $V$  is balanced, it is trivial.

ii) Let  $a \in A$ . Then  $a \in 1 \circ a \subseteq \mathcal{D} \circ A$ . Therefore,  $A \subseteq \mathcal{D} \circ A$ .

iii) Let  $t \in \mathcal{D} \circ A$ . Then, there exist  $r \in \mathcal{D}$  and  $a \in A$  such that  $t \in r \circ a$ , and so  $t \in \mathcal{D} \circ B$ , for all balanced sets  $B$  such that  $A \subseteq B$ . Hence,  $t \in B$ , for all balanced sets  $B$ , where  $A \subseteq B$ . So  $t \in \text{bal}(A)$ . Therefore,  $\mathcal{D} \circ A \subseteq \text{bal}(A)$ . For prove the inverse inclusion, first we show that  $\mathcal{D} \circ A$  is balanced. Let  $t \in \mathcal{D} \circ (\mathcal{D} \circ A)$ . So there exist  $r, r' \in \mathcal{D}$  and  $x \in A$  such that  $t \in r \circ (r' \circ x) = rr' \circ x$ . Thus  $t \in \mathcal{D} \circ A$ . Therefore  $\mathcal{D} \circ A$  is balanced and since  $A \subseteq \mathcal{D} \circ A$  we obtain that  $\text{bal}(A) \subseteq \mathcal{D} \circ A$ .  $\square$

**Definition 3.5.** If  $A$  is a nonempty subset of  $V$ . Then the convex hull of  $A$  is defined as follows

$$\text{con}(A) = \bigcap \{X : A \subseteq X, X \text{ is a convex subset of } V\}.$$

This is the smallest convex subset of  $V$  that contains  $A$ .

**Proposition 3.6.** Let  $A$  be a subset of  $V$ . The following statements are valid:

- i)  $A \subseteq \text{con}(A)$ ,
- ii)  $\text{con}(\text{con}(A)) = \text{con}(A)$ .

*Proof.* It is an immediate consequence of definition of  $\text{con}(A)$ .  $\square$

**Corollary 3.7.**  $A$  is a convex set if and only if  $A = \text{con}(A)$ .

*Proof.* By definition of  $\text{con}(A)$  and Proposition 3.6, it is obtained.  $\square$

**Theorem 3.8.** Let  $B$  be a balanced subset of a hypervector space  $V$  and for every  $x \in V$ , there exists  $\varepsilon > 0$  such that  $\varepsilon \circ x \subseteq B$ . Then  $B$  is an absorbing subset of  $V$ .

*Proof.* For every  $0 \leq t < \varepsilon$ , there exists  $0 \leq t_0 < 1$  such that  $t = t_0\varepsilon$ . Thus

$$t \circ x = t_0\varepsilon \circ x = t_0 \circ (\varepsilon \circ x) \subseteq t_0 \circ B \subseteq \mathcal{D} \circ B.$$

Since  $B$  is balanced, we obtain  $t \circ x \subseteq B$  and therefore  $B$  is an absorbing subset of  $V$ .  $\square$

**Proposition 3.9.** The image of a convex (balanced) set, under a strong homomorphism is a convex (balanced) set.

*Proof.* The proof is straightforward.  $\square$

**Theorem 3.10.** Let  $(V, +, \circ, K)$  and  $(W, \oplus, *, K)$  be two hypervector spaces. The image of an absorbing subset of  $V$  under an onto strong homomorphism  $f : V \rightarrow W$  is an absorbing subset of  $W$ .

*Proof.* Let  $x \in W$ . Hence there exists  $x_0 \in V$  such that  $f(x_0) = x$ . Suppose that  $A$  is an absorbing subset of  $V$ , it follows that there exists  $\varepsilon > 0$  such that  $t \circ x_0 \subseteq A$ , for all  $0 \leq t < \varepsilon$ . From here, we obtain that

$$f(t \circ x_0) = t * f(x_0) = t * x \subseteq f(A).$$

Hence,  $f(A)$  is an absorbing subset of  $W$ .  $\square$

**Definition 3.11.** [16] Let  $\rho$  be an equivalence relation on a hypervector space  $(V, +, \circ, K)$ . If  $A, B$  are non-empty subsets of  $V$ , then  $A\bar{\rho}B$  means that for every  $a \in A$ , there exists  $b \in B$  such that  $a\rho b$  and for all  $\acute{b} \in B$ , there exists  $\acute{a} \in A$  such that  $\acute{a}\rho\acute{b}$ . The equivalent relation  $\rho$  is called regular if for all  $r \in K$ , from  $a\rho b$ , it follows that  $(r \circ a)\bar{\rho}(r \circ b)$ .

**Theorem 3.12.** [16] Let  $(V, +, \circ, K)$  be a hypervector space,  $\rho$  be an equivalence relation on  $V$  and  $V/\rho = \{[v] : v \in V\}$ . Then  $V/\rho$  is a hypervector space with respect to actions

$$\oplus : V/\rho \times V/\rho \rightarrow V/\rho,$$

where  $[x] \oplus [y] = [x + y]$ , and

$$\odot : K \times V/\rho \rightarrow P^*(V/\rho)$$

$$k \odot [x] = \{[z] : z \in k \circ x\},$$

if and only if  $\rho$  is regular.

**Theorem 3.13.** [16] Let  $\rho$  be a regular equivalence relation on  $V$ . Then the canonical projection

$$\pi : V \rightarrow V/\rho$$

$$\pi(x) = [x],$$

is an onto strong homomorphism.

According to Proposition 3.9, Theorems 3.10, 3.12 and 3.13, we have the following Proposition:

**Proposition 3.14.** If  $A$  is a convex, balanced or an absorbing subset of  $V$ , then  $A/\rho = \{[a] : a \in A\}$  is respectively a convex, balanced or an absorbing subset of  $V/\rho$ , where  $\rho$  is a regular equivalence relation on  $V$ .

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