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Research Paper

THE STRONGLY ANNIHILATING-SUBMODULE GRAPH OF A MODULE

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Abstract. In this paper, we define the notion of strongly annihilating-submodule graph of modules. This graph is a straightforward common generalization of the annihilating-submodule graph and the annihilating-ideal graph. In addition to providing the properties of this graph in general, we investigate the behavior of the graph when modules are reduced or divisible.

1. Introduction

Throughout the paper R is a commutative ring with nonzero identity and M is a unitary right R-module. For a submodule N of M, denoted by $N \leq M$, the ideal $\{r \in R \mid Mr \subseteq N\}$ will be denoted by $(N :_R M)$ (briefly by $(N : M)$). Recall that M is *indecomposable* if it is nonzero and cannot be written as a direct sum of two nonzero submodules. A module is called *uniform* if the intersection of any two nonzero submodule is nonzero. Also a submodule N of M is called an *essential submodule* of M, denoted by $N \leq_e M$, if for any nonzero submodule K of

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M, $K\cap N \neq 0$. For $X \subseteq M$, the annihilator of X in R is the ideal $\text{ann}_R(X) = \{r \in R \mid Xr = 0\}$. We say that M has *uniform dimension* n (written u.dim$M = n$) if there exists an essential submodule $N \leq_e M$ which is a direct sum of n uniform submodules, i.e., u.dim$M$ is the supremum of the set $\{k \mid M$ contains a direct sum of k nonzero submodules$\}$, for more details see [14]. The definitions and notions of graph theory used throughout this paper can be found in [12].

For any ring R with the set of zero-divisors $Z(R)$, *the zero-divisor graph* of R, denoted by $\Gamma(R)$, is a simple graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$ (see for example [1, 2, 3, 4, 5]). An ideal I of a commutative ring R is called *annihilating-ideal* if $IJ = 0$, for a nonzero ideal J of R. Also the set of all annihilating-ideals of R is denoted by $\mathbb{A}(R)$. The notion of annihilating-ideal graph was introduced and studied in [9] and [10]. *The annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a simple graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R)\setminus\{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. Recently, the notions of *zero-divisor graph* and *annihilating-ideal graph* have been extended from rings to modules in different ways. For instance, we can refer to [8] and [15]. In [8], the authors introduced and studied the annihilating-submodule graph. By *the annihilating-submodule graph* of M, denoted by $\mathbb{AG}(M)$, we mean the simple graph with vertices $\{0 \neq N \leq M \mid M(N : M)(K : M) = 0$, for a nonzero submodule K of $M\}$ and two distinct vertices N and K are adjacent if and only if $M(N : M)(K : M) = 0$, see [7] and [8].

In this paper, we define and study the notion of strongly annihilating-submodule graph as a straightforward common generalization of two graphs $\mathbb{AG}(R)$ and $\mathbb{AG}(M)$. *The strongly annihilating-submodule graph* of M, denoted by $\mathbb{SAG}(M)$, is an undirected (simple) graph in which a nonzero submodule N of M is a vertex if $N(K : M) = 0$ or $K(N : M) = 0$, for a nonzero submodule $K \leq M$ and two distinct vertices N and K are adjacent if and only if $N(K : M) = 0$ or $K(N : M) = 0$. It is clear that if $M = R$, then $\mathbb{SAG}(R) = \mathbb{AG}(R)$ and if M is a multiplication R-module, then $\mathbb{SAG}(M) = \mathbb{AG}(M)$. We investigate the interplay between the graph theoretic properties of $\mathbb{SAG}(M)$ and some algebraic properties of a module M. In Section 2, some properties of $\mathbb{SAG}(M)$ is presented. For example, we show that $\mathbb{SAG}(M)$ is a connected graph with diam$(\mathbb{SAG}(M)) \leq 3$ (Theorem 2.4). Also, if $\mathbb{SAG}(M)$ contains a cycle, then gr$(\mathbb{SAG}(M)) \leq 4$ (Theorem 2.5). We prove that if M is a finitely generated semisimple R-module such that its homogeneous components are simple, then for any two submodules N, K of M we have N and K are adjacent if and only if $N \cap K = 0$ (Proposition 2.11). It is shown that $\mathbb{AG}(M) = \mathbb{SAG}(M) \cong \mathbb{AG}(R)$, for any finitely generated faithful multiplication R-module M (Theorem 2.20). In Section 3, we investigate the properties of $\mathbb{SAG}(M)$, when M is a reduced module. For instance, we show that if M is a reduced R-module such that $\mathbb{SAG}(M)$ is a bipartite graph and M is not a vertex in $\mathbb{SAG}(M)$, then $\mathbb{SAG}(M)$ is a complete bipartite graph and u.dim$M = 2$ (Theorem 3.3). Finally, in Section 4, we focus on divisible modules. For example, we prove that if M contains a nonzero divisible submodule, then $\mathbb{SAG}(M)$ is the empty graph or every nonzero submodule of M is a vertex in $\mathbb{SAG}(M)$ (Proposition 4.3).

2. Some properties of $\mathbb{SAG}(M)$

Throughout the paper M is a unitary right R-module and N, K are nonzero submodules of M. The following useful results will be used frequently in this paper.

Lemma 2.1. (1) *If N and K are adjacent in $\mathbb{SAG}(M)$, then N_1 and K_1 are adjacent in $\mathbb{SAG}(M)$ for every $0 \neq N_1 \leq N$ and $0 \neq K_1 \leq K$ with $N_1 \neq K_1$;*
(2) *If $N \cap K = 0$, then N and K are adjacent in $\mathbb{SAG}(M)$;*
(3) *If N is not a vertex of $\mathbb{SAG}(M)$, then $N \leq_e M$.*

Proof. Clear. □

Lemma 2.2. *If N and K are adjacent in $\mathbb{AG}(M)$, then either N and K are also adjacent in $\mathbb{SAG}(M)$ or there exists a nonzero submodule of $N \cap K$ such that is adjacent to both N and K in $\mathbb{SAG}(M)$. In particular; the set of all vertices of $\mathbb{AG}(M)$ is equal to the set of all vertices of $\mathbb{SAG}(M)$.*

Proof. Suppose that N and K are not adjacent in $\mathbb{SAG}(M)$. Then $N \cap K \neq 0$. Since $M(N : M)(K : M) = 0$, we have $M(N \cap K : M)(K : M) = 0$ and $M(N \cap K : M)(N : M) = 0$. Now one of the following cases holds.
Case 1: $M(N \cap K : M) = 0$. Then $N(N \cap K : M) = 0$ and $K(N \cap K : M) = 0$. Since by hypothesis, $N \cap K \neq N$ and $N \cap K \neq K$, $N \cap K$ is adjacent to both N and K in $\mathbb{SAG}(M)$.
Case 2: $M(N \cap K : M) \in \{N, K\}$. Then we have $N(K : M) = 0$ or $K(N : M) = 0$, a contradiction.
Case 3: $M(N \cap K : M) \notin \{0, N, K\}$. Then $M(N \cap K : M)$ is adjacent to both N and K in $\mathbb{SAG}(M)$. □

In the following result, we use the notations $d_A(N, K)$ and $d_S(N, K)$ for showing the distance of two vertices N and K in $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$, respectively.

Lemma 2.3. *Let N and K be two vertices in $\mathbb{AG}(M)$. Then we have the following statements.*
(1) *If $d_A(N, K) = 1$, then $d_S(N, K) \leq 2$.*
(2) *If $d_A(N, K) = 2$, then $d_S(N, K) = 2$.*
(3) *If $d_A(N, K) = 3$, then $d_S(N, K) = 3$.*

Proof. (1) follows from Lemma 2.2.

(2). Let $d_A(N,K) = 2$ and $N - L - K$ be a path in $\mathbb{AG}(M)$. Clearly, N and K are not adjacent in $\mathbb{SAG}(M)$ and so $d_S(N,K) \geq 2$. By Lemma 2.1(1) and Lemma 2.2, there exists $L_1 \leq L$ such that both N and K are adjacent to L_1 in $\mathbb{SAG}(M)$. Thus $N - L_1 - K$ is a path in $\mathbb{SAG}(M)$ and hence $d_S(N,K) = 2$.

(3). Let $d_A(N,K) = 3$ and $N - L - T - K$ be a path in $\mathbb{AG}(M)$. Clearly, $d_S(N,K) \geq 3$. Since $d_A(N,T) = 2$, $d_S(N,T) = 2$ and so $N - T_1 - T$ is a path in $\mathbb{SAG}(M)$ for some vertex T_1. Clearly, T_1 and K are not adjacent in $\mathbb{AG}(M)$ and since $T_1 - T - K$ is a path in $\mathbb{AG}(M)$, $d_A(T_1,K) = 2$ and so $d_S(T_1,K) = 2$. Therefore $T_1 - T_2 - K$ is a path in $\mathbb{SAG}(M)$, for some vertex T_2 and hence we have the path $N - T_1 - T_2 - K$ in $\mathbb{SAG}(M)$. □

For a given graph G, we use the notations $\mathrm{diam}(G)$ and $\mathrm{gr}(G)$ for the diameter and the girth of G, respectively. Also the vertex set of G is denoted by $V(G)$.

Theorem 2.4. *$\mathbb{SAG}(M)$ is a connected graph with $\mathrm{diam}(\mathbb{SAG}(M)) \leq 3$.*

Proof. Since $V(\mathbb{SAG}(M)) = V(\mathbb{AG}(M))$, for any two vertices N and K in $\mathbb{SAG}(M)$, by [8, Theorem 3.4], $d_A(N,K) \leq 3$. Now by Lemma 2.3, $d_S(N,K) \leq 3$ and the proof is complete. □

Theorem 2.5. *If $\mathbb{SAG}(M)$ contains a cycle, then $\mathrm{gr}(\mathbb{SAG}(M)) \leq 4$.*

Proof. Let $N_1 - N_2 - \cdots - N_n - N_1$ be a cycle in $\mathbb{SAG}(M)$ and set $L = N_1 \cap N_3$. If $L = 0$, then $N_1 - N_3$ is an edge and so $N_1 - N_2 - N_3 - N_1$ is a cycle. Thus we may assume $L \neq 0$ and consider the following cases:

(a) $L = N_1$. Then $N_1 \subseteq N_3$ and since $N_3 - N_4$ is an edge by Lemma 2.1(1), $N_1 - N_4$ is also an edge. Hence $N_1 - N_2 - N_3 - N_4 - N_1$ is a cycle of length 4.

(b) $L = N_2$. Then $N_2 \subseteq N_3$ and since $N_3 - N_4$ is an edge by Lemma 2.1(1), $N_2 - N_4$ is also an edge. Hence $N_2 - N_3 - N_4 - N_2$ is a cycle of length 3.

(c) $L = N_3$. Then $N_3 \subseteq N_1$ and since $N_1 - N_n$ is an edge by Lemma 2.1(1), $N_3 - N_n$ is also an edge. Hence $N_1 - N_2 - N_3 - N_n - N_1$ is a cycle of length 4.

(d) $L = N_4$. Then $N_4 \subseteq N_3$ and since $N_2 - N_3$ is an edge by Lemma 2.1(1), $N_2 - N_4$ is also an edge. Hence $N_2 - N_3 - N_4 - N_2$ is a cycle of length 3.

(e) $L \notin \{N_1, N_2, N_3, N_4\}$. Then L is adjacent to both N_2 and N_4. Thus $L - N_2 - N_3 - N_4 - L$ is a cycle of length 4. □

In the following result, we provide a sufficient condition for existence a cycle in $\mathbb{SAG}(M)$.

Proposition 2.6. *If $\mathbb{SAG}(M)$ contains a path of length 4, then $\mathbb{SAG}(M)$ contains a cycle.*

Proof. Let $N_1-N_2-N_3-N_4-N_5$ be a path of length 4. If $N_2 \cap N_4 = 0$, then $N_2-N_3-N_4-N_2$ is a cycle. Thus we assume that $N_2 \cap N_4 \neq 0$ and set $L = N_2 \cap N_4$. We consider the following cases:

Case 1: $L = N_1$. Then by Lemma 2.1(1), $N_1 - N_2 - N_3 - N_1$ is a cycle.
Case 2: $L = N_2$. Then by Lemma 2.1(1), $N_2 - N_3 - N_4 - N_5 - N_2$ is a cycle.
Case 3: $L = N_3$. Then by Lemma 2.1(1), $N_1 - N_2 - N_3 - N_1$ is a cycle.
Case 4: $L = N_4$. Then by Lemma 2.1(1), $N_1 - N_2 - N_3 - N_4 - N_1$ is a cycle.
Case 5: $L = N_5$. Then by Lemma 2.1(1), $N_3 - N_4 - N_5 - N_3$ is a cycle.
Case 6: $L \notin \{N_1, N_2, N_3, N_4, N_5\}$. Then by Lemma 2.1(1), L is adjacent to both N_1 and N_3. Thus $N_1 - N_2 - N_3 - L - N_1$ is a cycle. □

An R-module M is called *prime* if $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$, for any nonzero submodule N of M. Also the empty graph is denoted by K_0.

Proposition 2.7. *The following statements are equivalent:*
(1) $\mathbb{SAG}(M)$ *is the empty graph;*
(2) M *is a uniform R-module,* $\mathrm{ann}(M)$ *is a radical ideal and M is not a vertex;*
(3) $\mathrm{ann}_R(M)$ *is a prime ideal and M is not a vertex;*
(4) M *is a prime module and M is not a vertex.*

Proof. (1) ⇒ (2). Let $\mathbb{SAG}(M) = K_0$. Then by Lemma 2.1(2), $N \cap K \neq 0$, for all nonzero submodules N and K of M, This implies that M is a uniform R-module. Now suppose that I and J are two ideals of R such that $IJ \subseteq \mathrm{ann}(M)$, but $MI \neq 0$ and $MJ \neq 0$. Since $MI(MJ : M) \subseteq MIJ = 0$, MI and MJ must be vertices, a contradiction. Thus $MI = 0$ or $MJ = 0$ and so $\mathrm{ann}(M)$ is a prime ideal. It follows that $\mathrm{ann}(M)$ is a radical ideal.
(2) ⇒ (1). Suppose that N is a vertex of $\mathbb{SAG}(M)$. Then there exists a vertex K such that $N(K : M) = 0$ or $K(N : M) = 0$. If $K = N$, then $N(N : M) = 0$. Otherwise, $K \neq N$ and since M is uniform, $N \cap K \neq 0$ and hence $L(L : M) = 0$, where $L = N \cap K$. Thus in any case, there exists a vertex $L \leq N$ such that $L(L : M) = 0$. Now, $M(L : M)^2 = M(L : M)(L : M) \subseteq L(L : M) = 0$ and since $\mathrm{ann}(M)$ is radical, $M(L : M) = 0$. This means that M is a vertex, a contradiction.
(1) ⇒ (3). It is similar to the proof of (1) ⇒ (2).
(3) ⇒ (1). Suppose that N is a vertex of $\mathbb{SAG}(M)$. Then there exists a vertex K such that $N(K : M) = 0$ or $K(N : M) = 0$. If $K = N$, then $N(N : M) = 0$ and so $M(N : M)(N : M) = 0$. If $K \neq N$, then $M(N : M)(K : M) = 0$. In any case, since $\mathrm{ann}(M)$ is a prime ideal, $M(N : M) = 0$ or $M(K : M) = 0$ and hence M is a vertex , a contradiction.
(1) ⇒ (4). Suppose $NI = 0$, where N is a nonzero submodule of M and I is an ideal of R. If

$MI \neq 0$, then $MI(N:M)=0$ and so N is a vertex, a contradiction by (1). Thus $MI=0$ and so M is a prime R-module.

(4) $\Rightarrow$ (1). Suppose that N is a vertex of $\mathbb{SAG}(M)$. Then there exists a nonzero submodule K of M such that $N(K:M)=0$ or $K(N:M)=0$. Since M is prime, we have $M(K:M)=0$ or $M(N:M)=0$. Hence M is a vertex, respectively, a contradiction. We note that if $K=N$, then $N(N:M)=0$ and again since M is prime, $M(N:M)=0$, a contradiction. □

If $\mathbb{SAG}(R)=K_0$ and I is an ideal of R such that $I^2=0$, then I is a vertex in $\mathbb{SAG}(R)$, a contradiction. Thus R is an integral domain. Also if M is a simple R-module and $\mathbb{SAG}(M) \neq K_0$, then M is the only vertex in $\mathbb{SAG}(M)$. Thus we must have $0=M(M:M)=MR=M$, a contradiction; so $\mathbb{SAG}(M)=K_0$. Thus we have the following result.

Corollary 2.8. (1) $\mathbb{SAG}(R)=K_0$ *if and only if* R *is an integral domain.*
(2) *For any simple* R*-module* M, $\mathbb{SAG}(M)=K_0$.

Example 2.9. (1) $\mathbb{SAG}(\mathbb{Q})$ is the empty graph when we consider $\mathbb{Q}$ as a $\mathbb{Q}$-module. However, $\mathbb{SAG}(\mathbb{Q})$ is a complete graph when we consider $\mathbb{Q}$ as a $\mathbb{Z}$-module, because $(H:_{\mathbb{Z}}\mathbb{Q})=0$, for every $0 \neq H \lneqq \mathbb{Q}$.
(2) In $\mathbb{Z}_n$ as a $\mathbb{Z}$-module, every nonzero proper submodule is a vertex. To see this, let $n=p_1^{\alpha_1}\cdots p_t^{\alpha_t}$, where p_i's are distinct prime numbers. For every nonzero proper submodule $N=p_1^{\beta_1}\cdots p_t^{\beta_t}\mathbb{Z}_n$, we have $N(K:\mathbb{Z}_n)=0$, where $K=p_1^{\gamma_1}\cdots p_t^{\gamma_t}\mathbb{Z}_n$ with $\beta_i+\gamma_i=\alpha_i$, for $1\leq i\leq t$.
(3) Let $R=\mathbb{Z}$ and $M=\mathbb{Z}_p\oplus\mathbb{Z}_p$, where p be a prime number. For every $0\neq N \lneq M$, we have $(N:M)=p\mathbb{Z}$ and so $K(N:M)=0$ for every nonzero submodule K of M. Hence $\mathbb{SAG}(M)$ is a complete graph with $p+2$ vertices, for more details see [16].

Proposition 2.10. *The ring* R *is a field if and only if for every* R*-module* M*, either* $\mathbb{SAG}(M)=K_0$ *or* $\mathbb{SAG}(M)$ *is a nonempty complete graph.*

Proof. ($\Rightarrow$) Let R be a field and M be an R-module. If $dim(M_R)=1$, then by Corollary 2.8, $\mathbb{SAG}(M)=K_0$. If $dim(M_R)\geq 2$, then for every nonzero proper submodule N of M, $(N:M)=0$. Because $0\neq r\in(N:M)$ implies that $Mr\subseteq N$, and hence $M\subseteq Nr^{-1}\subseteq N\subseteq M$, a contradiction. Thus $K(N:M)=0$, for every nonzero submodule K of M and hence N is adjacent to each nonzero submodule of M.

($\Leftarrow$) Suppose that for every R-module M, $\mathbb{SAG}(M)=K_0$ or $\mathbb{SAG}(M)$ is a nonempty complete graph. Let I be a maximal ideal of R and put $M=\frac{R}{I}\oplus R$. Since $\mathbb{SAG}(M)\neq K_0$ and $\frac{R}{I}(R:M)=0$, we have $\frac{R}{I}(J:M)=0$, for every nonzero ideal J of R. Thus every nonzero ideal of R is a vertex. Hence by hypothesis, $J_1(J_2:M)=0$, for all distinct nonzero ideals J_1 and J_2 of R. Since $IJ_2\subseteq(J_2:M)$, we have $J_1IJ_2=0$. Thus for any $r\in R\setminus\{0,1\}$, we have $RIRr=0$ and $RIR(1-r)=0$. Because R and Rr are distinct nonzero ideals of R and two ideals R and $R(1-r)$ are as well. Therefore we conclude that $Ir = 0$ and $I(1-r) = 0$, for all $r \notin \{0, 1\}$. This implies that $I = 0$ and so R is a field. □

Let $M = \oplus_I S_i$ be a finitely generated semisimple R-module. If we set $M_\lambda = \Sigma_{i \in I_\lambda} S_i$, where $I_\lambda \subseteq I$ is maximal with respect to the property that $S_i \cong S_j$, for all $i, j \in I_\lambda$, then $M = \oplus_\lambda M_\lambda$ and each M_λ is said a homogenous component of M.

Proposition 2.11. *Let M be a finitely generated semisimple R-module such that its homogeneous components are simple and let N, K be two nonzero submodules of M. Then N and K are adjacent if and only if $N \cap K = 0$.*

Proof. By Lemma 2.1(2), the " if " part is obvious. Suppose that $M = \oplus_I S_i$, where S_i's are non isomorphic simple submodules of M and N, K are adjacent. On the contrary, assume that $N \cap K \neq 0$. By [6, Proposition 9.4], there exist subsets I_1 and I_2 of I such that $N \cong \oplus_{I_1} S_i$, $K \cong \oplus_{I_2} S_i$ and $M/K \cong \oplus_{I \setminus I_2} S_i$. Since N and K are adjacent, without loss of generality, we may assume $N(K : M) = 0$. Then $\Pi_{I \setminus I_2} \mathrm{ann}_R(S_i) \subseteq \cap_{I \setminus I_2} \mathrm{ann}_R(S_i) = (K : M) \subseteq \mathrm{ann}_R(N) = \cap_{I_1} \mathrm{ann}_R(S_i)$. We note that for any $i \in I$, $\mathrm{ann}_R(S_i)$ is a prime (maximal) ideal of R. Thus for any $j \in I_1$, there exists $i_j \in I \setminus I_2$ such that $\mathrm{ann}_R(S_{i_j}) \subseteq \mathrm{ann}_R(S_j)$ and hence $\mathrm{ann}_R(S_{i_j}) = \mathrm{ann}_R(S_j)$. This implies that $S_{i_j} \cong S_j$, because S_i is a simple R-module for any $i \in I$. On the other hand, $N \cap K$ contains a simple submodule, say, T. Again by [6, Proposition 9.4], there exist $\alpha \in I_1$ and $\beta \in I_2$ such that $T \cong S_\alpha \cong S_\beta$. Since $\alpha \in I_1$, there exists $i_\alpha \in I \setminus I_2$ such that $S_{i_\alpha} \cong S_\alpha$. Thus we have $S_{i_\alpha} \cong S_\beta$, a contradiction. □

Corollary 2.12. *If M is a finitely generated semisimple R-module such that its homogeneous components are simple, then $\mathbb{SAG}(M) = \mathbb{AG}(M)$.*

Proof. Suppose that N and K are adjacent in $\mathbb{AG}(M)$. By Lemma 2.2, N and K are adjacent in $\mathbb{SAG}(M)$ or there exists a nonzero submodule L of $N \cap K$ such that is adjacent to both N and K in $\mathbb{SAG}(M)$. If the latter case occurs, then by Proposition 2.11, we have $L \cap N = 0$, a contradiction. Thus N and K are adjacent in $\mathbb{SAG}(M)$ and the proof is complete. □

Remark 2.13. Let $M = \oplus_{i=1}^n S_i$, where S_i's be non isomorphic simple submodules of M. Then by Proposition 2.11, the vertex set of $\mathbb{SAG}(M)$ is $\{\oplus_{i=1}^n S'_i \mid S'_i = S_i \text{ or } S'_i = 0, \text{ for any } i\} \setminus \{0, M\}$. Thus we have

$$|V(\mathbb{SAG}(M))| = \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} = 2^n - 2.$$

Also the degree of the vertex $\oplus_{i=1}^{n} S'_i$ is equal to $2^{n-k}-1$, where k is the number of nonzero components in $\oplus_{i=1}^{n} S'_i$. Since S_i is adjacent to S_j, for each $i \neq j$, the graph $\mathbb{SAG}(M)$ includes a complete subgraph with n vertices. Thus if $n = 2$, $\mathrm{gr}(\mathbb{SAG}(M)) = \infty$ and if $n \geq 3$, $\mathrm{gr}(\mathbb{SAG}(M)) = 3$; in particular, it is easy to see that the clique number of $\mathbb{SAG}(M)$ is equal to n. Moreover $\mathrm{diam}(\mathbb{SAG}(M)) = 3$, because $S_1 \oplus S_2 \oplus \ldots \oplus S_{n-1} - S_n - S_1 - S_2 \oplus S_3 \oplus \ldots \oplus S_n$ is a path of length 3.

Proposition 2.14. *The following statements hold.*

(1) *If M is a prime R-module, then $\mathbb{SAG}(M) = K_0$ or M is a vertex. In particular;* $\mathrm{diam}(\mathbb{SAG}(M)) \leq 2$.

(2) *If M is a semisimple R-module, then $\mathbb{SAG}(M) = K_0$ or every nonzero proper submodule of M is a vertex.*

(3) *If M is a semisimple prime R-module, then $\mathbb{SAG}(M) = K_0$ or $\mathbb{SAG}(M)$ is a complete graph and M is a vertex.*

(4) *If M is a nonsimple homogenous semisimple R-module, then $\mathbb{SAG}(M)$ is a complete graph such that every nonzero submodule of M is a vertex.*

Proof. (1). Suppose that $\mathbb{SAG}(M) \neq K_0$ and N is a vertex in $\mathbb{SAG}(M)$. Then $N(T:M) = 0$ or $T(N:M) = 0$, for a nonzero submodule T of M. Since M is a prime R-module, $M(T:M) = 0$ or $M(N:M) = 0$. This implies that M is a vertex and moreover $\mathrm{diam}(\mathbb{SAG}(M)) \leq 2$.
(2). Suppose that $\mathbb{SAG}(M) \neq K_0$ and N is a nonzero proper submodule of M. Then $M = N \oplus K$, for a nonzero proper submodule K of M. Clearly, N is adjacent to K.
(3). Suppose that $\mathbb{SAG}(M) \neq K_0$ and $M = \oplus_I S_i$, where S_i's are simple submodule of M. Since M is prime, $\mathrm{ann}(M) = \mathrm{ann}(S_i)$, for any $i \in I$. Now if N is a nonzero submodule of M, then by [6, Proposition 9.4], $M/N \cong \oplus_J S_i$, for some $\emptyset \neq J \subseteq I$. Thus $\mathrm{ann}(M/N) = \mathrm{ann}(M)$ and so $M(N:M) = 0$. Therefore $K(N:M) = 0$, for any nonzero submodule K of M. It follows that $\mathbb{SAG}(M)$ is complete.
(4). Suppose that $M = \oplus_I S_i$, where S_i's are isomorphic simple R-modules and $|I| \geq 2$. It is clear that for any i, $\mathrm{ann}(M) = \mathrm{ann}(S_i)$ is a maximal ideal of R. Therefore for every nonzero proper submodule N of M, we have $(N:M) = \mathrm{ann}(M)$ and so $K(N:M) = 0$, for every nonzero submodule K of M. Thus N is adjacent to each nonzero submodule of M. □

If we set $R = \mathbb{Z}$ and $M = (\oplus_I \mathbb{Z}_2) \oplus (\oplus_J \mathbb{Z}_3)$ such that $|I| \geq 2$ and $|J| \geq 2$, then $(\oplus_I \mathbb{Z}_2) \oplus \mathbb{Z}_3$ and $\oplus_I \mathbb{Z}_2$ are vertices, but not adjacent. Thus the being homogenous property is required in Proposition 2.14(4). Also the following example shows that the converse of part (3) is not true.

Example 2.15. In $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module, since for every proper submodule H of M, $M/H \cong M$ and $\operatorname{ann}(M) = 0$, we have $(H : M) = 0$ and so $K(H : M) = 0$ for each submodule K of M. Thus $\mathbb{SAG}(M)$ is a complete graph and also M is a vertex.

An R-module M is called a *comultiplication* module if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$, i.e., $N = (0 :_M \operatorname{ann}(N))$.

Proposition 2.16. (1) *If $M = M_1 \oplus M_2$, where M_1, M_2 are nonzero submodule of M, then every nonzero submodule of M_1 is adjacent to every nonzero submodule of M_2.*
(2) *If $\mathbb{SAG}(M) = K_0$, then M is an indecomposable module.*
(3) *If M is a non simple semisimple R-module, then every nonzero proper submodule of M is a vertex.*
(4) *A nonzero submodule N of M is a vertex in $\mathbb{SAG}(M)$ if $\operatorname{ann}(N) \neq \operatorname{ann}(M)$ or $(0 :_M (N : M)) \neq 0$.*
(5) *If M is a multiplication module, then $0 \neq N \leq M$ is a vertex in $\mathbb{SAG}(M)$ if and only if $(0 :_M (N : M)) \neq 0$. In particular, if M is a cyclic module, then M is not a vertex.*
(6) *If M is a multiplication module, then every nonzero proper submodule of M is a vertex if MI is a vertex for every maximal ideal I of R.*
(7) *In a comultiplication module, every nonzero proper submodule is a vertex.*
(8) *If M is a not vertex, then a nonzero submodule N of M is a vertex in $\mathbb{SAG}(M)$ if and only if $(0 :_M (N : M)) \neq 0$.*

Proof. (1), (2) and (3) are easy.
(4). Since by Lemma 2.2, for any submodule N of M, we have N is a vertex in $\mathbb{SAG}(M)$ if and only if N is a vertex in $\mathbb{AG}(M)$, the result is obtained by [8, Proposition 3.2].
(5). In a multiplication module M, we have $\mathbb{AG}(M) = \mathbb{SAG}(M)$. Now the result is obtained by [8, Proposition 3.2]. The "in particular" statement follows from this fact that every cyclic module is multiplication.
(6). Assume that for every maximal ideal I of R, MI is a vertex and N is a nonzero proper submodule of M. Since M is multiplication, $N = MJ$ for some proper ideal J of R. Then $N = MJ \subseteq MI$, for some maximal ideal I of R. Now since MI is a vertex, N is also a vertex.
(7). If M is a comultiplication module, then $\operatorname{ann}(N) \neq \operatorname{ann}(M)$, for any nonzero proper submodule N of M. Now we are done by applying (4).
(8). It is clear by Lemma 2.2 and [8, Theorem 3.3]. □

Proposition 2.17. *Let R be an Artinian ring and M be a finitely generated R-module. Then every nonzero proper submodule N of M is a vertex in $\mathbb{SAG}(M)$.*

Proof. It is immediate from Lemma 2.2 and [8, Proposition 3.5]. □

Proposition 2.18. *Every vertex in* $\mathbb{SAG}(M)$ *has finite degree if and only if every vertex in* $\mathbb{AG}(M)$ *has finite degree.*

Proof. The " if " part is clear. For the "only if" part, we assume that every vertex in $\mathbb{SAG}(M)$ has finite degree and N is a vertex in $\mathbb{AG}(M)$ with infinite degree. We denote the set of neighbors of vertex N in graphs $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$ by $\mathrm{N}_A(N)$ and $\mathrm{N}_S(N)$, respectively. Suppose that $\mathrm{N}_S(N) = \{N_1, \cdots, N_t\}$ and $\{K_1, K_2, \cdots\} \subseteq \mathrm{N}_A(N) \setminus \mathrm{N}_S(N)$. By Lemma 2.2, for any i, there exists a nonzero submodule T_i of $N \cap K_i$ such that is adjacent to both N and K_i in $\mathbb{SAG}(M)$. Since $\mathrm{N}_S(N) = \{N_1, \cdots, N_t\}$, there exists $1 \leq m \leq t$ such that $N_m = T_i$, for infinite number of i. This implies that N_m is adjacent to K_i in $\mathbb{SAG}(M)$, for infinite number of i. Thus the degree of N_m in $\mathbb{SAG}(M)$ is infinite, a contradiction. □

Theorem 2.19. *Let R be a reduced ring, and M be a faithful R-module which is not prime. Then the following statements are equivalent:*

(1) $\mathbb{SAG}(M)$ *is a finite graph;*

(2) M *has only finitely many submodules;*

(3) *Every vertex of* $\mathbb{SAG}(M)$ *has finite degree;*

Consequently, if one of the these conditions holds, then $\mathbb{SAG}(M)$ *has n vertices if and only if M has only n nonzero proper submodules.*

Proof. The proof is obtained by [8, Theorem 3.7] and Proposition 2.18. □

Theorem 2.20. *For any finitely generated faithful multiplication R-module M,* $\mathbb{AG}(M) = \mathbb{SAG}(M) \cong \mathbb{AG}(R)$.

Proof. Suppose that $IJ = 0$, where I and J are two ideals of R. Then $MI(MJ : M) = M(MJ : M)I \subseteq MIJ = 0$. On the other hand, if $N(K : M) = 0$, where N and K are two submodules of M, then since M is multiplication, $N = MI$ and $K = MJ$ for some ideals I and J of R and we have $MIJ \subseteq MI(MJ : M) = N(K : M) = 0$. Now, since M is faithful, we conclude that $IJ = 0$. Also, we note that if $MI = MJ$, for some ideals I and J, then by [13, Theorem 3.1], $I = J$. Thus there exists a one to one correspondence between the set of all ideals of R and the set of all submodules of M. Therefore $\mathbb{SAG}(M) \cong \mathbb{AG}(R)$. □

Lemma 2.21. *Let N be a vertex of* $\mathbb{SAG}(M)$ *such that $(N : M)$ is a maximal ideal of R. Then $(N : M) \in Ass(M)$ or every nonzero proper submodule of M is a vertex.*

Proof. We note that every submodule of M is a vertex in $\mathbb{SAG}(M)$ if and only is a vertex in $\mathbb{AG}(M)$. Now the result follows from [8, Lemma 3.8]. $\square$

Let M be a right R-module. The *socle* of M, denoted by $soc(M)$, is the sum of all simple submodules of M and if there are no simple submodules, we write $soc(M) = 0$. The set of all nonzero submodules of M is denoted by $S(M)$ and we use α for the cardinality of $S(M)$. Also the complete graph with α vertices is denoted by K_α.

Proposition 2.22. *We have exactly one of the following assertions in* $\mathbb{SAG}(M)$*.*
(1) *Every nonzero proper submodule of M is a vertex.*
(2) *There exists a maximal ideal m of R such that Mm is a vertex if and only if $soc(M) \neq 0$.*

Proof. It is immediate by [8, Proposition 3.9] and this fact that $V(\mathbb{SAG}(M)) = V(\mathbb{AG}(M))$. $\square$

Proposition 2.23. *The following statements hold.*
(1) *Let M be a prime module with $soc(M) \neq 0$. Then $\mathbb{SAG}(M) = K_0$ or $\mathbb{SAG}(M) = K_\alpha$.*
(2) *Let M be a non simple with $soc(M) \neq 0$. Then $\mathbb{SAG}(M) \neq K_0$. In particular; $\mathbb{SAG}(M) \neq K_0$ when M be a non simple Artinian module.*
(3) *Let M be a non simple with $soc(M) \leq_e M$. Then every nonzero submodule of M contains a submodule that is a vertex.*

Proof. (1) If M is a simple module, then $\mathbb{SAG}(M) = K_0$. Otherwise, since $soc(M) \neq 0$, there exists a proper simple submodule L of M. As M is prime, $\text{ann}(M) = \text{ann}(L)$ and so $\text{ann}(M)$ is a maximal ideal of R. Thus $\text{ann}(M) = (N : M)$, for every nonzero proper submodule N of M. This implies that every two nonzero distinct submodules of M are adjacent.
(2) Suppose that M is a non simple module with $soc(M) \neq 0$. Then there exists a simple submodule xR of M, where $0 \neq x \in M$. Now $\text{ann}(x)$ is a maximal ideal of R and we have $M\text{ann}(x)(xR : M) \subseteq xR(\text{ann}(x)) = 0$. Thus if $M\text{ann}(x) \neq 0$, then xR is a vertex of $\mathbb{SAG}(M)$ and so $\mathbb{SAG}(M) \neq K_0$. Now, we assume that $M\text{ann}(x) = 0$. Then since $\text{ann}(x)$ is a maximal ideal of R, we have $\text{ann}(M) = \text{ann}(x)$. Therefore $\text{ann}(M) = (xR : M)$ and so $M(xR : M) = 0$. This shows that xR is vertex; so $\mathbb{SAG}(M) \neq K_0$.
(3) Let N be a nonzero submodule of M and $soc(M) \leq_e M$. Then $soc(M) \cap N = soc(N) \neq 0$. If $soc(N) \lneq soc(M)$, then $soc(N)$ is a vertex because $soc(M)$ is a semisimple R-submodule. If $soc(N) = soc(M)$, then $soc(M) \subseteq N$ and hence $N \leq_e M$. Now since $soc(M) \neq 0$, by part (2), there exists a submodule K of M such that K is a vertex. Thus $N \cap K \neq 0$ and so $N \cap K$ is a vertex contained N. $\square$

Proposition 2.24. *Let M be a non simple prime module. Then $\mathbb{SAG}(M) = K_\alpha$ if and only if every nonzero proper submodule of M is adjacent to M.*

Proof. Since M is prime, $\mathrm{ann}(M) = \mathrm{ann}(N)$, for any nonzero submodule N of M. Thus $M(N : M) = 0$ if and only if $K(N : M) = 0$, where N and K are nonzero proper submodule of M. □

3. $\mathbb{SAG}(M)$ and reduced modules

Recall that an R-module M is said to be *reduced* if $r^2m = 0$, for $r \in R$ and $m \in M$, then $rm = 0$.

Lemma 3.1. *Let M be a reduced R-module. If $N(N : M) = 0$, for some nonzero proper submodule N of M, then $M(N : M) = 0$; in particular, M is a vertex in $\mathbb{SAG}(M)$.*

Proof. Let $x \in M$ and $r \in (N : M)$. Then $xr \in N$ and so $xr^2 \in N(N : M) = 0$. Since M is reduced, $xr = 0$. This implies that $M(N : M) = 0$. □

Lemma 3.2. *Let M be a reduced R-module with $M \notin V(\mathbb{SAG}(M))$. If $\mathbb{SAG}(M)$ is a bipartite graph with parts V_1 and V_2, then $\overline{V_i} = \cup_{N \in V_i} N$ is a submodule of M, for $i = 1, 2$.*

Proof. Let $x_1, x_2 \in \overline{V_1}$ and $r \in R$. Then $x_1 \in N_1$ and $x_2 \in N_2$ for some $N_1 \in V_1$ and $N_2 \in V_1$; so $x_1r \in N_1 \subseteq \overline{V_1}$. Now we have to show that $x_1 + x_2 \in \overline{V_1}$. Since $N_1, N_2 \in V_1$, there exist $K_1, K_2 \in V_2$ such that N_i is adjacent to K_i for $i = 1, 2$. By Lemma 2.1(2), $L := K_1 \cap K_2 \neq 0$. If $L = N_1$, then $N_1(N_1 : M) = 0$ because N_1 is adjacent to K_1. By Lemma 3.1, we have $M \in V(\mathbb{SAG}(M))$, a contradiction. Similarly, if $L = N_2$, we get a contradiction. Since L is adjacent to N_1 and N_2, we must have $L \in V_2$. Now we show that $N_i \cap L = 0$, for $i = 1, 2$. If $N_1 \cap L \neq 0$, then by Lemma 2.1(1), $N_1 \cap L$ is adjacent to both L and N_1 (since M is reduced, by Lemma 3.1, it is easy to cheek that $N_1 \cap L \neq L$ and $N_1 \cap L \neq N_1$). Thus $N_1 \cap L \in V_1 \cap V_2 = \emptyset$, a contradiction. Similarly, $N_2 \cap L = 0$. Hence for $i = 1, 2$, we have $N_i(L : M) \subseteq N_i \cap L = 0$ and so $(N_1 + N_2)(L : M) = 0$ (again by Lemma 3.1, we note that $N_1 + N_2 \neq L$). Therefore $N_1 + N_2$ is adjacent to L and hence $x_1 + x_2 \in N_1 + N_2 \in V_1 \subseteq \overline{V_1}$. □

With notations as in above lemma, we have the following.

Theorem 3.3. *Let M be a reduced R-module with $M \notin V(\mathbb{SAG}(M)$. If $\mathbb{SAG}(M)$ is a bipartite graph, then the following statements hold.*
(1) $\mathbb{SAG}(M)$ *is a complete bipartite graph.*
(2) $\mathrm{u.dim} M = 2$.

Proof. (1) Let $V(\mathbb{SAG}(M)) = V_1 \cup V_2$ where $V_1 \cap V_2 = \emptyset$ and no two elements of V_i are adjacent for $i = 1, 2$. Assume that $N_1 \in V_1$ and $K_2 \in V_2$. Then there exist $K_1 \in V_2$ and $N_2 \in V_1$ such that N_i is adjacent to K_i, for $i = 1, 2$. If $N_1 \cap K_2 = 0$, then by Lemma 2.1(2), N_1 is adjacent to K_2 and the proof is complete. Now suppose that $N_1 \cap K_2 \neq 0$. First we show that $N_1 \cap K_2 \notin \{K_1, N_2\}$. If $N_1 \cap K_2 = K_1$, then $K_1 \subseteq N_1$ and since N_1 is adjacent to K_1, we have $K_1(K_1 : M) = 0$. Now by Lemma 3.1, M is vertex, a contradiction. Similarly, $N_1 \cap K_2 \neq N_2$. Since N_i is adjacent to K_i, for $i = 1, 2$, we have N_2 is adjacent to $N_1 \cap K_2$ and also K_1 is adjacent to $N_1 \cap K_2$. Thus $N_1 \cap K_2 \in V_1 \cap V_2$, a contradiction. Therefore $N_1 \cap K_2 = 0$ and hence N_1 and K_2 are adjacent; so $\mathbb{SAG}(M)$ is a complete bipartite graph.
(2) We first show that $\overline{V_1} \cap \overline{V_2} = 0$. Suppose that $x \in \overline{V_1} \cap \overline{V_2}$. Then $x \in N_1$ and $x \in K_2$, for some $N_1 \in V_1$ and $N_2 \in V_2$. Thus there exist $K_1 \in V_2$ and $N_2 \in V_1$ such that N_1 adjacent K_1 and N_2 adjacent K_2. Since M is reduced, by Lemma 3.1, we can check that $N_1 \cap K_2 \notin \{K_1, N_2\}$. On the other hand, by Lemma 2.1(1), $N_1 \cap K_2$ is adjacent to both K_1 and N_2. Thus $N_1 \cap K_2 \in V_1 \cap V_2$, a contradiction. Next we claim that $\overline{V_i}$'s are uniform submodules of M. For see this, if T_1 and T_2 are two nonzero submodules of $\overline{V_1}$ such that $T_1 \cap T_2 = 0$, then T_1 and T_2 are adjacent. Without loss of generality, we assume that $T_1 \in V_1$ and $T_2 \in V_2$. Then $T_2 \subseteq \overline{V_2}$. This implies that $T_2 \in \overline{V_1} \cap \overline{V_2}$, a contradiction. Therefore $\overline{V_1}$ is a uniform submodule of M and similarly, $\overline{V_2}$ is uniform. To the complete of proof, we show that $\overline{V_1} \oplus \overline{V_2}$ is essential in M. Suppose that K is a submodule of M such that $K \cap \overline{V_1} \oplus \overline{V_2} = 0$. Then by Lemma 2.1(1), K is adjacent to every element of V_1 and V_2. Thus $K \in V_1 \cap V_2$, a contradiction. □

Corollary 3.4. *Let M be a reduced R-module with $M \notin V(\mathbb{SAG}(M))$. Then $\mathbb{SAG}(M)$ contains no cycle if and only if $\mathbb{SAG}(M)$ is a star graph.*

Proof. The one direction is trivial. For the other direction, assume that $\mathbb{SAG}(M)$ has no cycles. Then $\mathbb{SAG}(M)$ is a tree and so it is a bipartite graph. Now by Theorem 3.3, $\mathbb{SAG}(M)$ is a complete bipartite graph. Since $\mathbb{SAG}(M)$ has no cycles, we conclude that at least one of the partitions of graph is singleton, as desired. □

Lemma 3.5. *If $\mathbb{SAG}(M)$ contains a cycle of odd length, then $\mathbb{SAG}(M)$ contains a triangle.*

Proof. Using induction, we show that for every cycle of odd length $2n + 1 \geq 5$, there exists a cycle with length $2k + 1$ such that $k < n$. Assume that $N_1 - N_2 - \cdots - N_{2n+1} - N_1$ is a cycle with odd length $2n + 1$. If two distinct non consecutive N_i and N_j are adjacent, the proof is complete. Otherwise, we set $0 \neq L = N_1 \cap N_3$. Then by Lemma 2.1(1), $L \neq N_i$

for all $1 \leq i \leq 2n+1$ and L is adjacent to both N_4 and N_{2n+1}. Hence we have the cycle $N_{2n+1} - L - N_4 - N_5 - \cdots - N_{2n+1}$, which is the desired cycle. □

Proposition 3.6. *For any R-module M, if $\mathrm{gr}(\mathbb{SAG}(M)) = 4$, then $\mathbb{SAG}(M)$ is a bipartite graph such that its parts are not singleton. The converse is true, if M is a reduced module with $M \notin V(\mathbb{SAG}(M))$.*

Proof. Let $\mathrm{gr}(\mathbb{SAG}(M)) = 4$. By Lemma 3.5, we observe that the length of any cycle in $\mathbb{SAG}(M)$ is even. Thus by [12, Proposition 1.6.1], $\mathbb{SAG}(M)$ is a bipartite graph and since has a cycle of length 4, the proof is immediate. The converse follows from Theorem 3.3. □

4. $\mathbb{SAG}(M)$ and divisible modules

Let M be an R-module. The submodule N of M is called *divisible* if $Nr = N$, for each $0 \neq r \in R$. Also M is called *second* if $MI = M$ or $MI = 0$, for each ideal I of R. It is easy to see that M is a second module if and only if $\mathrm{ann}(M) = (N : M)$, for every proper submodule N of M. Clearly, if M is a second R-module, then $\mathrm{ann}(M)$ is a prime ideal of R, for more details, see [11].

Theorem 4.1. *Consider the following statements.*
(1) $\mathrm{ann}(M)$ *is a prime ideal and M is a divisible $R/\mathrm{ann}(M)$-module.*
(2) *Every nonzero proper submodule of M is adjacent to M.*
(3) *M is a second module.*
(4) $\mathbb{SAG}(M) = K_\alpha$*, where $\alpha = |S(M)|$.*
(5) *M is a non simple homogeneous semisimple module.*
Then we have (1) ⇔ (2) ⇔ (3) ⇔ (4) *and* (5) ⇒ (3). *Moreover, if M is a finitely generated R-module or R is an Artinian ring, then we have* (3) ⇔ (5).

Proof. (1) ⇒ (3) It is sufficient to show that $(N : M) = \mathrm{ann}(M)$, for any proper submodule N of M. Assume that $r \in (N : M)$ and $Mr \neq 0$. Then $Mr \subseteq N$ and since M is a divisible $R/\mathrm{ann}(M)$-module $M = Mr \subseteq N$. Thus $M = N$, a contradiction. Hence $Mr = 0$ and so $(N : M) = \mathrm{ann}(M)$.
(3) ⇒ (4) Let N and K be two nonzero proper submodules of M. By (3), $(N : M) = \mathrm{ann}(M)$ and so $M(N : M) = 0$. Thus $K(N : M) = 0$ and hence $\mathbb{SAG}(M) = K_\alpha$.
(4) ⇒ (2) is clear.
(2) ⇒ (1) For any nonzero proper submodule N of M, we have $M(N : M) = 0$. Thus $(N : M) = \mathrm{ann}(M)$, i.e., M is a second R-module. This implies that $\mathrm{ann}(M)$ is a prime ideal of R. On the other hand, for any $r \notin \mathrm{ann}(M)$ we have $0 \neq Mr = MRr = MrR = M$, because

M is second.

(5) $\Rightarrow$ (3) Similar to proof of Proposition 2.23(1), we have $(N : M) = \text{ann}(M)$, for any proper submodule N of M, i.e., M is second.

(3) $\Rightarrow$ (5) Since M is second, $(N : M) = \text{ann}(M)$ for any nonzero proper submodule N of M and also $\text{ann}(M)$ is a prime ideal of R. Now if R is Artinian, $\text{ann}(M)$ is a maximal ideal of R. Also if M is a finitely generated R-module, $(N : M) = \text{ann}(M)$ is a maximal ideal of R, for some maximal submodule N of M. Thus in any case, $R/\text{ann}(M)$ is a field, and so M is a homogeneous semisimple module as an R/P-module and as an R-module, where $P = \text{ann}(M)$. $\square$

Proposition 4.2. *Let R be a ring and M be a divisible R-module. Then $\mathbb{SAG}(M) = K_0$ or $\mathbb{SAG}(M) = K_\alpha$, where $\alpha = |S(M)|$.*

Proof. If M is simple, then $\mathbb{SAG}(M) = K_0$. Thus we assume that M is not simple. For any nonzero proper submodule N of M, we have $(N : M) = 0$. Because if $(N : M) \neq 0$, then $M = M(N : M) \subseteq N$ and so $M = N$, a contradiction. Thus $(N : M) = 0$, and so $K(N : M) = 0$, for any submodule K of M. $\square$

Proposition 4.3. *Let M be an R-module which contains a nonzero divisible submodule, say N. Then $\mathbb{SAG}(M) = K_0$ or every nonzero submodule of M is a vertex in $\mathbb{SAG}(M)$.*

Proof. If there exists a nonzero proper submodule K of M such that $(K : M) = 0$, then $L(K : M) = 0$, for any $L \leq M$ and so every nonzero submodule of M is a vertex. Thus we assume that $(K : M) \neq 0$, for any $0 \neq K \leq M$. Now, if N is not minimal, then there exists $0 \neq K \lneqq N$ and since N is divisible, $N = N(K : M) \subseteq M(K : M) \subseteq K$, a contradiction. Thus we may assume that N is minimal. If $nr = 0$, for some $0 \neq n \in N$ and $0 \neq r \in R$, then $nRr = Nr = 0$, a contradiction. Thus $nr \neq 0$, for any $0 \neq n \in N$ and $0 \neq r \in R$. It follows that $R \cong nR = N$. For any $0 \neq r \in R$, since $Nr = N$, we conclude that $Rr = R$. Therefore R is a field and by Theorem 2.10, the proof is complete. $\square$

Corollary 4.4. *If M is a multiplication R-module which contains a nonzero divisible submodule, then $\mathbb{SAG}(M) = K_0$.*

Proof. If $\mathbb{SAG}(M) \neq K_0$, then by Proposition 4.3, M is a vertex in $\mathbb{SAG}(M)$ and so $M(N : M) = 0$, for a nonzero submodule N of M. Now, since M is multiplication, $N = M(N : M) = 0$, a contradiction. Thus $\mathbb{SAG}(M) = K_0$. $\square$

Proposition 4.5. *Let M be an R-module with $\mathbb{SAG}(M) = K_0$ and N be a nonzero submodule of M. Then:*
(1) *If N is a second submodule of M, then N is simple.*
(2) *If N is a divisible submodule of M, then R is a field.*

Proof. (1). On the contrary, assume that $0 \neq K \leq N$. If $(K : N) = 0$, then since $(K : M) \subseteq (K : N) = 0$, we have $(K : M) = 0$. This implies that K is a vertex, a contradiction. Thus $(K : N) \neq 0$. We note that $N(K : M) \subseteq N(K : N)$. Now if $N(K : N) = 0$, then $N(K : M) = 0$ and so K is a vertex, which again a contradiction. Therefore $N = N(K : N) \subseteq K$ because N is second. Thus $K = N$, and so N is a simple submodule of M.
(2). Let N be a divisible submodule of M. Since every divisible submodule is a second submodule, by part (1), N is simple. Thus $N = nR$, for any $0 \neq n \in N$. Now, clearly $r \to nr$ is an R-isomorphism of R into N. Since $Nr = N$, for any $0 \neq r \in R$, we have $R = Rr$ and hence R is a field. □

Proposition 4.6. *Let R be an integral domain and M be an R-module which contains a nonzero divisible submodule. If every submodule of M is cyclic, then $\mathbb{SAG}(M)$ is a complete graph.*

Proof. The proof follows from [15, Theorem 2.5]. □

Proposition 4.7. *Let S be an integral domain and R be a subring of S such that $|R| < |S|$. Then every nonzero R-submodule of S is a vertex in $\mathbb{SAG}(S_R)$.*

Proof. We show that $(sR :_R S) = 0$, for any $0 \neq s \in S$. If there exists $0 \neq r \in (sR :_R S)$, then $Sr \subseteq sR$. Since $|Sr| = |S|$ and $|sR| = |R|$, we must have $|S| \leq |R|$, a contradiction. □

We conclude the paper with the following result.

Corollary 4.8. *Let R be an integral domain and X be a set of commuting indeterminates over the ring R. If $|R| < |R[X]|$, then every nonzero R-submodule of $R[X]$ is a vertex in $\mathbb{SAG}(R[X]_R)$.*

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