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Research Paper

COMMUTATIVITY DEGREE AND NON-COMMUTING GRAPH IN FINITE GROUPS AND MOFANG LOOPS AND THEIR RELATIONSHIPS

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Abstract. Terms like commutativity degree, non-commuting graph and isoclinism are far well-known for much of the group theorists nowadays. There are so many papers about each of these concepts and also about their relationships in finite groups. Also, there are some recent researches about generalizing these notions in finite rings and their connexions.

The concepts of commutativity degree and non-commuting graph are also extended to non-associative structures such as Moufang loops and some part of the known results in group theory in these contexts have been expanded to them.

In this paper, we are going to generalize the notion of isoclinism in finite Moufang loops and then study the relationships between these three concepts. Among other results, we prove that two isoclinic finite Moufang loops have the same commutativity degree and if they have the same sizes of centers and commutants then they have isomorphic non-commuting graphs. Also, the converses of these results have been investigated. Furthermore, it has been proved that a finite simple group can be characterized by its non-commuting graph. We will prove the same is true for a finite simple Moufang loop by imposing one additional hypothesis, namely, the isoclinism of the regarding loops.

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1. Introduction

A set Q with one binary operation is called a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. A quasigroup Q is called a loop if Q possesses a neutral element e, i.e., if $ex = xe = x$ holds for every $x \in Q$. A loop Q is called a Moufang loop if any of the (equivalent) Moufang identities,

$$((xy)x)z = x(y(xz)), \quad (M1)$$
$$x(y(zy)) = ((xy)z)y, \quad (M2)$$
$$(xy)(zx) = x((yz)x), \quad (M3)$$
$$(xy)(zx) = (x(yz))x. \quad (M4)$$

holds for every $x, y, z \in Q$. Undoubtedly, Moufang loops are more studied loops than the other classes of loops. They appear naturally in algebra (as the multiplicative loop of octonions, and in projective geometry (Moufang planes) and other significant fields). They can be regarded as generalizations of groups and despite the lack of associativity, hold several properties of groups that everyone knows. For instance, each member x has a two-sided inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$; any two elements generate a subgroup (this property is called diassociativity); in finite Moufang loops, the order of an element divides the order of the loop; every finite Moufang loop of odd order is solvable; and as has been shown recently in [12], the order of a subloop divides the order of the loop (Lagrange's theorem in Moufang loop theory). Also, there are Sylow and Hall like theorems for finite Moufang loops. For more details see [6, 20, 9, 12, 13]. On the other hand, many basic and vital tools of group theory are not available for Moufang loops. For instance, the absence of associativity makes presentations very unusual and hard to calculate.

We give here some basic definitions in loop theory that we use later. For arbitrary elements x, y and z of a quasigroup Q, the commutator, $[x, y]$, and the associator, $[x, y, z]$, are defined by $xy = (yx)[x, y]$ and $((xy)z) = (x(yz))[x, y, z]$, respectively. By diassociativity and invertability in Moufang loops, in a Moufang loop M, we get the formal definitions of commutators and associators as $[x, y] = x^{-1}y^{-1}xy$ and $[x, y, z] = ((xy)z)^{-1}(x(yz))$. We define the commutant (or Moufang center, also called, centrum) $C(Q)$ of Q as $\{x \in Q \mid xy = yx, \quad \forall y \in Q\}$. The center $Z(Q)$ of a Moufang loop Q is the set of all elements of Q which commute and associate with all other elements of Q. A non-empty subset P of Q is called a subloop of Q if P be itself a loop under the binary operation of Q, in particular, if this operation is associative on P, then it is a subgroup of Q. A subloop N of a loop Q is said to be normal in Q if $xN = Nx$; $x(yN) = (xy)N$; $N(xy) = (Nx)y$; for every $x, y \in Q$. In Moufang loop Q, the subloops $Z(Q)$ and $C(Q)$ are normal subloops. The commutator (or derived) subloop, denoted by Q' is the least normal subloop of Q such that $\frac{Q}{Q'}$ is an abelian group. Hence Q' is the least normal subloop of Q containing all commutators $[x, y]$ and all associators $[x, y, z]$. A loop Q is said to be simple if it has no non-trivial normal subloops. For more details about loops and Moufang loops one may see [6, 20, 12].

Also, we recall some graph theory notions that we use in this paper. We will denote a complete graph with n vertices by K_n. All graphs considered in this paper are finite and simple and also don't have any loop or multiple edges. For a graph Γ, we denote its vertex and edge sets by $V(\Gamma)$ and $E(\Gamma)$, respectively. A clique in a graph Γ is an induced subgraph whose all vertices are pairwise adjacent. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. The vertex chromatic number of a graph Γ is denoted by $\chi(\Gamma)$ and it is the minimum k for which $k-$vertex coloring of a graph Γ such that no two adjacent vertices have the same color.

In a graph Γ, a path of edges and vertices wherein a vertex is reachable from itself is called a cycle. A graph cycle of length at least four in Γ that has no cycle chord (i.e., the graph cycle is an induced subgraph) is called a chordless cycle of Γ. A chordal graph is a simple graph possessing no chordless cycles. A perfect graph, Γ, is a graph in which for every induced subgraph its clique number is equal to its chromatic number. The class of chordal graphs is a subset of the class of perfect graphs. A graph Γ is called weakly perfect graph if $\omega(\Gamma) = \chi(\Gamma)$. So, all perfect graphs are weakly perfect. It has been proved that a graph is perfect if and only if neither the graph nor its complement has a chordless cycle of odd order (this theorem has been formally called *strongly perfect graph theorem* or *Berg theorem*, see [7], Theorem 1.2). A graph is called k-regular if the vertices of the graph are of the same degree k. Our other used notations about graphs are standard and for more details, one may see [5].

There are many papers on assigning a graph to a ring or a group in order to investigate their algebraic properties. For any non-abelian group G the non-commuting graph of G, $\Gamma = \Gamma_G$ is a graph with vertex set $G\backslash Z(G)$, where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. This graph is connected with diameter 2 and girth 3 for a non-abelian finite group and has received some attention in the existing literature. The order of groups in some classes of finite groups has been characterized by their non-commuting graphs (especially all finite simple groups and non-abelian nilpotent groups with irregular isomorphic non-commuting graphs), although the order of an arbitrary finite group cannot be characterized by its non-commuting graph. Also, although, in general, a finite group cannot be characterized by its non-commuting graph, however, it has been proved recently that a finite simple group can be characterized by its non-commuting graph. For more details, see [1, 8, 18, 21].

Similarly, the non-commuting graph of a finite Moufang loop has been defined by the second author of this paper in [4]. He has defined this graph as follows: *Let M be a Moufang loop, then $M\backslash C(M)$ as the vertex set of this graph and two vertices x and y joined by an edge whenever*

$[x, y] \neq 1$. He has shown that this graph is connected (as for groups) and obtained some results related to the non-commuting graph of a finite non-commutative Moufang loop. Then he has tried to characterize some finite non-commutative Moufang loops with their non-commuting graph. Also, he obtained some results related to the non-commuting graph of a finite non-commutative Moufang loop. Finally, he has given a conjecture stating that the above result (i.e. the characterization of finite simple groups by their non-commuting graphs) is true for all finite simple Moufang loops. In the sequel, we will prove this conjecture with an additional hypothesis.

For a finite algebraic structure A, with at least one binary operation like as "$\cdot$", the commutativity degree of A with respect to this operation is defined as:

$$Pr(A) = \frac{|\{(x, y) \in A^2 \mid x \cdot y = y \cdot x\}|}{|A^2|}.$$

In other words, the commutativity degree of an algebraic structure measures its amount of farness or closeness to be commutative. For a finite group A it is proved that $Pr(A) = \frac{k(A)}{|A|}$, where, $k(A)$ is the number of conjugacy classes of A (see [11, 16, 14] for example). The computational results on $Pr(A)$ are mainly due to Gustafson [11] who shows that $Pr(A) \leqslant \frac{5}{8}$ for a finite non-abelian group A, and MacHale [16] who proves this inequality for a finite non-abelian ring. Also, the second author of this paper and his colleagues have shown in [2] that the $\frac{5}{8}$ is not an upper bound for $Pr(A)$, where A is a finite non-commutative semigroup and/or monoid.

Now, let M be a finite non-commutative Moufang loop. The second author of this paper has extended this notion to finite Moufang loops and then for a finite Moufang loop M, tried to give the best upper bound for $Pr(M)$. Also, he has obtained some results related to the $Pr(M)$ and asked some similar questions raised and answered in group theory about the relations between the structure of a finite group and its commutativity degree in finite Moufang loops. It has been proved that for a well-known class of finite Moufang loops, called Chein loops, the best upper bound for the commutativity degree is $\frac{23}{32}$ and conjectured (by presenting considerable evidence) that it can be extended to all finite Moufang loops [3].

In this paper, we are going to generalize the notion of isoclinism in finite Moufang loops and then study the relationships between these three concepts. Among other results, we prove that two isoclinic finite Moufang loops have the same commutativity degree and if they have the same sizes of centers and commutants then they have isomorphic non-commuting graphs. Also, the converses of these results have been investigated.

2. Generalizing Some Known Facts About Isoclinism, Non-commuting Graph and Commutativity Degree of Groups to Moufang Loops

In this section, we are going to generalize some known facts about non-commuting graph and commutativity degree of finite groups to Moufang loops. Two groups are said to be isoclinic if there is an isoclinism between them, i.e., there is an isomorphism between their inner automorphism groups as well as an isomorphism between their derived subgroups such that the isomorphisms are compatible with the commutator map $Inn(G)\times Inn(G)\to G'$. It is known that the isoclinic groups have the same commutativity degree [14]. First, we generalize this definition to finite Moufang loops and deduce the same result for them, as follows:

Definition 2.1. Let M and L be two finite Moufang loops and M' and L' be their commutator subloops, respectively. Then we say that they are isoclinic if there are two isomorphisms $\varphi:\dfrac{M}{Z(M)}\to\dfrac{L}{Z(L)}$ and $\psi:M'\to L'$ such that the following diagram commutes:

$$\begin{array}{ccc}
\dfrac{M}{Z(M)}\times\dfrac{M}{Z(M)} & \xrightarrow{(\varphi,\varphi)} & \dfrac{L}{Z(L)}\times\dfrac{L}{Z(L)} \\
\alpha\downarrow & & \downarrow\beta \\
M' & \xrightarrow[\psi]{} & L'
\end{array}$$

(so, $\psi\alpha=\beta(\varphi,\varphi)$), where, $\alpha(xZ(M),yZ(M))=[x,y]$ and $\beta(uZ(L),vZ(L))=[u,v]$. Also, we say that the pair (φ,φ) is an isoclinism from M to L and write $M\overset{iso.}{\cong}L$.

Theorem 2.2. *Let M and L be two finite isoclinic Moufang loops. Then $Pr(M)=Pr(L)$.*

Proof. One can write:

$$\begin{aligned}
|\frac{M}{Z(M)}|^2\cdot Pr(M) &= \frac{1}{|Z(M)|^2}|M|^2\cdot Pr(M)\\
&= \frac{1}{|Z(M)|^2}|\{(x,y)\in M^2 \mid [x,y]=1\}|\\
&= \frac{1}{|Z(M)|^2}|\{(x,y)\in M^2 \mid \alpha(xZ(M),yZ(M))=1\}|\\
&= |\{(u,v)\in(\frac{M}{Z(M)})^2 \mid \alpha(u,v)=1\}|\\
&= |\{(u,v)\in(\frac{M}{Z(M)})^2 \mid \psi(\alpha(u,v))=1\}|\\
&= |\{(u,v)\in(\frac{M}{Z(M)})^2 \mid \beta(\varphi(u),\varphi(v))=1\}| \quad (\beta(\varphi,\varphi)=\psi\alpha)\\
&= |\{(w,z)\in(\frac{L}{Z(L)})^2 \mid \beta(w,z))=1\}| \qquad (\varphi \text{ is an isomorphism})\\
&= |\frac{L}{Z(L)}|^2\cdot Pr(L).
\end{aligned}$$

But $\dfrac{M}{Z(M)}\cong\dfrac{L}{Z(L)}$ and so $|\dfrac{M}{Z(M)}|=|\dfrac{L}{Z(L)}|$, which gives $Pr(M)=Pr(L)$. □

Remark 2.3. *Clearly, isoclinism is an equivalence relation between groups as well as Moufang loops. Also, it is clear that two isomorphic Moufang loops (groups) are also isoclinic, but the converse is not true. For example, if we denote the quaternion group and the dihedral group of order* 8 *with* Q_8 *and* D_8 *then they are isoclinic but non-isomorphic. Also, there are isoclinic but non-isomorphic non-associative Moufang loops of order* 16.

The next result shows that the isoclinism of Moufang loops preserves the perfectness of non-commuting graphs.

Proposition 2.4. *Let* M *and* L *be two isoclinic finite Moufang loops. Then* Γ_M *is a perfect graph if and only if* Γ_L *is perfect.*

Proof. Let $M \overset{iso.}{\cong} L$ with isomorphisms $\varphi : \frac{M}{Z(M)} \to \frac{L}{Z(L)}$ and $\psi : M' \to L'$. Suppose that Γ_M is not perfect. Then by Berg theorem, as mentioned in the introduction, it has a chordless cycle of odd order, say, $\{m_1, \ldots, m_r\}$. We have $\ell_i Z(L) = \varphi(m_i Z(M))$ for all $1 \leq i \leq r$. Now, since $\psi^{-1}(1_L) = 1_M$, it follows that $[m_i, m_j] \neq 1$ if and only if $[\ell_i, \ell_j] \neq 1$. So, $\{\ell_1, \ldots, \ell_r\}$ is also a chordless cycle of odd order for Γ_L and so it is not perfect. The converse is also true, since φ and ψ are isomorphisms. □

3. Some More Relations Between Isoclinism, commutativity Degree and Non-commuting Graphs in Finite Moufang Loops

In this section, we are going to obtain some more relationships between isoclinism and commutativity degree, on one hand, and non-commuting graphs and non-commuting graphs, on the other hand, in finite non-commutative Moufang loops. Our first tool is an easy result which has a key role in finding relationships between the stated concepts above.

Lemma 3.1. *Let* M *be a finite non-commmutative Moufang loop. Then* $Pr(M) = \dfrac{|M|^2 - 2e}{|M|^2}$ *and equivalently,* $e = \dfrac{|M|^2}{2}(1 - Pr(M))$*, where* e *is equal to the number of edges of the non-commuting graph of* M*.*

Proof. If Γ_M be the non-commuting graph of M then we have:

$$\begin{aligned} 2e &= 2|E(\Gamma_M)| = \sum_{x \in M \setminus C(M)} deg(x) \\ &= \sum_{x \in M \setminus C(M)} (|M| - |C_M(x)|) \\ &= \sum_{x \in M \setminus C(M)} |M| - \sum_{x \in M \setminus C(M)} |C_M(x)| - \sum_{x \in C(M)} |M| + \sum_{x \in C(M)} |M| \\ &= \sum_{x \in M} |M| - \sum_{x \in M} |C_M(x)| = |M|^2 - \sum_{x \in M} |C_M(x)|. \end{aligned}$$

But $Pr(M) = \dfrac{\sum_{x \in M} |C_M(x)|}{|M|^2}$ and so,

$$2e = |M|^2 - |M|^2 \cdot Pr(M)$$

and therefore,

$$e = \frac{|M|^2}{2}(1 - Pr(M))$$

or

$$Pr(M) = \frac{|M|^2 - 2e}{|M|^2}.$$

□

Theorem 3.2. *Let M and L be finite Moufang loops such that $Pr(M) = Pr(L)$ and also $\Gamma_M \cong \Gamma_L$. Then $|M| = |L|$. If M is centerless then L is too.*

Proof. We have $Pr(M) = Pr(L)$, whence by lemma 3.1,

$$\begin{aligned} & \frac{|L|^2 - 2e}{|L|^2} = \frac{|M|^2 - 2e}{|M|^2} \\ \Longrightarrow \quad & |M|^2(|L|^2 - 2e) = |L|^2(|M|^2 - 2e) \\ \Longrightarrow \quad & |M|^2|L|^2 - 2e|M|^2 = |L|^2(|M|^2 - 2e|L|^2) \\ \Longrightarrow \quad & |M|^2 = |L|^2 \Rightarrow |M| = |L|. \end{aligned}$$

Now, If M is centerless then since $|L| - |C(L)| = |M| - |C(M)| = |M| - 1$ thus $|C(L)| = 1$.

□

As a consequence of theorems 2.2 and 3.2, we have:

Corollary 3.3. *Let M and L be finite Moufang loops such that $M \overset{iso.}{\cong} L$ and also $\Gamma_M \cong \Gamma_L$. Then $|M| = |L|$.* □

Remark 3.4. *The condition of isoclinism of Moufang loops in the hypothesis of the above corollary is necessary, because if we drop it, then the assertion is not true, as it has been shown by a counterexample in* [17].

The next theorem shows that finite isoclinic Moufang loops have isomorphic non-commuting graphs under some additional conditions.

Theorem 3.5. *Let M and L be two finite isoclinic Moufang loops with the same sizes of centers and centrums (commutants) i.e., $|Z(M)| = |Z(L)|$ and $|C(M)| = |C(L)|$. Then $\Gamma_M \cong \Gamma_L$.*

Proof. Since $M \overset{iso.}{\cong} L$, by theorem 2.2, $Pr(M) = Pr(L)$, so by lemma 3.1, we have:

$$\frac{|M|^2 - 2e}{|M|^2} = \frac{|L|^2 - 2e'}{|L|^2},$$

where, e and e' be the numbers of edges of non-commuting graphs of M and L, respectively.

Now, Let (ϕ, ψ) be the isoclinism pair between M and L. Then by definition of isoclinism, we have:

$$\frac{M}{Z(M)} \overset{\varphi}{\cong} \frac{L}{Z(L)}$$

and $M' \overset{\psi}{\cong} L'$. So,

$$|\frac{M}{Z(M)}| = |\frac{L}{Z(L)}|$$

and $|M'| = |L'|$. But, $|Z(M)| = |Z(L)|$ and therfore, $|M| = |L|$. Also, by hypothesis $|C(M)| = |C(L)|$ and hence,

$$|V(\Gamma_M)| = |M| - |C(M)| = |L| - |C(L)| = |V(\Gamma_L)|.$$

Also, $e = e'$ by (1) and (2).

Now, let $\{x, y\}$ be an edge in Γ_M. Then $xy \neq yx$, i.e, $[x, y] \neq 1_M$ in M and so, $\psi([x, y]) \neq \psi(1_M)$ in L or $\psi([x, y]) = [x', y'] \neq 1_M$. Thus $\{x', y'\}$ is an edge in Γ_L. The converse is also true, since ψ^{-1} is an isomorphism between L and M. Therefore $\Gamma_M \cong \Gamma_L$. □

Remark 3.6. *The converse of theorem 3.5 is not true generally. By a counter-example in* [17]*, there are non-abelian groups with non-equal orders but having isomorphic non-commuting graphs, such as G and H with $|G| = 2^{10}$, $|H| = 5^6$ and $\Gamma_G \cong \Gamma_H$. But, since G and H are $p-$groups, so they have non-trivial centers and hence $|Z(G)| \neq |Z(H)|$. Therefore, $|\frac{G}{Z(G)}| \neq |\frac{H}{Z(H)}|$ and so $G \overset{iso.}{\ncong} H$.*

Also, if we impose even the equality of orders of centers and centrums to the isomorphism of the non-commuting graphs of two Moufang loops, as the following counterexample shows, we could not prove isoclinism of them.

Example 3.7. *The following facts and data have been computed and calculated by GAP codes* [10]*. There are exactly* 51 *groups of order* 32 *up to isomorphism. These groups are classified in* 8 *isoclinic classes as shown in the following table.*

Table 1. *Isoclinism classes of groups of order* 32

Class No.	*Size of Class*	*Nilpotency Class*	*Conjugacy Class Sizes*	$\lvert Z(G)\rvert$	$\lvert G'\rvert$	$Pr(G)$	$deg(g)$	$\lvert E(\Gamma_G)\rvert$
1	7	1	1(32)	32	1	1	0	0
2	15	2	1(8),2(24)	8	2	$\frac{5}{8}$	16(24)	192
3	10	3	1(4),2(12),4(16)	4	4	$\frac{7}{16}$	16(12),24(16)	288
4	9	2	1(4),2(12),4(16)	4	4	$\frac{7}{16}$	16(12),24(16)	288
5	2	2	1(2),16(30)	2	2	$\frac{17}{32}$	16(30)	240
6	2	3	1(2),2(6),4(24)	2	4	$\frac{11}{32}$	16(6),24(24)	336
7	3	3	1(2),2(6),4(24)	2	4	$\frac{11}{32}$	16(6),24(24)	336
8	3	4	1(2),2(6),4(24)	2	8	$\frac{11}{32}$	16(6),24(24)	336

As it is seen in table 1, all of the 19 *groups in the classes* 3 *and* 4 *have isomorphic non-commuting graphs, but each of the groups in class* 3 *is not isoclinic to each of the groups in class* 4 *(they have different nilpotency class and we know that nilpotency class is an invariant in isoclinism of groups). Although, their sizes of centers, derived subgroups and also commutativity degrees are the same. The same result is true for classes* 6*,* 7 *and* 8*.*

On the other hand, note that both isoclinic groups and groups with isomorphic non-commuting graphs have the same proportions of conjugacy class sizes. However, the conjugacy class sizes statistics need not determine the nilpotency class for groups of prime power orders.

As well, all of the groups in classes 2 *and* 5 *have regular non-commuting graphs. All of the non-commuting graphs of groups in class* 2 *are isomorphic pairwise and* $16-$*regular with* 24 *vertices. Also, all of the non-commuting graphs of groups in class* 5 *are isomorphic pairwise and* $16-$*regular with* 30 *vertices.*

Now, we are ready to derive the main relationships between non-commuting graphs and commutativity degrees of two finite Moufang loops. We show that given any two finite Moufang loops with isomorphic non-commuting graphs, by considering the equality of one of these three parameters: their orders, their sizes of commutants and their commutativity degree, the equality of the other two parameters follows.

Theorem 3.8. *Let M and L be two finite Moufang loops. Then the following statements hold:*

(i) If $\Gamma_M \cong \Gamma_L$ and $|M| = |L|$ then $Pr(M) = Pr(L)$ and $|C(M)| = |C(L)|$;

(ii) If $\Gamma_M \cong \Gamma_L$ and $Pr(M) = Pr(L)$ then $|M| = |L|$ and $|C(M)| = |C(L)|$;

(iii) If $\Gamma_M \cong \Gamma_L$ and $|C(M)| = |C(L)|$ then $|M| = |L|$ and $Pr(M) = Pr(L)$;

(iv) If $Pr(M) = Pr(L)$ and $|M| = |L|$ and $|C(M)| = |C(L)|$ then $|V(\Gamma_M)| = |V(\Gamma_L)|$, $|E(\Gamma_M)| = |E(\Gamma_L)|$.

Proof. (i) Since $\Gamma_M \cong \Gamma_L$,

$$|M| - |C(M)| = |V(\Gamma_M)| = |V(\Gamma_L)| = |L| - |C(L)|.$$

But $|M| = |L|$ and hence $|C(M)| = |C(L)|$. Also, $|E(\Gamma_M)| = |E(\Gamma_L)|$ and so by lemma 3.1, $Pr(M) = Pr(L)$.
(ii) It is true by theorem 2.2.
(iii) The proof is similar to the proof of part (i).
(iv) Since $Pr(M) = Pr(L)$ and $|M| = |L|$, so by lemma 3.1, $|E(\Gamma_M)| = |E(\Gamma_L)|$. Also, since $|C(M)| = |C(L)|$, we get $|V(\Gamma_M)| = |V(\Gamma_L)|$. □

Remark 3.9. *There is a lot of evidence that we can deduce the isomorphism of non-commuting graphs by the hypothesis in part* (iv) *of theorem 3.8 and we were not able to find a counterexample yet. So, we state it as a conjecture as below.*

Conjecture 3.10. Let M and L be two finite Moufang loops. If $Pr(M) = Pr(L)$ and $|M| = |L|$ and $|C(M)| = |C(L)|$ then $\Gamma_M \cong \Gamma_L$.

4. Characterization of finite simple Moufang loops by isoclinism and non-commuting graphs

A non-associative finite simple Moufang loop is called a Paige loop. The classification of finite simple Moufang loops has been completed by Paige and Liebeck (see [19] and [15]). In 2006, Abdollahi, Akbari, and Maimani proposed a conjecture as follows [1]:

Conjecture 4.1 (AAM's Conjecture)**.** Let P be a finite non-abelian simple group and G be a group such that $\Gamma_G \cong \Gamma_P$. Then $G \cong P$.

Thereafter, this conjecture is verified for all sporadic simple groups, the alternating groups in some papers by the first author of [1] and some others, and finally Solomon and Woldar proved it in [21]. So, coming back to finite Moufang loops, it follows that every associative finite simple Moufang loop is characterizable by its non-commuting graph. Now, it is a natural question that what happens about Paige loops? Can we characterize a Paige loop by its non-commuting graph? Formally, the second author of this paper has proposed it as a conjecture in [4]:

Conjecture 4.2. Let S be a finite non-commutative simple Moufang loop and L be a Moufang loop such that $\Gamma_L \cong \Gamma_S$. Then $L \cong S$.

So, to prove this conjecture it is enough to consider only "Paige loops". To study this problem, we start with the following key lemma.

Lemma 4.3. *Let S be a non-commutative simple Moufang loop. Then any Moufang loop M isoclinic to S is isomorphic to $S \times A$ for some commutative Moufang loop A.*

Proof. We have $M' \cong S' \cong S$ (since S is non-commutative and simple). Hence, M' is also non-commutative, and by its simplicity $Z(M') = 1$ and so, $M' \cap Z(M) \subseteq Z(M')$ ($x \in M' \cap Z(M) \Rightarrow x \in M',\ x \in Z(M), xy = yx\ (\forall y \in M) \Rightarrow x \in Z(M')$). So we get $M' \cap Z(M) \subseteq Z(M') = 1$. Now, $M \overset{iso.}{\cong} S$ and so $\dfrac{M}{Z(M)} \cong \dfrac{S}{Z(S)} \cong S \cong M'$ is perfect, and hence, $M = M'Z(M)$ Thus $M = M' \times Z(M)$ since $M' \cap Z(M) = 1$. Therefore, $M \cong S \times Z(M)$. If we let $A = Z(M)$ then the result follows. □

Corollary 4.4. *If S be a finite simple Moufang loop and M be any Moufang loop which is isoclinic to S with the same order, then they are isomorphic.*

Proof. By lemma 4.3, $M \cong S \times A$, where A is a commutative Moufang loop. Now, since $|M| = |S||A|$ and on the other hand $|M| = |S|$, so $|A| = 1$. Therefore $M \cong S$. □

The following result gives a condition under which a Moufang loop is isoclinic with its proper subloop. Its analogous is known for groups (see [14]).

Theorem 4.5. *Let M be a Moufang loop and L be a subloop of M such that $M = LZ(M)$. Then M and L are isoclinic. If L is finite then the converse is also true.*

Proof. If $M = LZ(M)$, then $Z(L)$ centeralizes L and $Z(M)$, hence centeralizes M, thus $Z(L) \subseteq L \cap Z(M) \subseteq Z(L)$ and

$$\frac{L}{Z(L)} = \frac{L}{Z(M) \cap L} \cong \frac{LZ(M)}{Z(M)} = \frac{M}{Z(M)}.$$

The isomorphism $i_1 : \dfrac{L}{Z(L)} \to \dfrac{M}{Z(M)}$ being induced by the inclusion $i : L \to M$. Furthermore, let $(x, y) \in M^2$. Then $x = l_1 z_1$, $y = l_2 z_2$, where , $l_1, l_2 \in L$, $z_1, z_2 \in Z(M)$ and we have

$$\begin{aligned} [x, y] &= x^{-1}y^{-1}xy = (l_1z_1)^{-1}(l_2z_2)^{-1}(l_1z_1)(l_2z_2) \\ &= l_1^{-1}l_2^{-1}l_1l_2z_1^{-1}z_1z_2^{-1}z_2 = [l_1, l_2] \in L' \end{aligned}$$

and $M' = L'$. So, we have actually proved that $(i_1, 1_{M'})$ is an isoclinism pair from L to M. Conversely, if L is isoclinic to M and is finite, then $\dfrac{M}{Z(M)} \cong \dfrac{L}{Z(L)}$ is also finite. But

$$|\frac{M}{Z(M)}| \geq |\frac{LZ(M)}{Z(M)}| = |\frac{L}{L \cap Z(M)}| = |\frac{L}{Z(L)}||\frac{Z(L)}{L \cap Z(M)}| \geq |\frac{L}{Z(L)}| = |\frac{M}{Z(M)}|.$$

Thus we have equality all along, and so, $M = LZ(M)$. □

As an immediate result of the above theorem, we have:

Corollary 4.6. *Let S be a finite simple Moufang loop and let M be a Moufang loop such that $M \cong T$, where T is a subloop of S. If $M \overset{iso.}{\cong} S$ then $M \cong S$.*

We can say much more about simple Moufang loops. In fact, all finite non-commutative simple Moufang loops can be characterized by isoclinism and isomorphism of non-commuting graphs, as the next result shows.

Corollary 4.7. *Let S be a finite simple Moufang loop and let M be a Moufang loop such that $\Gamma_M \cong \Gamma_S$ and M is isoclinic to S. Then $M \cong S$.*

Proof. Since M is isoclinic to S then by theorem 2.2, $Pr(M) = Pr(S)$ and by lemma 4.3, $M \cong S \times A$ where A is a commutative Moufang loop. Also, since $\Gamma_M \cong \Gamma_S$, by theorem 3.2, $|M| = |S|$ and so,

$$|M| = |S \times A| = |S||A| \Rightarrow |A| = 1 \Rightarrow M \cong S \times 1 \cong S.$$

□

The following result was proved by E. Artin. It classifies the finite simple groups with the same order, and only contains two classes of finite classic simple groups denoted by $B_n(q)$ and $C_n(q)$, a linear group $L_3(4)$ and an alternating group A_8:

Lemma 4.8. *Let G and M be finite simple groups, $|G| = |M|$. Then the following holds:*

(i) If $|G| = |A_8| = |L_3(4)|$, then $G \cong A_8$ or $G \cong L_3(4)$;
(ii) If $|G| = |B_n(q)| = |C_n(q)|$, where $n \geq 3$, and q is odd, then $G \cong B_n(q)$ or $G \cong C_n(q)$;
(iii) If M is not in the above cases, then $G \cong M$.

□

Corollary 4.9. *Let S be a finite non-commutative simple Moufang loop and M be a Moufang loop such that $C(M) = 1$. If $\Gamma_M \cong \Gamma_S$ then $|M| = |S|$ and $Pr(M) = Pr(S)$. In particular, if M is also simple then $M \cong S$.*

Proof. It is clear by above lemma and theorem 3.8. □

Remark 4.10. *In the case of S be a finite simple group, the above result is much weaker than the existing result that says* [21]*: "Let S be a finite non-abelian simple group and G be a group such that $\Gamma_G = \Gamma_S$. Then $G \cong S$." To prove this result, Solomon and Woldar have used many tools that are not available in finite Moufang loops in general. So, the corresponding statement for Moufang loops is still remaining a conjecture.*

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