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Research Paper

# $H$-SUPPLEMENTED MODULES AND SINGULARITY

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Abstract. Let $M$ be a module over a ring $R$. We call $M$, $\delta$-$H$-supplemented provided for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$ for every submodule $X$ of $M$ with $M/X$ singular. We prove that $M$ is $\delta$-$H$-supplemented if and only if for every submodule $N$ of $M$ there exists a direct summand $D$ of $M$ such that $(N + D)/N \ll_\delta M/N$ and $(N + D)/D \ll_\delta M/D$.

## 1. Introduction

All rings considered in this work are associative with identity and all modules are unitary right $R$-modules. Let $M$ and $N$ be $R$-modules. It is useful to indicate that by $N \leq M$, we mean $N$ is a submodule of $M$. A submodule $N$ of $M$ is said to be *small* in $M$ if $N + K \neq M$ for any proper submodule $K$ of $M$, and we denote it by $N \ll M$. As a generalization, Zhou in [19] introduced the concept of $\delta$-small submodules. A submodule $N$ of $M$ is said to be $\delta$-small in $M$ (denoted by $N \ll_\delta M$) provided $M \neq N + K$ for any proper submodule $K$ of $M$ with

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$M/K$ singular. General properties and some useful characterizations of $\delta$-small submodules of a module have been provided in [19].

Maybe, the most important concept in module theory which is closely related to smallness, is lifting modules. A module $M$ is called *lifting*, provided every submodule $N$ of $M$ contains a direct summand $D$ of $M$ such that $N/D \ll M/D$. A number of results concerning lifting modules have been appeared in the literature in recent years and many generalizations of the concept of lifting modules have been introduced and studied by several authors (see [6], [7] and [10]).

Recall from [3] that a module $M$ is called *$H$-supplemented* in case for every submodule $N$ of $M$, there exists a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$ for every submodule $X$ of $M$. Different definition's style, unusual properties and being a generalization of lifting modules, all led many researchers to study and investigate $H$-supplemented modules further than Mohamed and Müller's elementary introduction in [3]. After introducing this new notation, some authors tried to investigate them more. Maybe first serious effort has been done in [9]. Some general properties of $H$-supplemented modules such as homomorphic images and direct summands of these modules were investigated in [9]. After that in [8], the authors presented some equivalent conditions for a module to be $H$-supplemented that shows that this class of modules is closely related to the concept of small submodules. In fact in [8], the authors proved that a module $M$ is $H$-supplemented if and only if for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$. Talebi and his coauthors in [12] studied the concept of $H$-supplemented modules via preradicals. If $\tau$ indicates a preradical, they call a module $M$ is $\tau$-$H$-supplemented provided for every submodule $N$ of $M$, there is a direct summand $D$ of $M$ such that $(N + D)/N \subseteq \tau(M/N)$ and $(N + D)/D \subseteq \tau(M/D)$ ([12]). In fact this definition somehow develops the concept of $H$-supplemented modules. They are some works that are relevant to this strange class of modules containing some further properties of them and introducing some generalizations of $H$-supplemented modules ([15, 5, 16]).

Let $M$ be a module over a ring $R$. Following [14], $M$ is called *(non)cosingular* provided ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$, where $\overline{Z}(M) = \bigcap\{Kerf \mid f : M \to U\}$ in which $U$ is an arbitrary small right $R$-module. The author in [11] considered the class of right $\delta$-small $R$-modules in the definition of $\overline{Z}(M)$, and define $\overline{Z}_\delta(M)$ to be $\bigcap\{Kerg \mid g : M \to V\}$ where $V$ is a $\delta$-small module (i.e. there exists another module $U$ such that $V \ll_\delta U$). In [11], $M$ is said to be *(non-)$\delta$-cosingular* in case ($\overline{Z}_\delta(M) = M$) $\overline{Z}_\delta(M) = 0$. Since for a module $M$ we have $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$, every cosingular right $R$-module is $\delta$-cosingular and every non-$\delta$-cosingular module is noncosingular.

The work by Y. Zhou [19] became a pioneer in studying lifting modules, supplemented modules, $\oplus$-supplemented modules and the others using singularity. In fact [19] made a different impression on works have been done on supplemented modules and related concepts. This new approach produced some excellent works on delta version of supplemented modules. These nice papers can be [17, 10, 2, 18, 1]. The first person who worked on $\delta$-version is M. T. Koşan. He introduced $\delta$-lifting modules and $\delta$-supplemented modules and tried to investigate their natural properties. Halicioglu, Inankil, Harmanci, Ungor, Talebi and Hosseinpour created some nice perspectives on $\delta$-supplemented modules and their natural derivations ([1] and [2]). They generally focused on modules whose cyclic submodules have $\delta$-supplements that are direct summands. It is important to indicate that in [1], the authors introduced a new class of modules namely *principally Goldie*$^*$*-$\delta$-lifting* modules which is a proper generalization of principally $\delta$-lifting modules. In [17] a new approach on studying $\oplus$-supplemented modules via singularity has been introduced. The authors called a module $M$, $\oplus$-supplemented relative to an ideal $I$ of the ring $R$, $I$-$\oplus$-supplemented if for every submodule $N$ of $M$ there is a direct summand $K$ of $M$ such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$. They also compare $I$-$\oplus$-supplemented modules with $\oplus$-supplemented modules. The structure of $I$-$\oplus$-supplemented modules and $\oplus$-$\delta$-supplemented modules over a Dedekind domain is completely determined in [17].

Inspiring by works mentioned, in this manuscript we are interested in studying $H$-supplemented modules via singularity. We say that a module $M$ is $\delta$-$H$-supplemented in case for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$, for every submodule $X$ of $M$ with $M/X$ singular. We provide an equivalent condition for this definition impressing the close relation of $\delta$-$H$-supplemented modules to the concept of $\delta$-small submodules. Some conditions are presented to ensure that every $\delta$-$H$-supplemented module is $\delta$-lifting.

## 2. $\delta$-version of $H$-supplemented modules

As generalizations of lifting modules and supplemented modules, the author in [10] introduced $\delta$-lifting modules and $\delta$-supplemented modules. A module $M$ is called *$\delta$-lifting* in case every submodule $N$ of $M$ contains a direct summand $D$ of $M$ such that $N/D \ll_\delta M/D$. General properties of these new classes of modules had been naturally discussed. We naturally are interested in defining $\delta$-version of $H$-supplemented modules.

**Definition 2.1.** Let $M$ be a module. Then $M$ is said to be $\delta$-$H$-supplemented, provided for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$ for every submodule $X$ of $M$ with $M/X$ singular.

The following provides an equivalent condition for a module to be $\delta$-$H$-supplemented.

**Lemma 2.2.** *Let $M$ be a module. Then $M$ is $\delta$-$H$-supplemented if and only if for every submodule $N$ of $M$ there exists a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$.*

*Proof.* Let $M$ be $\delta$-$H$-supplemented and $N \leq M$. Then there is a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$, for every submodule $X$ of $M$ such that $M/X$ is singular. Suppose that $(N+D)/N + X/N = M/N$ for a submodule $X$ of $M$ containing $N$ with $M/X$ singular. Then $D + X = M$. By assumption $N + X = M$, which implies that $X = M$ as required. To verify the second $\delta$-small case, let $(N+D)/D + Y/D = M/D$ where $M/Y$ is singular. Then $N + Y = M$. Being $M$ a $\delta$-$H$-supplemented module implies $D + Y = M$. Therefore, $Y = M$. Conversely, let $N + X = M$ with $M/X$ singular. Then $(N+D)/D + (X+D)/D = M/D$. Note that $M/(X+D)$ is singular as well as $M/X$ is singular. Hence $X + D = M$, since $(N+D)/D \ll_\delta M/D$. Now, suppose that $D + Y = M$ for a submodule $Y$ of $M$ such that $M/Y$ is singular. Then $(N+D)/N + (N+Y)/N = M/N$ and $M/(N+Y)$ as a homomorphic image of $M/Y$ is singular. Being $(N+D)/N$ a $\delta$-small submodule of $M/N$ combining with last equality implies $N + Y = M$. □

**Proposition 2.3.** *Let $M$ be a module. Then in each of the following cases $M$ is $H$-supplemented if and only if $M$ is $\delta$-$H$-supplemented.*

(1) *$M$ is a singular module.*

(2) *$M$ has no simple projective submodule.*

*Proof.* (1) This follows from the fact that every homomorphic image of a singular module is singular. In fact, every $\delta$-small submodule of a singular module is a small submodule of that module.

(2) Let $M$ be a $\delta$-$H$-supplemented module that has no simple projective submodule. Suppose that $N$ is a submodule of $M$. Then there is a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$. Let $(N+D)/N + T/N = M/N$ for a submodule $T/N$ of $M/N$. By [19, Lemma 1.2], $(N+D)/N$ contains a semisimple projective direct summand $Y/N$ of $M/N$ such that $Y/N \oplus T/N = M/N$. Then there is a submodule $N'$ of $Y$ such that $Y = N \oplus N'$. since $Y/N$ is projective. It follows that $N'$ contains a simple projective simple submodule as $N'$ is semisimple. Now we can conclude that $Y = N$ and therefore $T/N = M/N$ implying that $(N+D)/N \ll M/N$. By a same argument we can verify $(N+D)/D \ll M/D$. Therefore, $M$ is $H$-supplemented. □

**Corollary 2.4.** *Let $R$ be a ring such that every simple right $R$-module is singular. Then a right $R$-module $M$ is $H$-supplemented if and only if $M$ is $\delta$-$H$-supplemented. In particular, an $\mathbb{Z}$-module $M$ is $H$-supplemented if and only if $M$ is $\delta$-$H$-supplemented.*

**Proposition 2.5.** *Let $M$ be a non-$\delta$-cosingular module. Then $M$ is $\delta$-lifting if and only if $M$ is $\delta$-$H$-supplemented.*

*Proof.* Let $M$ be $\delta$-$H$-supplemented and $N \leq M$. Then there exists a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$. Note that $D$ is non-$\delta$-cosingular as well as $M$. Hence $(N+D)/N$ is both $\delta$-small and non-$\delta$-cosingular ([11, Proposition 2.4]) which implies $N + D = N$. In fact $N/D$ is a $\delta$-small submodule of $M/D$ showing that $M$ is $\delta$-lifting. The other implication is easy to check. □

**Proposition 2.6.** *Let $M$ be an indecomposable module. Then $M$ is $\delta$-$H$-supplemented if and only if every proper submodule of $M$ is $\delta$-small in $M$ or $M$ is a simple module.*

*Proof.* Let $M$ be indecomposable and $\delta$-$H$-supplemented. Consider an arbitrary proper submodule $N$ of $M$. Then there is a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$. Suppose $D = 0$. Then clearly $N \ll_\delta M$. Otherwise, $D = M$ implies $M/N \ll_\delta M/N$. Now [19, Lemma 1.2] yields that $M/N$ is projective and semisimple (it is sufficient in [19, Lemma 1.2] that we set $M = M/N$, $N = M/N$ and $X = 0$). It follows now that $N$ must be a direct summand of $M$. Being $M$ indecomposable implies $N = 0$ which shows $M$ is a simple module. The converse is straightforward to check. □

The last proposition shows that an indecomposable module is $\delta$-$H$-supplemented if and only if it is $\delta$-lifting.

We next present some examples of $\delta$-$H$-supplemented modules.

**Example 2.7.** (1) It is not hard to check by definitions that every $\delta$-lifting module is $\delta$-$H$-supplemented. But we may indicate that the converse does not hold. To verify this assertion, suppose that $M_1$ is a $H$-supplemented module with a unique composition series $M_1 \supset U \supset V \supset 0$ (we may choose the $\mathbb{Z}$-module $M_1 = \mathbb{Z}_8$). Now, let $M = M_1 \oplus M_1/U \oplus U/V \oplus V/0$. Then $M$ is a $H$-supplemented module by [8, Corollary 4.5(2)] and clearly a $\delta$-$H$-supplemented module. Note also that $M_1 \oplus U/V$ is not a $\delta$-lifting module by [10, Example 2.2(2)]. This implies $M$ is not a $\delta$-lifting module by [10, Lemma 2.3(2)].

(2) Every $H$-supplemented module is $\delta$-$H$-supplemented as well as every small submodule of a module is $\delta$-small in that module. The converse does not hold in general. Now let $F = \mathbb{Z}_2$ which is a field and $S = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each $i$. Let $R$ be the subring of $S$ generated by $\oplus_{i=1}^{\infty} F_i$ and $1_S$. It is well-known that $R$ is not a semiperfect ring which yields that $R_R$ is not a $H$-supplemented module. By [19, Example 4.1], $R$ is a $\delta$-semiperfect ring. Now [10, Theorem 3.3] implies that $R_R$ is $\delta$-lifting and consequently $R_R$ is $\delta$-$H$-supplemented.

Let $M$ be a module. The $M$ is said to be *principally Goldie*$^*$*-*$\delta$*-lifting* in case for every $x \in M$, there is a direct summand $D$ of $M$ such that $(xR + D)/D \ll_\delta M/D$ and $(xR + D)/xR \ll_\delta M/xR$. Recall from [18] that $M$ is *principally* $\oplus$*-*$\delta$*-supplemented* provided for every cyclic submodule $xR$ of $M$, there is a direct summand $D$ of $M$ such that $M = xR + D$ and $xR \cap D \ll_\delta D$. It is easy to verify that every principally $\delta$-lifting module is principally Goldie$^*$-$\delta$-lifting and every principally Goldie$^*$-$\delta$-lifting module is principally $\oplus$-$\delta$-supplemented.

The following introduces some principally Goldie$^*$-$\delta$-lifting modules which are not $\delta$-$H$-supplemented. So the class of principally Goldie$^*$-$\delta$-lifting modules contains properly the class of $\delta$-$H$-supplemented modules.

**Example 2.8.** (1) Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. Since $\mathbb{Z}$ has no nonzero projective simple module, then $M$ is not $\delta$-$H$-supplemented by Corollary 2.4. In other words, since every cyclic submodule of $M$ is small in $M$, then $M$ is principally Goldie$^*$-$\delta$-lifting.

(2) Let $M = \mathbb{Q} \oplus \mathbb{Z}/2\mathbb{Z}$ as an $\mathbb{Z}$-module. Then $M$ is not a supplemented module since $\mathbb{Q}$ is not a supplemented module. Hence $M$ is not an $H$-supplemented $\mathbb{Z}$-module. Now from Corollary 2.4, $M$ is not a $\delta$-$H$-supplemented $\mathbb{Z}$-module while $M$ is principally Goldie$^*$-$\delta$-lifting by [1, Example 4.7]. Note also that by [18, Example 3.1(2)], $M$ is a principally $\oplus$-$\delta$-supplemented $\mathbb{Z}$-module.

Let $M$ be a right $R$-module. Then $\delta(M)$ is the reject in $M$ of the class of all simple singular right $R$-modules ([19, Defenition 1.4]). It should be noted by [19, Lemma 1.5] that $\delta(M)$ is the sum of all $\delta$-small submodules of $M$. The following says that a module $M$ with no nonzero $\delta$-small submodule is $\delta$-$H$-supplemented if and only if $M$ is semisimple.

**Proposition 2.9.** *Let $M$ be a $\delta$-$H$-supplemented module. Then:*

(1) *If $\delta(M) = 0$, then $M$ is a semisimple module.*

(2) *Every non-$\delta$-cosingular submodule of $M$ is a direct summand of $M$.*

*Proof.* (1) To verify this assertion, let $N \leq M$. Then there is a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$. Being $D$ a direct summand of $M$ implies $\delta(M/D) = 0$. It follows that $N + D = D$ which shows that $N$ is contained in $D$. Therefore, $D/N \ll_\delta M/N$. Note also that $D/N \oplus (D' + N)/N = M/N$. Hence $D/N$ is a semisimple projective submodule of $M/N$. In fact, $N$ will be a direct summand of $D$ and of course a direct summand of $M$.

(2) Let $N$ be a non-$\delta$-cosingular submodule of $M$. Then there is a direct summand $D$ of $M$ such that $(N+D)/N \ll_\delta M/N$ and $(N+D)/D \ll_\delta M/D$. Since $(N+D)/D$ is a homomorphic image of $N$, then it is non-$\delta$-cosingular by[11, Proposition 2.4]. Now $(N + D)/D \ll_\delta M/D$ implies $N + D = D$. The rest is the same as the proof of (1). □

Recall that a submodule $N$ of a module $M$ is called *(projection) fully invariant* provided for every (idempotent) endomorphism $h$ of $M$ we have $h(N) \subseteq N$. It is clear by definitions that every fully invariant submodule of a module is projection invariant. Note also that a module $M$ is called *(weak) duo* in case every (direct summand) submodule of $M$ is fully invariant.

**Proposition 2.10.** *Let $M$ be a module and $N$ a projection invariant submodule of $M$. If $M$ is $\delta$-$H$-supplemented, then $M/N$ so is.*

*Proof.* Let $K/N$ be an arbitrary submodule of $M/N$. Then there exists a direct summand $D$ of $M$ such that $M = K + X$ if and only if $M = D + X$ for every submodule $X$ of $M$ such that $M/X$ is singular. Put $M = D \oplus D'$. As $N$ is a projection invariant submodule of $M$, we conclude that $(N + D)/N \oplus (N + D')/N = M/N$. Now, suppose $K/N + Y/N = M/N$ for a submodule $Y/N$ of $M/N$ with $M/Y$ singular. Then $K + Y = M$ and by assumption $M = D + Y$. Clearly now $M/N = (D + N)/N + Y/N$. Now for the other implication, let $M/N = (D + N)/N + T/N$ with $M/T$ singular. Hence $M = D + T$ and again by assumption $M = K + T$. Obviously $M/N = K/N + T/N$. □

It is known that a module $M$ is said to be *distributive* in case the lattice of submodules of $M$ is distributive, i.e. for each submodules $N, K, L$ of $M$ the equalities $(N \cap L) + (N \cap K) = N \cap (L + K)$ and $N + (K \cap L) = (N + K) \cap (N + L)$ hold.

**Corollary 2.11.** (1) *Every homomorphic image of a distributive $\delta$-$H$-supplemented module is $\delta$-$H$-supplemented.*

(2) *Every direct summand of a weak duo $\delta$-$H$-supplemented module is $\delta$-$H$-supplemented.*

**Theorem 2.12.** *Let $M = M_1 \oplus M_2$ be a distributive module. Then $M$ is $\delta$-$H$-supplemented module if and only if $M_1$ and $M_2$ are $\delta$-$H$-supplemented.*

*Proof.* Let $M_1$ and $M_2$ be $\delta$-$H$-supplemented and $N \leq M$. Set $N_1 = N \cap M_1$ and $N_2 = N \cap M_2$. Then $N = N_1 + (N_2$. Now, there are direct summands $D_i$ of $M_i$ for $i = 1, 2$, such that $(N_i + D_i)/N_i \ll_\delta M_i/N_i$ and $(N_i + D_i)/D_i \ll_\delta M_i/D_i$. We shall prove that $(N + D)/N \ll_\delta M/N$ and $(N + D)/D \ll_\delta M/D$ where $D = D_1 \oplus D_2$ which is a direct summand of $M$. Suppose that $(N + D)/N + X/N = M/N$ for a submodule $X$ of $M$ containing $N$ with $M/X$ singular. Then $D + X = M$. It follows that $D_1 + (X \cap M_1) = M_1$. Now $(N_1 + D_1)/N_1 + (X \cap M_1)/N_1 = M_1/N_1$ and $M_1/(X \cap M_1) \cong X + M_1/X \leq M/X$ is a singular module. Therefore, $X \cap M_1 = M_1$ which implies that $M_1$ is contained in $X$. Now consider again the equality $D + X = M$. So $D_2 + (X \cap M_2) = M_2$. As $(N_2 + D_2) + X \cap M_2/N_2 = M_2/N_2$ and $(N_2 + D_2)/N_2 \ll_\delta M_2/N_2$ and also $M_2/X \cap M_2 \cong X + M_2/X \leq M/X$ is singular, we conclude that $X \cap M_2 = M_2$. So that $M_2$ is contained in $X$ which implies that $X = M$. For the other $\delta$-small case, let

$(N+D)/D + T/D = M/D$ where $T/D \leq M/D$ and $M/T$ is singular. Now $N + T = M$ and hence $N_1 + (T \cap M_1) = M_1$. Being $(N_1 + D_1)/D_1$ a $\delta$-small submodule of $M_1/D_1$ combining with the fact that $M_1/(T \cap M_1)$ is singular and the last equality imply that $T \cap M_1 = M_1$ and therefore $M_1 \subseteq T$. By a same process, $T$ will contain $M_2$. Hence $T = M$ as required. It follows now that $M$ is $\delta$-$H$-supplemented. The converse follows from Proposition 2.10. □

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