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Research Paper

ON THE SCHUR PAIR OF GROUPS

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Abstract. In this paper, it is shown that $(\mathcal{V}, \mathfrak{X})$ is a Schur pair if and only if the Baer-invariant of an $\mathfrak{X}$-group with respect to $\mathcal{V}$ is an $\mathfrak{X}$-group. Also, it is proved that a locally $\mathfrak{X}$ class inherited the Schur pair property of , whenever $\mathfrak{X}$ is closed with respect to forming subgroup, images and extensions of its members. Subsequently, many interesting predicates about some generalizations of Schur's theorem and Schur multiplier of groups will be concluded.

1. Introduction

Let F_∞ be a free group on a countably set $\{x_1, x_2, \ldots\}$ and V be a non-empty subset of it. Every non-identity element of V is called a *word* and has a unique representation as $v(x_{i_1}, x_{i_2}, \cdots, x_{i_k}) = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$, where $i_j \neq i_{j+1}$, for $1 \leq j \leq k-1$ and the α_j are non-zero integers. Each mapping of $\{x_1, x_2, \ldots\}$ into a group G induces a homomorphism $\alpha : F_\infty \to G$ and for all $v = v(x_{i_1}, x_{i_2}, \cdots, x_{i_k}) \in V$, the element $\alpha(v) \in G$ is called the *value* of the word v in G and it is computed by substituting the images $\alpha(x_{i_1}) = g_1, \alpha(x_{i_2}) = g_2$ and

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$\alpha(x_{i_k}) = g_k$ for the letters x_i in the representation $v = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_n}^{\alpha_n} (n \geq k)$. We usually write $\alpha(v) = v(g_1, g_2, \cdots, g_k)$. The subgroup generated by all values in G of words in V is called the *verbal subgroup* of G determined by V,

$$V(G) = \langle v(g_1, g_2, \ldots, g_k) \mid v \in V, g_i \in G, 1 \leq i \leq k, k \in \mathbb{N} \rangle .$$

Also, the *marginal subgroup* $V^*(G)$ of G consists of all $a \in G$ such that

$$v(g_1, \ldots, g_i a, \ldots, g_k) = v(g_1, \ldots, g_i, \ldots, g_k),$$

for all $1 \leq i \leq k$ and $v \in V$. A verbal (marginal) subgroup is always fully-invariant (characteristic) subgroup. For a normal subgroup N of G, the subgroup $V(N, G)$ is defined as

$$\left\langle v(g_1, \ldots, g_i n, \ldots, g_k) v(g_1, \ldots, g_k)^{-1} \mid v \in V, g_r \in G, n \in N, 1 \leq r \leq k, k \in \mathbb{N} \right\rangle .$$

A *variety* of groups, $\mathcal{V}$, is the class of all groups whose verbal subgroup is trivial.

Example 1.1. (i) If $V = \{[x_1, x_2]\}$, then $V(G) = G'$ is the derived subgroup and $V^*(G) = Z(G)$ is the center subgroup of G. Hence $V(N, G) = [N, G]$ and $\mathcal{V} = \mathcal{A}$ is the variety of abelian groups.

(ii) If $V = \{[x_1, x_2, \ldots, x_{k+1}]\} (k \in \mathbb{N})$, then $V(G) = \gamma_{k+1}(G)$ is the $(k + 1)$-th item of the lower central series and $V^*(G) = Z_k(G)$ is k-th item of the upper central series of G. Also, $V(N, G) = [N,_k G]$, in which $[N,_k G]$ is an abbreviation for $[N, \underbrace{G, \ldots, G}_{k}]$ and $\mathcal{V} = \mathcal{N}_k$ is the variety of nilpotent groups of class at most k.

(iii) If $V = \left[[x_{1_1}, \ldots, x_{1_{l+1}}], \cdots, [x_{k+1_1}, \ldots, x_{k+1_{l+1}}]\right]$, for some natural numbers k, l, then $V(G) = \gamma_{k+1}(\gamma_{l+1}(G))$. In 1981, M. R. R. Moghaddam (see [9]) proved $V(N, G) = [N,_l G,_k \gamma_l(G)]$, and $\mathcal{V} = \mathcal{N}_{l,k}$ is the variety of polynilpotent groups of class row (l, k). In general case, the variety of polynilpotent groups of class row $(k_1, k_2, \ldots, k_n)$, $\mathcal{N}_{k_1, k_2, \cdots, k_n}$, is defined inductively. For more details about varieties we refer the readers to [10].

The following lemma is due to P. Hall (1957) which indicates a connection between these three subgroups. Its proof is straightforward.

Lemma 1.2. *Let $\mathcal{V}$ be a variety of groups and G be a group with a normal subgroup N. Then*

(i) $V(G) = 1$ *if and only if* $V^*(G) = G$.

(ii) $V(G/N) = V(G)N/N$.

(iii) $V(N, G)$ *is the least normal subgroup of* G *contained in* N *such that*

$$N/V(N, G) \subseteq V^*(G/V(N, G)) .$$

(iv) $[N, V(G)] \subseteq V(N, G) \subseteq N \cap V(G)$.

Definition 1.3. Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of a group G where F is a free group and $R = \ker \pi$ and $\mathcal{V}$ be a variety of groups. The **Baer-invariant of** G, denoted by $\mathcal{V}M(G)$, is defined as

$$\frac{R \cap V(F)}{V(R,F)},$$

where is always abelian and independent of the choice of the free presentation of G (see [6]). In 1907, Schur proved if $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of the group G will be $(R \cap F')/[R,F]$ which is the **Schur-multiplier** of G denoted by $\mathcal{M}(G)$ [13]. Also, if $\mathcal{V} = \mathcal{N}_k$ is the veriety of nilpotent groups of class at most k, then **k-th multiplicator** of G is denoted by $\mathcal{N}_k M(G)$ and will be $(R \cap \gamma_{k+1}(F))/[R,_k F]$ (see [6]).

During the last century a large amount of researchers have been devoted to the study of various properties of the Baer-invariant of a group. There exists a long history of interaction between Baer-invariant and other mathematical concepts such as algebraic number theory, block theory of group algebras, and classification of finite simple groups, to mention only a few. It all started with the fundamental work of Schur (1904) in which multipliers were introduced in order to study projective representations of groups [13]. Historically, it was Fröhlich (1954) and next Gorenstein (1982) who first pointed out a role of the Schur multiplier of the Galois groups and in finite group theory, respectively (see [2] and [3]).

2. **Preliminaries**

In 1904, I. Schur asserted his classical result:

Theorem 2.1. [12] ***(Schur)*** *Let Z be a central subgroup of finite index n in a group G. Then G' is finite.*

But it was Wiegold to give for the order of G' the best upper bound $n^{\frac{1}{2}(\log_p n - 1)}$ in which p is the least prime dividing n, (see [15]). After him, Green proved this assertion for finite p-groups as follows.

Theorem 2.2. [4] ***(Green)*** *Let G be a group such that $[G : Z(G)] = p^n$. Then G' is finite and $|G'| \leq p^{\frac{1}{2}n(n-1)}$.*

Later Baer generalized Schur's theorem to higher terms of the upper and lower central series.

Theorem 2.3. [1] ***(Baer)*** *Let G be a group such that $|G/Z_n(G)|$ is finite. Then $|\gamma_{n+1}(G)|$ is finite.*

The following problem is generally attributed to P. Hall [5]: *If π is a set of primes and $[G : V^*(G)]$ is a finite π-group, is $V(G)$ also finite π-group*? In 1964, Turner-Smith generalized this concept to an arbitrary class of groups instead of finiteness [14].

Definition 2.4. Let $\mathcal{V}$ be a variety of groups and $\mathfrak{X}$ a class of groups. The pair $(\mathcal{V}, \mathfrak{X})$ is called **Schur pair**, if $G/V^*(G) \in \mathfrak{X}$ always implies that $V(G) \in \mathfrak{X}$.

If $\mathfrak{X}$ is a class of groups, $\mathfrak{L}\mathfrak{X}$ is the class of locally $\mathfrak{X}$-groups, consisting of all groups G such that every finite subset of G is contained in an $\mathfrak{X}$-subgroup. If $\mathfrak{X}$ is closed with respect to subgroup, it is equivalent to every finitely generated subgroup of G is an $\mathfrak{X}$-group. The first main theorem extends Schur pair property to a bigger class of groups.

Theorem 2.5. *Let $\mathfrak{X}$ be closed with respect to forming subgroups, images and extension. If $(\mathcal{V}, \mathfrak{X})$ is Schur pair, then $(\mathcal{V}, \mathfrak{L}\mathfrak{X})$ is also Schur pair.*

Proof. Suppose G is a group such that $G/V^*(G) \in \mathfrak{L}\mathfrak{X}$. We claim that $V(G) \in \mathfrak{L}\mathfrak{X}$. For, assume that H is a finitely generated subgroup of $V(G)$. Without loss generality, we can suppose that $H = \left\langle v_1(g_{1_1}, \cdots, g_{1_{k_1}}), \cdots, v_n(g_{n_1}, \cdots, g_{n_{k_n}})\right\rangle$. Put $K = \left\langle g_{1_1}, \cdots, g_{1_{k_1}}, \cdots, g_{n_1}, \cdots, g_{n_{k_n}}\right\rangle$. The quotient group $K/(K \cap V^*(G))$ can be considered as a finitely generated subgroup of $G/V^*(G)$ and so is an $\mathfrak{X}$-group. On the other hand,

$$\frac{K}{V^*(K)} \cong \frac{K/(K \cap V^*(G))}{V^*(K)/(K \cap V^*(G))} \in \mathfrak{X}.$$

Now, since $(\mathcal{V}, \mathfrak{X})$ is Schur pair, one can conclude that $H \leq V(K) \in \mathfrak{X}$. $\square$

With this theorem, one can see many interesting conclusions. Since the class of finite groups is closed under the desired conditions in theorem 2.5, by the Schur's theorem (Theorem 2.1) we have the following easy but important corollary which was proved by Mann, in 2007.

Corollary 2.6. [8] ***(Mann)*** *If G is a group whose central factor is locally finite group, then G' is locally finite.*

On the other hand, Green's theorem (theorem 2.3) implies the next assertion.

Corollary 2.7. *Suppose G is a group such that $G/Z(G)$ is locally finite p-group, then $\gamma_2(G)$ is a locally finite p-group.*

Also, by theorem 2.5 and Baer's theorem, one conclude:

Corollary 2.8. *Suppose G is a group such that $G/Z_n(G)$ is locally finite, then $\gamma_{n+1}(G)$ is a locally finite group.*

The following theorem presents an equivalent condition with Schur pair property.

Theorem 2.9. *Let $\mathfrak{X}$ be a class of groups which is closed with respect to forming subgroups, images and extensions. $(\mathcal{V}, \mathfrak{X})$ is Schur pair if and only if $G \in \mathfrak{X}$ implies that $\mathcal{V}M(G) \in \mathfrak{X}$.*

Proof. Suppose $(\mathcal{V}, \mathfrak{X})$ is Schur pair and G is an $\mathfrak{X}-$group. If $G \cong F/R$ is a free presentation of it and $S \trianglelefteq F$ such that $V^*(F/V(R, F)) = S/V(R, F)$, then

$$\frac{F/V(R,F)}{V^*(F/V(R,F))} \cong \frac{F}{S} \cong \frac{F/R}{SR/R} \in \mathfrak{X}.$$

Since $(\mathcal{V}, \mathfrak{X})$ is Schur pair, by Lemma 1.2 we conclude

$$\mathcal{V}M(G) \cong \frac{R \cap V(F)}{V(R,F)} \leq \frac{V(F)}{V(R,F)} = V\left(\frac{F}{V(R,F)}\right) \in \mathfrak{X}.$$

Conversely, let G be a group with $G/V^*(G) \in \mathfrak{X}$. We will prove $V(G) \in \mathfrak{X}$. Consider the free presentation $G \cong F/R$. There exists $S \trianglelefteq F$ such that $V^*(G) \cong S/R$. Hence $F/S \in \mathfrak{X}$ and by assumption, $S \cap V(F)/V(S,F) \cong \mathcal{V}M\,(F/S) \in \mathfrak{X}$. Now, $S/R = V^*(F/R)$ implies $V(S,F) \subseteq R$ and so

$$\frac{S \cap V(F)/V(S,F)}{R \cap V(F)/V(S,F)} \cong \frac{S \cap V(F)}{R \cap V(F)} \cong \frac{(S \cap V(F))R}{R} \in \mathfrak{X}.$$

Therefore $V(G)$ as an extension of $(S \cap V(F))R/R$ by $V(F/S)$ belongs to $\mathfrak{X}$, note that

$$\frac{V(F)R/R}{(S \cap V(F))R/R} \cong \frac{V(F)R}{S \cap V(F)R} \cong \frac{V(F)RS}{S} = \frac{V(F)S}{S} = V\left(\frac{F}{S}\right) \in \mathfrak{X}. \ \square$$

A group is called *Černikov* or *extermal*, if it is a finite extension of an abelian group satisfying Min. It is proved that for the class of Černikov groups, $(\mathcal{A}, \mathfrak{X})$ is a Schur pair. For details see [11], Chapter 3. Hence we conclude the following statement.

Corollary 2.10. *If G is a Černikov group, then the Schur-multiplier of G too is Černikov.*

Let π be a finite set of prime numbers. As a generalization of Baer's theorem (2.3), we know if the order of $G/Z_n(G)$ is a π-number, then $|\gamma_{n+1}(G)|$ is a finite π-number, too. It means $(\mathcal{N}_c, \mathfrak{X})$ is a Schur pair, whenever $\mathfrak{X}$ is the class of finite π-groups. Now, we have the following result.

Corollary 2.11. *If G is a finite π-group, then $\mathcal{N}_cM(G)$ is a finite π-group.*

In 1981, M. R. R. moghaddam proved $(\mathcal{N}_{k_1,k_2,\cdots,k_n}, \mathfrak{P})$ is a Schur pair, in which $\mathfrak{P}$ is the class of finite p-groups, [9]. Hence we can deduce

Proposition 2.12. *If G is a finite p-group, then $\mathcal{N}_{k_1,k_2,\cdots,k_n}M(G)$ is a finite p-group.*

Note that it is a vast generalization of Green's theorem, (theorem 2.3).

The class of locally finite groups is closed under subgroups, quotients, and extensions ([10], p. 429). As we said in the corollary 2.6, Mann proved $(\mathcal{A}, \mathfrak{X})$ is a Schur pair, in which $\mathfrak{X}$ is the class of locally finite group with finite exponent and with a long argument shew the exponent of Schur multiplier of a locally finite group with finite exponent, is finite (see [8]). Whereas it is a fast corollary of theorem 2.9.

Corollary 2.13. *If G is a locally finite group with finite exponent, then $\mathcal{M}(G)$ is locally finite and it has finite exponent.*

With the next lemma we can generalize more above results.

Lemma 2.14. *Suppose $\mathfrak{X}$ is a class of groups which is closed with respect to forming subgroups, homomorphic images and abelian tensor product of groups. Also $\mathcal{U}$ and $\mathcal{V}$ are two varieties of groups defined by two words u and v, respectively. If $(\mathcal{A}, \mathfrak{X}), (\mathcal{U}, \mathfrak{X})$ and $(\mathcal{V}, \mathfrak{X})$ are Schur pair, then $([\mathcal{U}, \mathcal{V}], \mathfrak{X})$ is Schur pair, too.*

Theorem 2.15. *Let $\mathfrak{X}$ be a class of groups which is closed with respect to forming subgroups, homomorphic images and abelian tensor product of groups. If a group G belongs to $\mathfrak{X}$, then $\mathcal{N}_{k_1,k_2,\cdots,k_n}M(G)$ is a $\mathfrak{X}$-group.*

Proof. It is enough to use several times the Lemma 2.14 for $\mathcal{U} = \mathcal{A} = \mathcal{V}$. □

A group G is said to have finite *rank (special rank)* $r = r(G)$, if every finitely generated subgroup can be generated by r elements and if r is the least positive integer with this property. If there is no such integer r, the group has rank ∞. The groups of rank 1 are just the locally cyclic groups and these are well-known to be the groups that are isomorphic with a subgroup of either the additive group of rational numbers or the multiplicative group of complex roots of unity. The class of finite rank is closed with respect to subgroups, homomorphic images and extensions. In 2013, Kurdachenko and Shumyatsky proved if G is a finite group such that $G/Z(G)$ has rank r, then the rank of G is r-bounded, [7]. Equivalently, by considering $\mathfrak{R}$ as the class of finite special rank, we have $(\mathcal{A}, \mathfrak{R})$ is Schur pair. Hence we can write the next useful assertion.

Corollary 2.16. *If G is a finite group with finite rank $r = r(G)$, then the rank of $\mathcal{M}(G)$ is r-bounded.*

Also, by the Lemma 2.14 and Theorem 2.9 we have

Proposition 2.17. *Let G be a locally finite group, then $\mathcal{N}_{k_1,k_2,\cdots,k_n}M(G)$ is locally finite.*

With similar argument of corollary 2.11 we can generalize the above proposition as follows.

Corollary 2.18. *If G is a locally finite π-group, then $\mathcal{N}_{k_1,k_2,\cdots,k_n}M(G)$ is a locally finite π-group, too.*

Further class of groups can be found whom are closed with respect to forming subgroups, homomorphic images and extensions. For those class, one can obtain a lot of more results.

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