Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 7 No. 1 (2020) pp 29-40.

Research Paper

FINITENESS OF CERTAIN LOCAL COHOMOLOGY MODULES

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Abstract. Cofiniteness of the generalized local cohomology modules $H^i_{\mathfrak{a}}(M,N)$ of two R-modules M and N with respect to an ideal $\mathfrak{a}$ is studied for some i's with a specified property. Furthermore, Artinianness of $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is investigated by using the above result, in certain graded situations, where $\mathfrak{b}_0$ is an ideal of R_0 and $\mathfrak{a} = \mathfrak{a}_0 + R_+$ such that $\mathfrak{b}_0 + \mathfrak{a}_0$ is an $\mathfrak{m}_0$-primary ideal.

1. Introduction

Generalized local cohomology has been introduced by J. Herzog in the local case [5] and in the more general case by Bijan-Zadeh [1]. Let R be a commutative Noetherian ring (not necessarily local) with identity, $\mathfrak{a}$ an ideal of R and let M, N be two R-modules. For an integer $i \geq 0$, the ith generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ is defined by $H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$. With $M = R$, we obtain the ordinary local cohomology module

DOI: 10.29252/as.2020.1683

MSC(2010): Primary:13D45

Keywords: Local cohomology, Finiteness, Serre subcategory.

Received: 27 April 2018, Accepted: 03 Sept 2019.

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$H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$ which was introduced by Grothendieck.

Recall that a class S of R-modules is a Serre subcategory of the category of R-modules, when it is closed under taking submodules, quotients and extensions. It can be observed that the subcategories of minimax $\mathfrak{a}$-cofinite R-modules, weakly Laskerian R-modules, and R-modules with finite support are examples of Serre classes.

This paper is divided into four sections. In the second section of the paper, we study some results on Serre classes by using spectral sequences. In Section 3, we investigate the cofiniteness and minimaxness property of generalized local cohomology modules.

Throughout Section 4, $R = \bigoplus_{n\geq 0} R_n$ is a graded commutative Noetherian ring, where the base ring R_0 is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}_0$. Moreover, we use $\mathfrak{a}_0$ to denote a proper ideal of R_0 and we set $R_+ = \bigoplus_{n>0} R_n$, the irrelevant ideal of R, $\mathfrak{a} = \mathfrak{a}_0 + R_+$, and $\mathfrak{m} = \mathfrak{m}_0 + R_+$ and $\mathfrak{b}_0$ is an ideal of R_0 such that $\mathfrak{a}_0 + \mathfrak{b}_0$ is $\mathfrak{m}_0$-primary ideal. Also, we use $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ to denote non-zero, finitely generated graded R-modules. It is well known that, the ith generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ inherits natural grading for each $i \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of all non-negative integers). In Section 4, using the results of Section 3, we study the Artinianness and cofiniteness of R-modules $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$.

Throughout this paper, R will always be a commutative Noetherian ring. For a representable R-module T, we denote by $\mathrm{Att}_R(T)$ the set of attached prime ideals of T. We will use $\mathrm{Max}(R)$ to denote the set of all maximal ideals of R.

For any unexplained notation and terminology we, refer the reader to [2, 3].

2. **Serre classes and Spectral sequences**

This section begins with the following theorem.

Theorem 2.1. *([12], Theorem 10.47). Let $A \xrightarrow{G} B \xrightarrow{F} C$ be covariant additive functors, where A, B, and C are abelian categories with enough injectives. Assume that F is left exact and that GE is right F-acyclic for every injective object E in A. Then, for every object T in A, there is a third quadrant spectral sequence with*

$$E_2^{p,q} = (R^pF)(R^qG)T \underset{p}{\Longrightarrow} R^n(FG)T = E^n.$$

Theorem 2.2. *Let the situation be as in Theorem 2.1. If $(R^pF)(R^qG)T = 0$ for all $p > r$ or $q > s$ (resp. $p < c$ or $q < d$), then $(R^rF)(R^sG)T \cong R^{r+s}(FG)T$, (resp. $(R^cF)(R^dG)T \cong R^{c+d}(FG)T$.)*

Proof. *It follows from Theorem 2.1 there is a Grothendieck's spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG)T \underset{p}{\Longrightarrow} R^{p+q}(FG)T = E^n.$$

We consider the exact sequence $E_k^{r-k,s+k-1} \longrightarrow E_k^{r,s} \longrightarrow E_k^{r+k,s+1-k}$. *Since* $E_k^{r-k,s+k-1} = E_k^{r+k,s-k+1} = 0$ *for all* $k \geq 2$, *we get* $E_2^{r,s} = E_3^{r,s} = \cdots = E_\infty^{r,s}$.
On the other hand, there is a filtration φ *of* E^{r+s} *with*

$$0 = \varphi^{r+s+1}E^{r+s} \subseteq \varphi^{r+s}E^{r+s} \subseteq \cdots \subseteq \varphi^1 E^{r+s} \subseteq \varphi^0 E^{r+s} = E^{r+s}$$

such that $E_\infty^{i,r+s-i} = \frac{\varphi^i E^{r+s}}{\varphi^{i+1}E^{r+s}}$ *for all* $0 \leq i \leq r+s$. *As* $E_2^{i,r+s-i} = 0$ *for all* $i \neq r$, *we have* $\varphi^{r+1}E^{r+s} = \varphi^{r+2}E^{r+s} = \cdots = \varphi^{r+s+1}E^{r+s} = 0$ *and* $\varphi^r E^{r+s} = \varphi^{r-1}E^{r+s} = \cdots = \varphi^0 E^{r+s} = E^{r+s}$. *It follows* $E_\infty^{r,s} = \frac{\varphi^r E^{r+s}}{\varphi^{r+1}E^{r+s}} \cong \varphi^r E^{r+s} \cong E^{r+s}$. *So* $(R^rF)(R^sG)T \cong R^{r+s}(FG)T$.
For the last part of Theorem, with an argument similar to the first part of Theorem, we get $E_2^{c,d} = E_3^{c,d} = \cdots = E_\infty^{c,d}$. *We have a filtration* φ *of* $E^{c+d} = R^{c+d}(FG)T$ *with* $0 = \varphi^{d+c+1}E^{c+d} \subseteq \varphi^{d+c}E^{d+c} \subseteq \cdots \subseteq \varphi^1 E^{c+d} \subseteq \varphi^0 E^{c+d} = E^{c+d}$ *such that* $\varphi^{c+1}E^{d+c} = \varphi^{c+2}E^{d+c} = \cdots = \varphi^{c+d+1}E^{c+d} = 0$ *and* $\varphi^c E^{c+d} = \varphi^{c-1}E^{c+d} = \cdots = \varphi^0 E^{c+d} = E^{c+d}$. *It follows* $E_\infty^{c,d} = \frac{\varphi^c E^{c+d}}{\varphi^{c+1}E^{c+d}} = \varphi^c E^{c+d} = E^{c+d}$. *This proves the claim.*

Theorem 2.3. *Let the situation be as in Theorem 2.1. If* S *is Serre class and* $(R^pF)(R^qG)T$ *is in* S *for all* $q < r$, *then* $R^n(FG)T$ *is in* S *for* $n < r$.

Proof. *By Theorem 2.1, there is a Grothendieck's spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG)T \underset{p}{\Longrightarrow} R^{p+q}(FG)T = E^n.$$

For all $i \geq 2$, $t < r$ *and* $p \geq 0$, *we consider the exact sequence*

$$0 \longrightarrow Kerd_i^{p,t} \longrightarrow E_i^{p,t} \longrightarrow E_i^{p+i,t-i+1}. \quad (1)$$

Since $E_i^{p,t} = \dfrac{Kerd_{i-1}^{p,t}}{Imd_{i-1}^{p-i+1,t+i-2}}$ *and* $E_i^{p,j} = 0$ *for all* $j < 0$, *we use (1) to obtain* $Kerd_{t+2}^{i,t-i} \cong E_{t+2}^{i,t-i} \cong \ldots \cong E_\infty^{i,t-i}$ *for all* $0 \leq i \leq t$. *In addition, there exists a finite filtration*

$$0 = \varphi^{t+1}E^t \subseteq \varphi^t E^t \subseteq \ldots \subseteq \varphi^1 E^t \subseteq \varphi^0 E^t = E^t$$

such that $E_\infty^{i,t-i} = \frac{\varphi^i E^t}{\varphi^{i+1}E^t}$ *for all* $0 \leq i \leq t$.
Now, the exact sequence

$$0 \longrightarrow \varphi^{i+1}E^t \longrightarrow \varphi^i E^t \longrightarrow E_\infty^{i,t-i} \longrightarrow 0 \quad (0 \leq i \leq t)$$

in conjunction with $E_\infty^{i,t-i} \cong Kerd_{t+2}^{i,t-i} \subseteq Kerd_2^{i,t-i} \subseteq E_2^{i,t-i}$ *yields* E^i *in* S *for all* $0 \leq i < r$.

Theorem 2.4. *Let the situation be as in Theorem 2.1. If* S *is Serre class and* $(R^pF)(R^qG)T$ *is in* S *for all* $q < t$ *and* $R^i(FG)T$ *is in* S *for all* $i \geq 0$, *then* $(R^pF)(R^tG)T$ *is in* S *for all* $p = 0, 1$.

Proof. *By Theorem 2.1, there is a Grothendieck's spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG)T \underset{p}{\Longrightarrow} R^{p+q}(FG)T = E^n.$$

Also, there is a bounded filtration $0 = \varphi^{t+1}E^t \subseteq \varphi^t E^t \subseteq \cdots \subseteq \varphi^1 E^t \subseteq \varphi^0 E^t = E^t$ *such that* $E_\infty^{i,t-i} \cong \frac{\varphi^i E^t}{\varphi^{i+1}E^t}$ *for all* $0 \leq i \leq t$ *and hence* $E_\infty^{p,q}$ *is in* S *for all* p, q*. Note that* $E_\infty^{p,q} = E_r^{p,q}$ *for large* r *and each* p *and* q*. It follows that there is an integer* $\ell \geq 2$ *such that* $E_r^{p,q}$ *is in* S *for all* $r \geq \ell$*. We argue by descending induction on* ℓ*. Now, assume that* $2 < \ell < r$ *and that the claim holds for* ℓ*. Since* $E_r^{p,q}$ *is in a subquotient of* $E_2^{p,q}$ *for all* $p, q \in \mathbb{N}_0$*, the hypotheses give* $E_r^{p+r,t-r+1}$ *is in* S *for all* $r \geq 2$*. In addition,* $E_\ell^{p,t} = \dfrac{Kerd_{\ell-1}^{p,t}}{Imd_{\ell-1}^{p-\ell+1,t+\ell-2}}$ *and* $Imd_{\ell-1}^{p-\ell+1,t+\ell-2} = 0$ *for* $p = 0, 1$*, it follows that* $Kerd_{\ell-1}^{p,t}$ *is in* S *for all* $\ell > 2$ *and* $p = 0, 1$*. Let* $r \geq 2$ *and* $p = 0, 1$*, we consider the sequence*

$$0 \longrightarrow Kerd_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both $Kerd_{\ell-1}^{p,t}$ *and* $E_{\ell-1}^{p+r,t-r+1}$ *are in* S*, it follows that* $E_{\ell-1}^{p,t}$ *is in* S *for* $p = 0, 1$*. This completes the inductive step.*

3. **Minimax and Cofinite modules**

We keep the notation and hypotheses given in the introduction and continue with the following definition and remark:

Definition and Remark 3.1. (i) An R-module T is said to be $\mathfrak{a}$ -cofinite if Supp$T \subseteq V(\mathfrak{a})$ and $Ext_R^i(\frac{R}{\mathfrak{a}}, T)$ is finitely generated R-module for all $i \geq 0$.
Moreover , if R is local ring with maximal ideal $\mathfrak{m}$, then an R-module is $\mathfrak{m}$-cofinite if and only if it is artinian R-module (see[10]).

(ii) T is called a minimax R-module if there is a finitely generated submodule L such that T/L is Artinian R-module.

(iii) Recall that an R-module T is said to be $\mathfrak{a}$-cominimax if Supp$(T) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}_R^i(R/\mathfrak{a}, T)$ is minimax for all i.

(iv) We say that T is a weakly Laskerian R-module if the set of associated primes of any quotient module of T is finite.

(v) T is $\mathfrak{a}$-weakly cofinite R-module if Supp$M \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}_R^i(\frac{R}{\mathfrak{a}}, T)$ is weakly Laskerian R-module for all $i \geq 0$.

In addition, we denote by $\dim \mathrm{Supp} H_\mathfrak{a}^i(M, N)$ the maximum of numbers $\dim(R/\mathfrak{p})$, where $\mathfrak{p}$ runs over the support of $H_\mathfrak{a}^i(M, N)$.

Theorem 3.2. *Let* t *and* k *be non-negative integers,* M *finitely generated* R*-module and* N *an arbitrary* R*-module. If* $\dim Supp H_\mathfrak{a}^i(M, N) \leq k$ *for all* $i < t$*, and* $\mathfrak{a} + Ann(M) = I$*, then* $\dim Supp H_\mathfrak{a}^i(M, N/\Gamma_I(N)) \leq k$*.*

Proof. From the short exact sequence $0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0$, we get the long exact sequence $H_\mathfrak{a}^i(M, \Gamma_I(N)) \longrightarrow H_\mathfrak{a}^i(M, N) \longrightarrow H_\mathfrak{a}^i(M, N/\Gamma_I(N)) \longrightarrow H_\mathfrak{a}^{i+1}(M, \Gamma_I(N))$

for all i. We assume that there exists an integer $i < t$ and $\mathfrak{p} \in \mathrm{Supp} H^i_{\mathfrak{a}}(M, N/\Gamma_I(N))$ such that $\dim(R/\mathfrak{p}) > k$ and $\mathfrak{p} \notin \mathrm{Supp} H^j_{\mathfrak{a}}(M, N/\Gamma_I(N))$ for all $j < i$. Thus by the long exact sequence as above, in conjunction with the fact that $H^i_{\mathfrak{a}}(M, \Gamma_I(N)) \cong \mathrm{Ext}^i_R(M, \Gamma_I(N))$, we obtain the following exact sequence

$$\mathrm{Ext}^i_R(M, \Gamma_I(N))_{\mathfrak{p}} \longrightarrow H^i_{\mathfrak{a}}(M, N)_{\mathfrak{p}} \longrightarrow H^i_{\mathfrak{a}}(M, N/\Gamma_I(N))_{\mathfrak{p}} \longrightarrow \mathrm{Ext}^{i+1}_R(M, \Gamma_I(N))_{\mathfrak{p}}. \quad (2)$$

Note that $H^j_{\mathfrak{a}}(M, N)_{\mathfrak{p}} = 0$ for all $j \leq i$, while $H^j_{\mathfrak{a}}(M, N/\Gamma_I(N))_{\mathfrak{p}} = 0$ for all $j < i$, and $H^i_{\mathfrak{a}}(M, N/\Gamma_I(N))_{\mathfrak{p}} \neq 0$. So, by the exact sequence (2), we have $\mathrm{Ext}^j_R(M, \Gamma_I(N))_{\mathfrak{p}} = 0$ for all $j \leq i$, and $\mathrm{Ext}^{i+1}_R(M, \Gamma_I(N))_{\mathfrak{p}} \neq 0$. It implies that $\Gamma_I(N)_{\mathfrak{p}} \neq 0$ and $\mathrm{depth}(M_{\mathfrak{p}}, \Gamma_I(N)_{\mathfrak{p}}) = i + 1 \geq 1$. Hence $\mathrm{Ann}(M)_{\mathfrak{p}} \nsubseteq \mathfrak{q}R_{\mathfrak{p}}$ for all $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}}(\Gamma_I(N))_{\mathfrak{p}}$. This contradicts the fact that $\mathrm{Ass}_{R_{\mathfrak{p}}}(\Gamma_I(N))_{\mathfrak{p}} = \mathrm{Ass}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \bigcap V(I)_{\mathfrak{p}}$.

Lemma 3.3. *Let T be a representable R-module and let Hom$(\frac{R}{\mathfrak{a}}, T)$ be of finite length. Then $V(\mathfrak{a}) \cap Att(T) \subseteq Max(R)$.*

Proof. Let $\mathfrak{p} \in V(\mathfrak{a}) \cap \mathrm{Att}(T)$, and let $T = S_1 + S_2 + \cdots + S_n$, where S_r is $\mathfrak{p}$-secondary. It follows that there is an integer k such that $\mathfrak{p}^k S_r = 0$, and so $\mathfrak{a}^k S_r = 0$. As $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, T)$ is of finite length, $\mathrm{Hom}(\frac{R}{\mathfrak{a}^k}, T)$ is of finite length. Since $S_r \subseteq \mathrm{Hom}(\frac{R}{\mathfrak{a}^k}, T)$, it follows that S_r has finite length. Therefore $\mathfrak{p} \in \mathrm{Max}(R)$ (see [[2] Corollary 7.2.12]), as required.

Lemma 3.4. *Let T be a representable R-module, and $\mathfrak{q} \in Spec(R)$. Let $x \notin \bigcup_{i=1}^n \mathfrak{p}_i$, where $\mathfrak{p}_i \in Att(T) - \{\mathfrak{q}\}$. Then $Att(T/xT) \subseteq \{\mathfrak{q}\}$.*

Proof. Let $S_1 + \cdots + S_n + S$ be a minimal secondary representation of T, where S is $\mathfrak{q}$-secondary and S_i is $\mathfrak{p}_i$-secondary for all $i = 1, \cdots, n$. Since $x \notin \bigcup_{i=1}^n \mathfrak{p}_i$ for all $1 \leq i \leq n$, so $xS_i = S_i$. Note that $(T/xT) = S/(S \cap (S_1 + \cdots + S_n + xS))$. Therefore $\mathrm{Att}(T/xT) \subseteq \mathrm{Att} S = \{\mathfrak{q}\}$. On other hand, if $\mathfrak{q} \notin \mathrm{Att}(T)$, then $\mathrm{Att}(T/xT) = \mathrm{Att}(0) = \emptyset$, our claim is clear.
For the prove Theorem 3.6, we need the following proposition:

Proposition 3.5. *Let T be R-module. The following conditions hold:*

(i) *Let $(R, \mathfrak{m})$ be local ring and let T be a minimax R-module such that $Supp(T) \subseteq \{\mathfrak{m}\}$. Then T is Artinian.*

(ii) *Let T be an R-module. Suppose $x \in \mathfrak{a}$ and $Supp(T) \subseteq V(\mathfrak{a})$. If $(0 :_T x)$ and T/xT are $\mathfrak{a}$-cominimax, then T is $\mathfrak{a}$-cominimax.*

Proof. i) In view of the assumtion and Definition and Remark 3.1(i), we have need only to show that $\mathrm{Ext}^i_R(R/\mathfrak{m}, T)$ is finitely generated for all $i \geq 0$. To do this, since T is a minimax R-module, there exists an exact sequence $0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$ where T' is finitely generated and T'' is Artinian R-module. using 3.1(i), T'' is $\mathfrak{m}$-cofinite, then $\mathrm{Ext}^i_R(R/\mathfrak{m}, T'')$ is finitely generated for all $i \geq 0$. So, in view of the long exact sequence

$\mathrm{Ext}^i_R(R/\mathfrak{m},T') \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{m},T) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{m},T'')$, we deduce that $\mathrm{Ext}^i_R(R/\mathfrak{m},T)$ is finitely generated for all $i \geq 0$, as required.

ii) Since $\mathrm{Supp}T \subseteq V(\mathfrak{a})$, we need just prove that $\mathrm{Ext}^i_R(R/\mathfrak{a},T)$ is minimax for all $i \geq 0$. So, we assume that r is a non-negative integer and prove $\mathrm{Ext}^r_R(R/\mathfrak{a},T)$ is minimax. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (0:_T x) & \longrightarrow & T & \xrightarrow{x} & xT & \longrightarrow & 0 \\ & & & & \downarrow x & \searrow x & & & \\ 0 & \longrightarrow & xT & \longrightarrow & T & \longrightarrow & T/xT & \longrightarrow & 0 \end{array}$$

implies the following long exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{Ext}^r_R(R/\mathfrak{a},(0:_T x)) & \longrightarrow & \mathrm{Ext}^r_R(R/\mathfrak{a},T) & \xrightarrow{\varphi} & \mathrm{Ext}^r_R(R/\mathfrak{a},xT) & \longrightarrow & \cdots \\ & & & & \varphi\downarrow & \searrow & & & \\ \cdots & \longrightarrow & \mathrm{Ext}^{r-1}_R(R/\mathfrak{a},T/xT) & \longrightarrow & \mathrm{Ext}^r_R(R/\mathfrak{a},xT) & \xrightarrow{\lambda} & \mathrm{Ext}^r_R(R/\mathfrak{a},T) & \longrightarrow & \cdots \end{array}.$$

As T/xT is $\mathfrak{a}$-comonimax, $\mathrm{Ext}^{r-1}_R(R/\mathfrak{a},T/xT)$ is minimax. Then $\ker\lambda$ is minimax. Also, $\varphi(0:_{\mathrm{Ext}^r_R(R/\mathfrak{a},T)} x)$ is minimax, since $\varphi(0:_{\mathrm{Ext}^r_R(R/\mathfrak{a},T)} x) \subseteq \ker\lambda$. Note that $\mathrm{Ext}^r_R(R/\mathfrak{a},(0:_T x))$ is minimax, so $\ker\varphi$ is minimax. Therefore, by using the exact sequence $0 \longrightarrow \ker\varphi \longrightarrow \mathrm{Ext}^r_R(R/\mathfrak{a},T) \longrightarrow \mathrm{Im}\varphi \longrightarrow 0$, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}(R/x,\ker\varphi) \longrightarrow \mathrm{Hom}(R/x,\mathrm{Ext}^r_R(R/\mathfrak{a},T)) \longrightarrow \varphi(0:_{\mathrm{Ext}^r_R(R/\mathfrak{a},T)} x) \longrightarrow 0.$$

It follows that $\mathrm{Hom}(R/x,\mathrm{Ext}^r_R(R/\mathfrak{a},T))$ is minimax. Since $x \in \mathfrak{a}$, $\mathrm{Hom}(R/x,\mathrm{Ext}^r_R(R/\mathfrak{a},T)) \cong \mathrm{Ext}^r_R(R/\mathfrak{a},T)$. This completes the proof.

Theorem 3.6. *Let $(R,\mathfrak{m})$ be local, M a finitely generated R-module and N minimax R-module. Let t be non-negative integer such that* $\dim$ *Supp*$H^i_{\mathfrak{a}}(M,N) \leq 1$ *for all $i < t$. Then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cominimax for all $i < t$ and Hom$(R/\mathfrak{a},H^t_{\mathfrak{a}}(M,N))$ is minimax.*

Proof. As $H^i_{\mathfrak{a}+\mathrm{Ann}(M)}(M,N) \cong H^i_{\mathfrak{a}}(M,N)$ for all i, we can assume that $\mathrm{Ann}(M) \subseteq V(\mathfrak{a})$. Now, we prove the claim by induction on $t \geq 0$. The case of $t = 0$ is clear. If $t = 1$, then it can be observed $H^0_{\mathfrak{a}}(M,N) \cong \mathrm{Hom}(M,\Gamma_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-cominimax and $\mathrm{Hom}(R/\mathfrak{a},H^1_{\mathfrak{a}}(M,N))$ is minimax by ([4], Theorem 2.7). Assume that $t > 1$ and the result holds true for the case $t-1$. From the short exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0$, we get the long exact sequence

$$H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)) \xrightarrow{f_i} H^i_{\mathfrak{a}}(M,N) \xrightarrow{g_i} H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) \xrightarrow{h_i} H^{i+1}_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)).$$

For each $i \geq 0$, we split the above exact sequence into the two following exact sequences

$$0 \longrightarrow \mathrm{Im} f_i \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{Im} g_i \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \mathrm{Im} g_i \longrightarrow H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) \longrightarrow \mathrm{Im} h_i \longrightarrow 0.$$

Note that $\mathrm{Im} f_i$ and $\mathrm{Im} h_i$ are minimax R-modules for all $i \geq 0$. Then, for each $i < t$, we obtain that $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cominimax module if and only if so is $H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N))$. On the other hand, we get by Theorem 3.2 that $\dim \mathrm{Supp} H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) \leq 1$ for all $i < t$. Therefore, in order to prove the theorem for the case of $t > 1$, we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. In addition, put $X = \bigcup_{i=0}^{t-1} \mathrm{Supp}(H^i_{\mathfrak{a}}(M,N))$, $S = \{\mathfrak{p} \in X \mid \dim(R/\mathfrak{p}) = 1\}$. Thus $S \subseteq \bigcup_{i=0}^{t-1} \mathrm{Ass}(H^i_{\mathfrak{a}}(M,N))$. By using the inductive hypothesis, the R-module $\mathrm{Hom}(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N))$ is minimax and R-module $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cominimaxe for all $i < t-1$; so $\mathrm{Ass} H^i_{\mathfrak{a}}(M,N)$ is finite set for all $i < t$. It implies that $\bigcup_{i=0}^{t-1} \mathrm{Ass}(H^i_{\mathfrak{a}}(M,N))$ is a finite set, and so S is a finite set. Assume that $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n\}$. Since $\mathfrak{p}_k$ is a non-maximal prime of R, $\mathrm{Hom}(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M,N))_{\mathfrak{p}_k}$ is finitely generated for all $i < t$. Now, it is straightforward to see that $\mathrm{Supp}_{R_{\mathfrak{p}_k}} H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k} \subseteq \mathrm{Max}(R_{\mathfrak{p}_k})$. It follows that $\mathrm{Hom}(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M,N))_{\mathfrak{p}_k}$ is Artinian for all $i < t$. As $H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k}$ is $\mathfrak{a}R_{\mathfrak{p}_k}$-torsion, it yields from Melkerssons' theorem ([9], Theorem 1.3) $H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k}$ is Artinian for all $i < t$ and $k = 1, \cdots, n$. Therefore $\bigcup_{i=0}^{t-1} \mathrm{Att} H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k}$ is finite set. Now, we choose an element $x \in \mathfrak{a}$ such that $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(N)} \mathfrak{p} \bigcup_{i=0}^{t-1} \bigcup_{\mathfrak{p} \in Y_i} \mathfrak{p}$, where $Y_i = \bigcup_{k=1}^{n} \{\mathfrak{q} \mid \mathfrak{q}R_{\mathfrak{p}_k} \in \mathrm{Att} H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k}\} - V(\mathfrak{a})$. It follows that x is an N-sequence. Therefore, we may consider the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ to obtain the exact sequence

$$H^i_{\mathfrak{a}}(M,N) \xrightarrow{x} H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{a}}(M,N/xN) \longrightarrow H^{i+1}_{\mathfrak{a}}(M,N)$$

for all $i \geq 0$. It implies the following exact sequence

$$0 \longrightarrow H^i_{\mathfrak{a}}(M,N)/xH^i_{\mathfrak{a}}(M,N) \xrightarrow{\alpha_i} H^i_{\mathfrak{a}}(M,N/xN) \xrightarrow{\beta_i} (0 :_{H^{i+1}_{\mathfrak{a}}(M,N)} x) \longrightarrow 0 \quad (3)$$

for all $i \geq 0$. Using the exact sequence (3) in conjunction with the hypothesis, yields the $\dim \mathrm{Supp}(H^i_{\mathfrak{a}}(M,N/xN)) \leq 1$ for all $i < t-1$. Hence we get by the inductive hypothesis that for all $i < t-1$ the R-modules $H^i_{\mathfrak{a}}(M,N/xN)$ is $\mathfrak{a}$-cominimax and the R-module $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, H^{t-1}_{\mathfrak{a}}(M,N/xN))$ is minimax.

In other hand, $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, H^i_{\mathfrak{a}}(M,N))_{\mathfrak{p}_k}$ is of finite length for all $i < t$ and $k = 1, \cdots, n$. Therefore, by Lemma 3.3, $V(\mathfrak{a})_{\mathfrak{p}_k} \bigcap \mathrm{Att} H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}_k} \subseteq \{\mathfrak{p}_k R_{\mathfrak{p}_k}\}$. By the choice of x and Lemma 3.4 and [[2] Corollary 7.2.12], we conclude that $(\frac{H^i_{\mathfrak{a}}(M,N)}{xH^i_{\mathfrak{a}}(M,N)})_{\mathfrak{p}_k}$ has finite length for all $i < t$ and all $k = 1, \cdots, n$. Then, there exists a finitely generated submodule L_{i_k} of $L_i = (H^i_{\mathfrak{a}}(M,N)/xH^i_{\mathfrak{a}}(M,N))$ such that $(L_{i_k})_{\mathfrak{p}_k} = (L_i)_{\mathfrak{p}_k}$. Put $V_i = L_{i_1} + L_{i_2} + \cdots + L_{i_n}$. Then V_i is a finitely generated submodule of L_i and $\mathrm{Supp}(\frac{L_i}{V_i}) \subseteq X - \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\} \subseteq \mathrm{Max}(R)$

for all $i < t$. Since $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, H^i_{\mathfrak{a}}(M, N/xN))$ is minimax for all $i < t$, it follows from exact sequence (3) that $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, L_i)$ is minimax for all $i < t$. Consideration of the exact sequence $0 \longrightarrow V_i \longrightarrow L_i \longrightarrow L_i/V_i \longrightarrow 0$ shows that $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, \frac{L_i}{V_i})$ is a minimax R-module for all $i < t$. Using Proposition 3.5(i), it is also Artinian module, because it is supported only at maximal ideals. As $\frac{L_i}{V_i}$ is $\mathfrak{a}$-torsion, it is concluded from Melkersson' theorem ([9], Theorem 1.3) that R-module $\frac{L_i}{V_i}$ is Artinian, and so L_i is minimax for all $i < t$. Again, consideration of the exact sequence (3) shows that $(0 :_{H^{t-1}_{\mathfrak{a}}(M,N)} x)$ is $\mathfrak{a}$-cominimax and that $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, (0 :_{H^t_{\mathfrak{a}}(M,N)} x)$ is minimax R-module. Since $x \in \mathfrak{a}$ and $\mathrm{supp} H^{t-1}_{\mathfrak{a}}(M, N) \subset V(\mathfrak{a})$, the R-modules $(0 :_{H^{t-1}_{\mathfrak{a}}(M,N)} x)$ and $H^{t-1}_{\mathfrak{a}}(M, N)/xH^{t-1}_{\mathfrak{a}}(M, N)$ are $\mathfrak{a}$-cominimax, it implies that $H^{t-1}_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cominimax by Proposition 3.5(ii). The following completes the proof: $\mathrm{Hom}(\frac{R}{\mathfrak{a}}, (0 :_{H^t_{\mathfrak{a}}(M,\, N)} x)) \cong \mathrm{Hom}(\frac{R}{\mathfrak{a}} \otimes \frac{R}{(x)}, H^t_{\mathfrak{a}}(M, N)) \cong \mathrm{Hom}(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M, N)$.
Theorem 3.6 yields the following:

Proposition 3.7. *Let $(R, \mathfrak{m})$ be local and M, N two finitely generated R-modules. Let t be non-negative integer such that $\dim Supp H^i_{\mathfrak{a}}(M, N) \leq 1$ for all $i < t$. Then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is finite.*

Proof. Using an argument similar to the proof of Theorem 3.6 the result follows.

Theorem 3.8. *Let $(R, \mathfrak{m})$ be local ring, and let M be a finitely generated R-module and N be a minimax R-module. Let t be a non-negative integers such that $\dim Supp H^i_{\mathfrak{a}}(M, N) \leq 2$ for all $i < t$. Then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-weakly cofinite for $j \geq 0$ and $i < t$.*

Proof. Since $\mathrm{Supp} H^i_{\mathfrak{a}}(M, N) \subseteq V(\mathfrak{a})$, it is sufficient to show that $\mathrm{Ext}^j_R(\frac{R}{\mathfrak{a}}, H^i_{\mathfrak{a}}(M, N))$ is weakly Laskerian for all $j \geq 0$ and $i < t$. Using ([2]), Theorem 4.3.2) and ([7], Exercice 7.7), without lost of generality, we may assume that R is complete. Now, suppose, contrary to our claim, that there are fixed integers $i < t$ and $j \geq 0$ such that $\mathrm{Ext}^j_R(\frac{R}{\mathfrak{a}}, H^i_{\mathfrak{a}}(M, N))$ is not weakly Laskerian. In view of definition, let T' be a submodule of $T = \mathrm{Ext}^j_R(\frac{R}{\mathfrak{a}}, H^i_{\mathfrak{a}}(M, N))$ such that there exists a countably infinite subset $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ of $\mathrm{Ass}(T/T')$ such that $\mathfrak{p}_i \neq \mathfrak{m}$ for all i. By ([8], Lemma 3.2), there exists $x \in \mathfrak{m}$ such that $x \notin \bigcup_{i=1}^{\infty} \mathfrak{p}_i$. Let $S = \{x^k \mid 0 \leq k \in \mathbb{Z}\}$. Then $S \cap \mathfrak{m} \neq \emptyset$. So, $\dim \mathrm{Supp} S^{-1}(H^i_{\mathfrak{a}}(M, N)) \leq 1$. It follows from Theorem 3.6, $\mathrm{Ext}^j_{S^{-1}R}(\frac{S^{-1}R}{S^{-1}\mathfrak{a}}, S^{-1}(H^i_{\mathfrak{a}}(M, N)))$ is minimax $S^{-1}R$-module. Therefore, $\mathrm{Ass}\frac{S^{-1}T}{S^{-1}T'}$ is finite set, and so by ([7], Theorem 6.2) $\mathrm{Ass} S^{-1}(T/T') = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass} T/T', \mathfrak{p} \cap S = \varnothing\}$, which is a contradiction.

4. **Artinianness of composed graded local cohomology**

The concept of tameness is the most fundamental concept related to the asymptotic behaviour of local cohomology modules. A graded R-module $T = \bigoplus T_n$ is said to be tame or asymptotic gap-free if either $T_n = 0$ for all $n \ll 0$ or else $T_n \neq 0$ for all $n \ll 0$. It is known that any graded Artinian R-module is tame. In this section, we keep the notation and hypotheses introduced in the introduction and we examine the Artinianness of graded module $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ for some $i \geq 0$ and $j \geq 0$.

Theorem 4.1. *Let T be an $\mathfrak{a}$-torsion minimax graded R-module and $\sqrt{\mathfrak{a}_0+\mathfrak{b}_0}=\mathfrak{m}_0$. Then $H^i_{\mathfrak{b}_0}(T)$ and $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T)$ are Artinian for all $i \geq 0$.*

Proof. As T is graded minimax R-module, there is an exact sequence of graded R-modules

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T/T' \longrightarrow 0 \qquad (4)$$

such that T' is graded finitely generated and T/T' is graded Artinian. The application of local cohomology with respect to $\mathfrak{b}_0$ to it leads to an exact sequence of graded R-modules $0 \longrightarrow \Gamma_{\mathfrak{b}_0}(T') \longrightarrow \Gamma_{\mathfrak{b}_0}(T) \longrightarrow \Gamma_{\mathfrak{b}_0}(T/T') \longrightarrow H^1_{\mathfrak{b}_0}(T') \longrightarrow H^1_{\mathfrak{b}_0}(T) \longrightarrow 0$ and the isomorphisms $H^i_{\mathfrak{b}_0}(T') \cong H^i_{\mathfrak{b}_0}(T)$ for all $i \geq 2$. We note that $\Gamma_{\mathfrak{b}_0}(T/T')=T/T'$ is Artinian. Since $\Gamma_{\mathfrak{a}}(T)=T$, the submodule T' is $\mathfrak{a}$-torsion. Then $H^i_{\mathfrak{b}_0}(T') \cong H^i_{\mathfrak{m}}(T')$ for all $i \geq 0$. This proves $H^i_{\mathfrak{b}_0}(T)$ is Artinian for all $i \geq 0$. For the second claim, if we apply the functor $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},\quad)$ to the short exact sequence (4), we have the following exact sequence of graded R-modules

$$Tor_{i+1}^{R_0}(\frac{R_0}{\mathfrak{b}_0},T/T') \longrightarrow Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T') \longrightarrow Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T) \longrightarrow Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T/T'). \quad (5)$$

According to [[4], Lemma 2.1], the graded module $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T/T')$ is Artinian for each i. Since $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T')$ is an $\mathfrak{a}$-torsion finitely generated graded R-module, $\mathfrak{m}^{k_i}Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T')=0$ for some $k_i \in \mathbb{N}$. It follows that $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T')$ is Artinian for all $i \geq 0$. In view of exact sequence (5), $Tor_i^{R_0}(\frac{R_0}{\mathfrak{b}_0},T)$ is Artinian.

Theorem 4.2. *Let T be an $\mathfrak{a}$-torsion and $\mathfrak{a}$-cofinite graded R-module and $\sqrt{\mathfrak{a}_0+\mathfrak{b}_0}=\mathfrak{m}_0$. Then $H^i_{\mathfrak{b}_0}(T)$ is Artinian and $\mathfrak{a}$-cofinite R-module for all $i \geq 0$.*

Proof. We proceed by induction on i. If $i=0$, then $\Gamma_{\mathfrak{b}_0}(T)=\Gamma_{\mathfrak{m}}(T)$ is Artinian and $\mathfrak{a}$-cofinite by ([10], Corollary 1.8). Let $i=1$. As T is $\mathfrak{a}$-torsion, there is a short exact sequence of $\mathfrak{a}$-torsion graded modules $0 \longrightarrow T \longrightarrow E \xrightarrow{d} C \longrightarrow 0$ such that E is injective and C is $\mathfrak{a}$-cofinite as well. Application of the functor $\Gamma_{\mathfrak{b}_0}(\quad)$ induces the following exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}_0}(T) \longrightarrow \Gamma_{\mathfrak{b}_0}(E) \xrightarrow{\Gamma_{\mathfrak{b}_0}(d)} \Gamma_{\mathfrak{b}_0}(C) \longrightarrow H^1_{\mathfrak{b}_0}(T) \longrightarrow 0 \ .$$

Consider $X=\operatorname{Im}(\Gamma_{\mathfrak{b}_0}(d))$. In view of the case $i=0$, we deduce that $\Gamma_{\mathfrak{b}_0}(T)$ and $\Gamma_{\mathfrak{b}_0}(C)$ are Artinian and $\mathfrak{a}$-cofinite. So $\operatorname{Hom}(\frac{R}{\mathfrak{a}},X)$ is finitely generated and hence the isomorphism $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}},X) \cong \operatorname{Ext}^{i+1}_R(\frac{R}{\mathfrak{a}},\Gamma_{\mathfrak{b}_0}(T))$, for each $i \geq 1$, implies that X is Artinian and $\mathfrak{a}$-cofinite. Now, the cofiniteness and Artinianness of X and $\Gamma_{\mathfrak{b}_0}(C)$ imply that $H^1_{\mathfrak{b}_0}(T)$ is Artinian and $\mathfrak{a}$-cofinite. On the other hand, for each $i>1$, there is an isomorphism $H^{i-1}_{\mathfrak{b}_0}(C) \cong H^i_{\mathfrak{b}_0}(T)$. Therefore the induction completes the proof.

Theorem 4.3. *Let M and N two finitely generated graded R-modules. Let t be a non-negative integer such that $\dim SuppH^i_{\mathfrak{a}}(N) \leq 1$ for all $i < t$ and $\sqrt{\mathfrak{a}_0 + \mathfrak{b}_0} = \mathfrak{m}_0$. Then $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M, N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i < t$ and $j \geq 0$. In addition, $H^j_{\mathfrak{b}_0}(H^t_{\mathfrak{a}}(M, N))$ is Artinian for all $j = 0, 1$.*

Proof. In view of Theorem 2.1, there is a Grothendieck's spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M, N).$$

Using Theorem 3.2, $\dim \operatorname{Supp} H^i_{\mathfrak{a}}(M, N) \leq 1$ for all $i < t$. It follows from Proposition 3.7 that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i < t$. Then, from Theorem 4.2, $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M, N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i < t$ and all $j \geq 0$. On the other, using the spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{b}_0}(H^q_{\mathfrak{a}}(M, N) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M, N)$$

in conjunction with Theorem 2.4 and the fact that $H^i_{\mathfrak{m}}(M, N)$ is Artinian for all $i \geq 0$, the result follows.

Corollary 4.4. *Let M and N two finitely generated graded R-modules. Let t be a non-negative integer such that $SuppH^i_{\mathfrak{a}}(M, N)$ is finite set for all $i < t$ and $\sqrt{\mathfrak{a}_0 + \mathfrak{b}_0} = \mathfrak{m}_0$. Then $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M, N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i < t$ and $j \geq 0$. In addition, $H^j_{\mathfrak{b}_0}(H^t_{\mathfrak{a}}(M, N))$ is Artinian for all $j = 0, 1$.*

Proof. Since the number of prime ideals between two given ones in Noetherian ring is zero or infinite, $\dim \operatorname{Supp} H^i_{\mathfrak{a}}(M, N) \leq 1$ for all $i < t$. So, it follows immediately by using Theorem 4.3.

Definition 4.5. A sequence $x_1, \cdots, x_n$ of elements of $\mathfrak{a}$ is said to be a generalized regular sequence of T if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(T/(x_1, \cdots, x_{i-1})T)$ satisfying $\dim(R/\mathfrak{p}) > 1$ for all $i = 1, 2, \cdots, n$. It is clear that if $\dim(T/\mathfrak{a}T) > 1$, each generalized regular sequence of T in $\mathfrak{a}$ has finite length. The length of a maximal regular sequence of T in $\mathfrak{a}$ is denoted by $\operatorname{gdepth}(\mathfrak{a}, T)$. (see [11]
Also, $\operatorname{gdepth}(M/\mathfrak{a}M, N)$ is a non-negative integer and is equal to the length of any maximal generalized regular N-sequence in $\mathfrak{a} + (0 : M)$.

Theorem 4.6. *Let M and N two finitely generated graded R-modules. If $gdepth(M/\mathfrak{a}M, N) = cd_{\mathfrak{a}}(M, N)$, where $cd_{\mathfrak{a}}(M, N) = sup\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M, N) \neq 0\}$, then $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M, N))$ is Artinian and cofinite for all $i \geq 0$ and $j \geq 0$. In particular, $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M, N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i \geq 0$ and $j \geq 0$ if N is $\mathfrak{a}$-cofinite.*

Proof. Using ([13], Theorem 3.2), Supp$H^i_{\mathfrak{a}}(M,N)$ is finite set for all $i < \text{gdepth}(M/\mathfrak{a}M,N)$. It follows from Corollary 4.4 that R-module $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i < \text{gdepth}(M/\mathfrak{a}M,N)$ and all $j \geq 0$. If $i > cd_{\mathfrak{a}}(M,N)$, then, in view of the definition of $cd_{\mathfrak{a}}(M,N)$, $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N)) = 0$. Thus we consider the case where $i = t = \text{gdepth}(M/\mathfrak{a}M,N)$. To this end, consider the Grothendieck's spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{b}_0}(H^q_{\mathfrak{a}}(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

According to Theorem 2.2, we have $H^j_{\mathfrak{b}_0}(H^t_{\mathfrak{a}}(M,N)) \cong (H^{t+j}_{\mathfrak{m}}(M,N))$. In view of ([2], Theorem 7.1.3), $H^i_{\mathfrak{m}}(N)$ is Artinian R-module and by ([6], Lemma 2.7), $H^i_{\mathfrak{m}}(N)$ is Artinian and $\mathfrak{a}$-cofinite R-module if N is $\mathfrak{a}$-cofinite R-module. Therefore, by using Grothendieck 's spectral sequence

$$E_2^{p,q} = \text{Ext}^p_R(N, H^q_{\mathfrak{m}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N),$$

the result follows from Theorem 2.3.

5. Acknowledgments

The author is deeply grateful to the referees for their comments the improvement of this paper. Also, I would like to express my particular thank to Professor Kamran Divaani-Azaar and Professor Amir Mafi for having carefully read the primary draft and also for his helpful contributions and suggestions.

References

[1] M. H. Bijan-Zadeh, *A common generalization of local cohomology theories*, Glasgow Math. J., **21** (1980) 173-181.

[2] M. P. Brodmann, R. Y. Sharp, *Local cohomology: An Algebraic introduction with geometric applications*, Cambridge Univ. Press, zb1 0903.13006, 1998.

[3] W. Bruns and J. Herzog, *Cohen Macaulay rings,* Cambridge Univ. Press, 1993.

[4] F. Dehghani-Zadeh, *On the finiteness properties of generalized local cohomology modules*, Int. Electronic. J. Algebra.

[5] J. Herzog, *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1974, htt://dx.doi.org/.

[6] A. Mafi, *Some results on local cohomology modules*, Arch. Math., **87**(2006) 211-216.

[7] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, 1986.

[8] T. Marley and J. Vassilev, *Cofiniteness and associated primes of local cohomology modules*, J. Alegebra., **256**(2002) 180-193.

[9] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambridge Philos. Soc., **107**(1990) 267-271.

[10] L. Melkersson, *Properties of cofinite modules and applications to local cohomology*, Math. Proc. Cambridge Phil. Soc., **125** (1999) 417-423.

[11] L. T. Nhan. *On generalized regular sequence and the finiteness for associated primes of local cohomology modules*, Comm. Algebra., **33** (2005) 793-806.

[12] J. Rotman, *An introduction to homological algebra*, 2nd edn., Springer, 2009.

[13] N. Zamani, *Finiteness Results on Generalized Local Cohomology Modules*, Colloq. Math., **16** No. 1 (2009) 65-70.

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