A NEW LOWER BOUND FOR COHOMOLOGICAL DIMENSION

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Abstract. Let \((R, m)\) be a Noetherian local ring, \(M\) a finitely generated \(R\)-module, and \(a\) an ideal of \(R\). We define the \(a\)-minimum dimension \(d(a, M)\) of \(M\) by

\[
d(a, M) = \min \{ \dim \frac{R}{p + a} : p \in \Assh_R(M) \}.
\]

In this paper, we show that \(cd(a, M) \geq \dim M - d(a, M)\) and we give some sufficient conditions and characterization for the equality to hold true.

1. Introduction

Throughout this paper, let \((R, m)\) be a commutative Noetherian local ring (with identity) and let \(M\) be a finitely generated \(R\)-module. For an \(R\)-module \(M\), the \(i\)-th local cohomology module of \(M\) with respect to \(a\) is defined as

\[
H^i_a(M) = \lim_{n \to \infty} \Ext^i_R(\frac{R}{a^n}, M).
\]

For the basic properties of local cohomology the reader can refer to [11] of Brodmann and Sharp.

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Recall that the cohomological dimension of $M$ with respect to $a$ is defined as
\[ \text{cd}(a, M) := \max\{i \in \mathbb{Z} : H^i_a(M) \neq 0\}. \]

The cohomological dimension has been studied by several authors; see, for example, Faltings [5], Hartshorne [6], Huneke-Lyubeznik [7] and Varbaro [10]. In particular in [5] and [7], several upper bounds for cohomological dimension were obtained. It follows from [1], Theorem 6.2.7 that $\text{cd}(a, M)$ is greater than or equal to the grade($a, M$). A natural question to ask is under what conditions one can obtain a better lower bound for $\text{cd}(a, M)$. The main aim of this article is to establish a new lower bound for cohomological dimension of finitely generated modules over a local ring.

Throughout this article, we denote $\{p \in \text{Spec} \ R : p \supseteq a\}$ by $V(a)$, $\text{Min} V(a)$ by $\text{Min}(a)$, and $\{p \in \text{Assh}_R(M) : \dim \frac{R}{p} = \dim M\}$ by $\text{Assh}_R(M)$. The radical of $a$, denoted by $\sqrt{a}$, is defined to be the set $\{x \in R : x^n \in a\text{ for some } n \in \mathbb{N}\}$. Recall that an $R$-module $M$ is called $a$-cofinite if $\text{Supp}(M) \subseteq V(a)$ and $\text{Ext}_R^i(\frac{R}{a}, M)$ is finitely generated for all $i \geq 0$. For any unexplained notation and terminology, we refer the reader to [1] and [8].

2. Main results

Definition 2.1. Let $M$ be a finitely generated $R$-module, and let $a$ be an ideal of $R$. We define the $a$-minimum dimension $d(a, M)$ of $M$ by
\[ d(a, M) = \text{Min}\{\dim \frac{R}{p + a} : p \in \text{Assh}_R(M)\}. \]

To prove the main results of this paper, we need the following lemmas.

Lemma 2.2. (see [3, Lemma 2.5]) Let $M$ be a finitely generated $R$-module, and let $a$ be an ideal of $R$. Then
\[ \text{cd}(a + Rx, M) \leq \text{cd}(a, M) + 1 \]
for any element $x \in m$.

Lemma 2.3. Let $M$ be a finitely generated $R$-module and $a$ be an ideal of $R$ with $d(a, M) > 0$. Then there exists an element $x \in m$ such that $\dim \frac{M}{xM} = \dim M - 1$ and $d(a, \frac{M}{xM}) \leq d(a, M) - 1$.

Proof. Since $d(a, M) > 0$, we have $\sqrt{p + a} \neq m$ for all $p \in \text{Assh}_R(M)$, and so there exists
\[ x \in m - \bigcup_{q \in \text{Min}(p + a), p \in \text{Assh}_R(M)} q. \]

By the definition, there exists $p \in \text{Assh}_R(M)$ such that $d(a, M) = \dim \frac{M}{(p + a)M}$. Now let $q \in \text{Assh}_R(\frac{M}{(p + a)M})$, then by the choice of $x$ we have
\[ \dim \frac{R}{q} = \dim \frac{M}{(p + Rx)M} = \dim M - 1 = \dim \frac{M}{xM}. \]
As $\text{Assh}_R(M/xM) \subseteq \text{Supp} M/xM$, we have $q \in \text{Supp} M/xM$, and so by the above equalities we have $q \in \text{Assh}_R(M/xM)$. It follows that
\[
d(a, M/xM) \leq \dim \frac{M}{(q + a)M} \leq \dim \frac{M}{(p + a + Rx)M} = \dim \frac{M}{(p + a)M} - 1 = d(a, M) - 1.
\]
This element $x$ has the requested properties. $\square$

**Theorem 2.4.** Let $M$ be a finitely generated $R$-module, and let $a$ be an ideal of $R$. Then cd$(a, M) \geq \dim M - d(a, M)$.

**Proof.** We prove this by induction on $n = d(a, M)$. If $d(a, M) = 0$ then we have $\dim \frac{M}{(p+a)M} = 0$ for some $p \in \text{Assh}_R(M)$ and so $\sqrt{p + a} = m$ for some $p \in \text{Assh}_R(M)$. It follows from [1, Exercise 6.1.9] and Non-vanishing Theorem [1, 6.1.4] that
\[
H^\dim M_a(M) \otimes \frac{R}{p} \cong H^\dim M_a(M/pM) \cong H^\dim M_a(M/pM) \cong H^\dim M_a + p(M/pM) \neq 0,
\]
and so $H^\dim M_a(M) \neq 0$.

Now suppose, inductively, that $d(a, M) > 0$, and the result has been proved for all finitely generated $R$-modules $N$ with $d(a, N) < d(a, M)$. By Lemma 2.3, there exists an element $x \in m$ such that $\dim M = \dim \frac{M}{xM} + 1$ and $d(a, M) \geq d(a, M/xM) + 1$. So by induction hypothesis we have $\text{cd}(a, M/xM) \geq \dim \frac{M}{xM} - d(a, M/xM)$. It follows that
\[
\dim M - d(a, M) = \dim \frac{M}{xM} + 1 - d(a, M) \leq \dim \frac{M}{xM} - d(a, M/xM) \leq \text{cd}(a, M/xM) \leq \text{cd}(a, M).
\]
This completes the proof. $\square$

The following examples shows that the equality does not hold in general.

**Example 2.5.** Let $M$ be a finitely generated $R$-module such that $\bigcap_{p \in \text{Assh}_R(M)} p \not\subseteq q$ for some $q$ in $\text{Assh}_R(M)$. Then for $x \in \bigcap_{p \in \text{Assh}_R(M)} p - q$ we have
\[
\text{cd}(Rx, M) = 1 > 0 = \dim(M) - d(Rx, M).
\]
For example, let $R = K[[X, Y, Z]]$, $M = \frac{K[[X,Y,Z]]}{(X,Y,G,Y,Z)}$, and $x = X$, where $K$ is a field and $X, Y, Z$ are independent indeterminates.
Example 2.6. Let $K$ be a field of characteristic 0. Let $R' := K[ X_1, X_2, X_3 ]$, $m' := ( X_1, X_2, X_3 )$ and $b := ( X_2^2 - X_1^2 - X_3^2 )$. Set $R := ( \frac{ R' }{ b } )_{ m' }$ and let $p$ be the extension of the ideal

$$( X_1 + X_2 - X_2 X_3, ( X_3 - 1 )^2 ( X_1 + 1 ) - 1 )$$

of $R'$ to $R$. Then $R$ is a 2-dimensional local domain, and $p$ is a prime ideal of $R$ with $\dim \frac{ R }{ p } = 1$ (see [1, Exercise 8.2.9]), and we have

$$\text{cd}( p, R ) = 2 > 1 = \dim( R ) - d( p, R ).$$

Therefore, it is natural to ask, under what conditions does the equality hold?

Our second aim is to find such conditions. The following theorem gives us a characterization for the equality $\text{cd}( a, M ) = \dim M - d( a, M )$.

Theorem 2.7. Let $M$ be a finitely generated $R$-module, and let $a$ be an ideal of $R$. Then the following statements are equivalent:

(i) $\text{cd}( a, M ) = \dim M - d( a, M )$;
(ii) There exists a sequence $x_1, x_2, ..., x_l$, where $l = d( a, M )$, such that for each $i = 1, 2, ..., l$

$$x_i \in m - \bigcup_{ q \in \text{Min}( p + a + R x_1 + ... + R x_{i-1} ) } q$$

and $H^1_{R x_1}( H^{c+i-1}_{a+R x_1+...+R x_{i-1}}( M ) ) \neq 0$, where $c = \text{cd}( a, M )$.

Proof. (i)$\Rightarrow$(ii) We use induction on $l = d( a, M )$. When $l = 0$, there is nothing to prove. So suppose that $d( a, M ) = l > 0$ and that the result has been proved for each ideal $b$ with $d( b, M ) < l$. Choose $x_1 \in m - \bigcup_{ q \in \text{Min}( p + a ) } q$; then we have

$$\dim M - d( a, M ) = \dim M - d( a + R x_1, M ) - 1$$

$[\text{by lemma 2.2}]$

$$\leq \text{cd}( a + R x_1, M ) - 1 \leq \text{cd}( a, M ).$$

So $\text{cd}( a + R x_1, M ) = \dim M - d( a + R x_1, M )$ and $d( a + R x_1, M ) = l - 1$. Therefore, by the inductive hypothesis, there exists a sequence $x_2, x_3, ..., x_l \in m$ such that, for each $i = 2, 3, ..., l$,

$$x_i \in m - \bigcup_{ q \in \text{Min}( p + a + R x_1 + ... + R x_{i-1} ) } q$$

and $H^1_{R x_1}( H^{c+i-1}_{a+R x_1+...+R x_{i-1}}( M ) ) \neq 0$. 

On the other hand, we have \( \text{cd}(a + Rx_1, M) = \text{cd}(a, M) + 1 \) and so \( \text{H}^{c+1}_{a + Rx_1}(M) \neq 0 \). By \([1]\), Proposition 8.1.2 (i)], there is an exact sequence
\[
\text{H}^{c}_{a}(M) \longrightarrow \text{H}^{c}_{a}(Mx_1) \longrightarrow \text{H}^{c+1}_{a + Rx_1}(M) \longrightarrow 0.
\]
It follows that the natural homomorphism \( \text{H}^{c}_{a}(M) \longrightarrow \text{H}^{c}_{a}(Mx_1) \) is not surjective. So \( \text{H}^{1}_{Rx_1}(\text{H}^{c}_{a}(M)) \neq 0 \) by \([1], \text{Remark 2.2.17}\). This completes the proof of (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (i) For \( d(a, M) = 0 \) the result is obvious. Now suppose, inductively, that \( d(a, M) = l > 0 \) and the result has been proved for each ideal \( b \) with \( d(b, M) < l \). Assume that there exists a sequence \( x_1, x_2, ..., x_l \in \mathfrak{m} \) such that, for each \( i = 1, 2, ..., l \),
\[
x_i \in \mathfrak{m} - \bigcup_{q \in \text{Min}(p + a + Rx_1 + ... + Rx_{i-1})} q
\]
and \( \text{H}^{1}_{Rx_1}(\text{H}^{c+i-1}_{a + Rx_1 + ... + Rx_{i-1}}(M)) \neq 0 \).

Note that \( d(a + Rx_1, M) = d(a, M) - 1 = l - 1 \), and so, by the inductive hypothesis, we have \( \text{cd}(a + Rx_1, M) = \dim(M) - d(a + Rx_1, M) \). It follows that \( \text{cd}(a + Rx_1, M) - 1 = \dim(M) - d(a, M) \). Since \( \text{H}^{1}_{Rx_1}(\text{H}^{c}_{a}(M)) \neq 0 \), the natural homomorphism \( \text{H}^{c}_{a}(M) \longrightarrow \text{H}^{c}_{a}(Mx_1) \) is not surjective by \([1], \text{Remark 2.2.17}\) and so \( \text{H}^{c+1}_{a + Rx_1}(M) \neq 0 \) by \([1], \text{Proposition 8.1.2 (i)}\]. Hence \( \text{cd}(a + Rx_1, M) = \text{cd}(a, M) + 1 \) and the result follows.

Recall that a sequence \( x_1, x_2, ..., x_l \in \mathfrak{a} \) is called an \( \mathfrak{a} \)-filter regular sequence of \( M \) if \( x_i \notin \mathfrak{p} \) for all \( \mathfrak{p} \in \text{Ass}(\frac{M}{(x_1, x_2, ..., x_{i-1})M}) - V(\mathfrak{a}) \) and all \( i = 1, 2, ..., l \). For an \( R \)-module \( M \), we shall denote \( \frac{M}{\Gamma_a(M)} \) by \( \overline{M} \).

**Lemma 2.8.** Let \( M \) be a finitely generated \( R \)-module, and let \( \mathfrak{a} \) be an ideal of \( R \) such that \( \Gamma_a(M) \neq M \). If \( M \) is an equidimensional \( R \)-module, then

(i) \( \overline{M} \) is an equidimensional \( R \)-module and we have \( \dim M = \dim \overline{M} \), and \( d(\mathfrak{a}, M) = d(\mathfrak{a}, \overline{M}) \).

(ii) If \( R \) is a catenary ring then \( \frac{\overline{M}}{x \overline{M}} \) is an equidimensional \( R \)-module for each \( \mathfrak{a} \)-filter regular element \( x \) of \( M \).

**Proof.** (i) This is immediate from the fact that
\[
\text{Min Ass}_R(\overline{M}) = \text{Min Ass}_R(M) - V(\mathfrak{a}) = \text{Assh}_R(M) - V(\mathfrak{a}) = \text{Assh}_R(\overline{M}).
\]

(ii) Let \( q \in \text{Min Ass}_R(\frac{\overline{M}}{x \overline{M}}) \). So we have \( q \in \text{Min(Ann}_R(\overline{M}) + Rx) \). It follows that there exists \( \mathfrak{p} \in \text{Min(Ann}_R(\overline{M})) = \text{Assh}_R(\overline{M}) \) such that \( q \in \text{Min}(p + Rx) \). As \( R \) is a catenary ring,
we have $h(q) = h(p) + 1$ and so $\dim(R/q) = \dim(R/p) - 1$.

It follows that

$$\dim(R/q) = \dim(R/p) - 1 = \dim(M) - 1 = \dim(\frac{M}{xM}).$$

Hence $q \in \text{Ass}_R(\frac{M}{xM})$ and so the claim follows. \(\square\)

**Theorem 2.9.** Let $R$ be a catenary ring, $M$ a finitely generated equidimensional $R$-module, and $\ell = \dim(M) - d(a, M)$. If there exists an $a$-filter regular sequence $x_1, x_2, \ldots, x_\ell$ of $M$ such that $d(a, M_{i-1}) = d(a, M_i)$, where $M_0 = M$ and $M_i = \frac{M_{i-1}}{x_i M_{i-1}}$, for all $i = 1, 2, \ldots, \ell$, then

$$\text{cd}(a, M) = \dim(M) - d(a, M).$$

**Proof.** By Theorem 2.4, it is enough for us to show that $\text{cd}(a, M) \leq \ell$. We argue by induction on $\ell$. When $\ell = 0$, since $M$ is equidimensional, by the definition of $d(a, M)$ we have $M = \Gamma_a(M)$ and so $\text{cd}(a, M) = \dim(M) - d(a, M)$.

Now suppose, inductively, that $\ell > 0$ and the result has been proved for smaller values of $\ell$. By the pervious lemma, in this case, $M$ is an equidimensional $R$-module, and we have $\dim(M) = \dim(M)$, $d(a, M) = d(a, M)$, and $\text{cd}(a, M) = \text{cd}(a, M)$. So in view of the inductive hypothesis we can replace $M$ by $\overline{M}$, and assume that $M$ is $a$-torsion free. The exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M_1 \longrightarrow M_2 \longrightarrow 0$$

induces an exact sequence

$$H_a^{\ell-1}(M_1) \longrightarrow H_a^{\ell}(M) \xrightarrow{x_1} H_a^{\ell}(M).$$

Since $d(a, M) = d(a, M_1)$ and $\dim(M_1) = \dim(M) - 1$, we obtain

$$\dim(M_1) - d(a, M_1) = \ell - 1.$$

By the pervious lemma, $M_1$ is an equidimensional $R$-module, so by induction hypothesis $H_a^{\ell}(M_1) = 0$ for all $i > \ell - 1$. Therefore, in view of the above exact sequence, $(0 : x_1) = 0$ for all $i > \ell$. But $x_1 \in a$ and $\overline{H}_a^{\ell}(M)$ is an $a$-torsion $R$-module, and so $\overline{H}_a^{\ell}(M) = 0$ for all $i > \ell$. This complete the inductive step, and the proof. \(\square\)

The following is an example to illustrate Theorem 2.9.

**Example 2.10.** Let $R = K[[X_1, X_2, X_3, X_4, X_5]]$ denote the formal power series ring in five variables over a field $K$. Put $M = \frac{K[[X_1, X_2, X_3, X_4, X_5]]}{(X_2)(X_3)}$ and $a = \langle X_1, X_2, X_3 \rangle$. In this case we have $\dim(M) - d(a, M) = 2$, and $x_1, x_2 + x_3$ is an $a$-filter regular sequence of $M$ which has the property mentioned in Theorem 2.9. It follows that $\text{cd}(a, M) = 2$ and $H_a^2(\langle x_1, x_2, x_3 \rangle) \frac{K[[X_1, X_2, X_3, X_4, X_5]]}{(X_2)(X_3)(X_5)} = 0$. 
Before proving Theorem 2.12, we need the following lemma which is proved in [2].

Lemma 2.11. (see [2, Lemma 4.3]) Let $M$ be a finitely generated $R$-module, and let $q \in V(\text{Ann}_R(\text{H}^{\dim M}_m(M)))$ such that $\dim M_q = \dim M - \dim \frac{R}{q}$. Then $\text{Ann}_R(0 : \text{H}^{\dim M}_m(M) q) = q$.

Theorem 2.12. Let $R$ be a catenary ring, and let $a$ be an ideal of $R$ such that $\dim \frac{R}{a} = 1$. Then the following statements are equivalent:

(i) $\text{cd}(a, M) = \dim(M) - \text{d}(a, M)$ for each finitely generated $R$-module $M$;

(ii) $\text{cd}(a, \frac{R}{p}) = \dim(R_p) - \text{d}(a, \frac{R}{p})$ for each prime ideal $p$ of $R$;

(iii) $\text{Ann}_R(0 : \text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q) = q$ for each prime ideal $p$ of $R$ and each prime ideal $q \in V(\text{Ann}_R(\text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q))$ with $\dim \frac{R}{q} = 1$.

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii) For $p \in \text{Spec}(R)$, let $q$ be a prime ideal of $R$ such that $q \supseteq \sqrt{p + a} = m$ and so $\text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) = \text{H}^{\dim \frac{R}{p}}_m(\frac{R}{p})$. Therefore the proof is complete if we show that

$$\text{Ann}_R(0 : \text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q) = q.$$ 

Since $R$ is catenary, we have

$$\dim(R_p)_q = \dim(R_p) - 1 = \dim(R_p) - \dim \frac{R}{q}.$$ 

The result now follows from Lemma 2.11.

(iii)⇒(i) It is enough, in order to prove this part, to show that, if $\text{cd}(a, M) = \dim(M)$, then there exists $p \in \text{Assh}_R(M)$ such that $\sqrt{p + a} = m$. By [3, Corollary 2.2], there exists $p \in \text{Assh}_R(M)$ such that $\text{cd}(a, \frac{R}{p}) = \dim(R_p)$. We show that for this $p$, we have $\sqrt{p + a} = m$. Suppose, on the contrary, that $\sqrt{p + a} \neq m$. Then there exists a prime ideal $q$ of $R$ such that $q \supseteq \sqrt{p + a}$ and $\dim \frac{R}{q} = 1$. Since $q \supseteq p = \text{Ann}_R(\text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q)$, by assumption (iii), we have $\text{Ann}_R(0 : \text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q) = q$. It follows that $(0 : \text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p}) q)$ is not finitely generated. But by [3, Theorem 3], Artinian local cohomology module $\text{H}^{\dim \frac{R}{p}}_a(\frac{R}{p})$ is $a$-cofinite, and this is a contradiction. □

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