COFINITELY WEAK GENERALIZED $δ$-SUPPLEMENTED MODULES

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Abstract. We will study modules whose cofinite submodules have weak generalized-$δ$-supplements. We attempt to investigate some properties of cofinitely weak generalized $δ$-supplemented modules. We will prove for a module $M$ and a semi-$δ$-hollow submodule $N$ of $M$ that, $M$ is cofinitely weak generalized $δ$-supplemented if and only if $\frac{M}{N}$ is cofinitely weak generalized $δ$-supplemented. Also we show that any $M$-generated module is cofinitely weak generalized $δ$-supplemented module, where $M$ is cofinitely weak generalized $δ$-supplemented. We obtain some other results about this kind of modules.

1. Introduction

Throughout the paper $R$ will be an associative ring with identity and we will consider only left unital $R$-modules. All definition not given here can be found in [1, 3, 5, 10].

A submodule $K$ of $M$ is called small in $M$ (denoted by $K \ll M$) if, $L + K \neq M$ for every proper submodule $L$ of $M$. The sum of all small submodules of the module $M$ is denoted by $\text{Rad}(M)$.

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A submodule $N$ of $M$ is called cofinite if $\frac{M}{N}$ is finitely generated.

For two submodules $N$ and $K$ of the module $M$, $N$ is called a supplement of $K$ in $M$ if $N$ is minimal with respect to the property $M = K + N$, equivalently $M = K + N$ and $N \cap K \ll N$. $N$ is called a weak supplement of $K$ in $M$ if $N + K = M$ and $N \cap K \ll M$.

The module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. $M$ is called weakly supplemented if every submodule of $M$ has a weak supplement in $M$.

2. A background of $\delta$-supplemented modules

In this section we introduce the $\delta$-small submodule of a module and then some preliminary lemmas and propositions about the class of $\delta$-supplemented modules are given. We develop to get some suitable results about the class cofinitely weak generalized $\delta$-supplemented modules in the section 3.

The singular submodule of a module $M$ (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0$ for some ideal $I \leq R\}$. A module $M$ is called singular (nonsingular) if $Z(M) = M$ (resp. $Z(M) = 0$).

$\delta$-small submodules were defined as a generalization of small submodules by Zhou in [11]. Let $M$ be a module and $L \leq M$. Then $L$ is called $\delta$-small in $M$ (denoted by $L \ll_{\delta} M$) if, for any submodule $N$ of $M$ with $M/N$ singular, $M = N + L$ implies that $M = N$. The sum of all $\delta$-small submodules of $M$ is denoted by $\delta(M)$.

It is easy to see that every small submodule of a module $M$ is $\delta$-small in $M$, so $\text{Rad}(M) \subseteq \delta(M)$ and, if $M$ is singular, all $\delta$-small submodules of $M$ are small and so $\text{Rad}(M) = \delta(M)$ in this case. Also any non-singular semisimple submodule of $M$ is $\delta$-small in $M$.

**Example 2.1.** Let $R$ be a semisimple ring and $M = R_R$. Since $R$ is the only essential ideal of $R$, so there is no nonzero singular factor module of $M$. Finally we conclude that all submodules of $M$ (even $M$) are $\delta$-small in $M$.

In the other hand since $M$ is semisimple, $0$ is the only small submodule of $M$. In this case $\text{Rad}(M) = 0$ and $\delta(M) = M$.

Especially let $R = M = \mathbb{Z}_6$. Then two non-trivial submodule of $M$, $M_1 = \{0, 3\}$ and $M_2 = \{0, 2, 4\}$ are $\delta$-small in $M$, but neither $M_1$ nor $M_2$ is small in $M$. Moreover $M \ll_{\delta} M$. Finally we have $\text{Rad}(M) = 0$ but $\delta(M) = M$.

The above example also shows that the inclusion $\text{Rad}(M) \subseteq \delta(M)$ can be strict.

Let $M$ be any module and $B \leq A$ be submodules of $M$. Then $B$ is called a $\delta$-cosmall submodule of $A$ in $M$ if $A/B \ll_{\delta} M/B$. A submodule $N$ of $M$ is called $\delta$-coclosed in $M$ if
$N$ has no proper $\delta$–cosmall submodule in $M$, that is, if $B \leq N$ such that $N/B \ll_{\delta} M/B$, then $N = B$. A submodule $A$ of $M$ is weak $\delta$–coclosed in $M$ if, given $B \leq A$ such that $A/B$ is singular and $A/B \ll_{\delta} M/B$, then $A = B$. For a submodule $N$ of $M$, $A \leq N$ is called a $\delta$–closure of $N$ in $M$ if $A$ is $\delta$–coclosed in $M$ and $N/A \ll_{\delta} M/A$ and $A$ is called a weak $\delta$–coclosure of $N$ in $M$ if $A$ is weak $\delta$–coclosed in $M$ and $N/A \ll_{\delta} M/A$. (for more information see [6]).

Let $K, N$ be submodules of module $M$. Then $N$ is called a $\delta$–supplement of $K$ in $M$ if $M = N + K$ and $N \cap K \ll_{\delta} N$. $N$ is called a weak $\delta$–supplement of $K$ in $M$ if $M = N + K$ and $N \cap K \ll_{\delta} M$. The module $M$ is called $\delta$–supplemented if every submodule of $M$ has a $\delta$–supplement in $M$. $M$ is called weak $\delta$–supplemented if every submodule of $M$ has a weak $\delta$–supplement in $M$.

Here we present two lemmas that state some properties of $\delta$–small submodules which we will use throughout the section 3.

**Lemma 2.2.** Let $M$ and $N$ be modules. Then

1. $\delta(M) = \sum \{ L \leq M \mid L \ll_{\delta} M \} = \bigcap \{ K \leq M \mid M/K \text{ is singular simple } \}$.
2. If $f: M \to N$ is an $R$-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$. Therefore $\delta(M)$ is a fully invariant submodule of $M$. In particular, if $K \leq M$, then $\delta(K) \subseteq \delta(M)$.
3. If $M = \bigoplus_{i \in I} M_i$, then $\delta(M) = \bigoplus_{i \in I} \delta(M_i)$.
4. If every proper submodule of $M$ is contained in a maximal submodule of $M$, then $\delta(M)$ is the unique largest $\delta$–small submodule of $M$. In particular if $M$ is finitely generated, then $\delta(M)$ is $\delta$–small in $M$.

*Proof.* See [11, Lemma 1.2]. □

**Lemma 2.3.** Let $M$ be a module and $\delta(M) \leq K \leq M$. Then the following hold:

1. If $\delta(M)$ is $\delta$–small in $M$ and $\delta(M)$ is a $\delta$–cosmall submodule of $K$ in $M$, then $K$ is $\delta$–small in $M$.
2. $\delta(M/\delta(M)) = 0$.

*Proof.* See [11, Lemma 1.3]. □
3. Cofinitely weak generalized $\delta$–supplemented modules

The submodules $\text{Rad}(M)$ and $\delta(M)$ of a module $M$ in the category of modules play important roles. Many authors studied some generalizations of supplemented, weakly supplemented, $\delta$–supplemented and weakly $\delta$–supplemented modules by us of these two functors. We refer to [2, 3, 8] for some of them.

Here we study and investigate a generalization of weakly $\delta$–supplemented modules.

**Definition 3.1.** Let $M$ be a module and $N, K$ two submodules of $M$. $N$ is called a generalized $\delta$–supplement of $K$ in $M$ if, $N + K = M$ and $N \cap K \subseteq \delta(N)$. $N$ is called a weak generalized $\delta$–supplement of $K$ in $M$ if, $N + K = M$ and $N \cap K \subseteq \delta(M)$. The module $M$ is called (cofinitely) generalized $\delta$–supplemented (briefly (C)G-$\delta$-S) if every (cofinite) submodule of $M$ has a generalized $\delta$–supplement in $M$. $M$ is called (cofinitely) weak generalized $\delta$–supplemented (briefly (C)WG-$\delta$-S) if every (cofinite) submodule of $M$ has a weak generalized $\delta$–supplement in $M$.

$G$-$\delta$-S modules are defined and investigated by Talebi and current author in [3]. Here a generalization of $G$-$\delta$-S modules namely CWG$\delta$-S modules and some other kind of modules related to these modules will be defined and investigated. First we present an elementary lemma.

**Lemma 3.2.** Let $M$ be a module and $V, U$ submodules of $M$. If $V$ is a weak generalized $\delta$–supplement of $U$ in $M$, then $\frac{V+L}{L}$ is a weak generalized $\delta$–supplement of $\frac{U}{L}$ in $\frac{M}{L}$ for every $L \leq U$.

**Proof.** We have $V + U = M$ and $V \cap U \leq \delta(M)$. So $\frac{M}{L} = \frac{V+L}{L} + \frac{U}{L}$. Let $\pi : M \rightarrow \frac{M}{L}$ be the natural epimorphism. Then by Lemma 2.1 (2), $\pi(V \cap U) \subseteq \pi(\delta(M)) \subseteq \delta(\frac{M}{L})$, where $\pi(V \cap U) = \frac{V+U+L}{L} = \frac{V+L}{L} \cap \frac{U}{L}$ by modularity. Hence $\frac{V+L}{L}$ is a weak generalized $\delta$–supplement of $\frac{U}{L}$ in $\frac{M}{L}$. \[ \square \]

**Proposition 3.3.** Every homomorphic image of a CWG$\delta$-S module is again CWG-$\delta$-S.

**Proof.** Let $M$ be a CWG-$\delta$-S module and $\frac{U}{N}$ a cofinite submodule of $\frac{M}{N}$ where $N \leq U \leq M$. Then $U$ is a cofinite submodule of $M$ and so there exists a submodule $V$ of $M$ such that $V + U = M$ and $V \cap U \leq \delta(M)$. By Lemma 3.2, $\frac{V+N}{N}$ is a weak generalized $\delta$–supplement of $\frac{U}{N}$ in $\frac{M}{N}$ and this completes the proof. \[ \square \]

Recall that a module $M$ is called semi–hollow if every proper finitely generated submodule of $M$ is small in $M$ ([3, 2.12]). Here we call a module $M$, semi-$\delta$-hollow if every proper
finitely generated submodule of \( M \) is \( \delta \)-small in \( M \). It is clear that if \( M \) is semi-\( \delta \)-hollow, then \( \delta(M) = M \). \( \frac{Z}{p^rZ} \) as \( \mathbb{Z} \)-modules, are semi-\( \delta \)-hollow.

**Proposition 3.4.** Let \( M \) be a module and \( N \) a semi-\( \delta \)-hollow submodule of \( M \). Then \( M \) is CWG-\( \delta \)-S iff \( \frac{M}{N} \) is.

**Proof.** The necessity follows from Proposition 3.3.

For converse suppose that \( N \) is a semi-\( \delta \)-hollow submodule of \( M \) and \( \frac{M}{N} \) is a CWG-\( \delta \)-S module. If \( U \) is a cofinite submodule of \( M \), then \( \frac{U+N}{N} \) is a cofinite submodule of \( \frac{M}{N} \). Let \( \frac{V}{N} \) be a weak generalized \( \delta \)-supplement of \( \frac{U+N}{N} \) in \( \frac{M}{N} \). So

\[
\frac{U+N}{N} + \frac{V}{N} = \frac{M}{N} \text{ and } \frac{U+N}{N} \cap \frac{V}{N} \subseteq \delta \left( \frac{M}{N} \right)
\]

This implies \( U + V = M \) and \( \frac{U+N}{N} \subseteq \delta \left( \frac{M}{N} \right) \). Since \( N \) is semi-\( \delta \)-hollow, we have \( N = \delta(N) \subseteq \delta(M) \) and so \( \frac{\delta(M)}{M} \) = \( \delta \left( \frac{M}{N} \right) \). Finally we get \( U \cap V \subseteq \delta(M) \), as desired. \( \square \)

**Proposition 3.5.** Let \( M \) be a WG-\( \delta \)-S module and \( N \) be a submodule of \( M \) such that \( \delta \left( \frac{M}{N} \right) = 0 \). Then \( \frac{M}{N} \) is semisimple.

**Proof.** Suppose that \( N \leq K \leq M \). There exists a submodule \( V \) of \( M \) such that \( K + V = M \) and \( K \cap V \leq \delta(M) \). According to Lemma 3.2, \( \frac{V+N}{N} \) is a weak generalized \( \delta \)-supplement of \( \frac{K}{N} \) in \( \frac{M}{N} \), so that

\[
\frac{K}{N} + \frac{V+N}{N} = \frac{M}{N} \text{ and } \frac{V+N}{N} \cap \frac{K}{N} \subseteq \delta \left( \frac{M}{N} \right) = 0
\]

That is \( \frac{M}{N} \) is semisimple. \( \square \)

**Corollary 3.6.** Let \( M \) be a CWG-\( \delta \)-S module and \( N \leq M \) such that \( \delta \left( \frac{M}{N} \right) = 0 \). Then every cofinite submodule of \( \frac{M}{N} \) is a direct summand.

**Proof.** If \( \frac{K}{N} \) is a cofinite submodule of \( \frac{M}{N} \), then \( K \) is a cofinite submodule of \( M \). Now apply the proof of Proposition 3.5 to complete the proof. \( \square \)

By Proposition 3.3 and Proposition 3.4 we have the next two corollaries.

**Corollary 3.7.** If \( M \) is a WG-\( \delta \)-S module, then \( \frac{M}{\delta(M)} \) is semisimple.

**Corollary 3.8.** Let \( M \) be a CWG-\( \delta \)-S module. Then every cofinite submodule of \( \frac{M}{\delta(M)} \) is a direct summand.

**Proposition 3.9.** Let \( f : M \to N \) be a homomorphism and \( L \) a weak generalized \( \delta \)-supplement submodule of \( M \) containing \( \ker(f) \). Then \( f(L) \) is a weak generalized \( \delta \)-supplement of \( f(M) \).
Proof. Suppose that $L$ is a weak generalized $\delta$-supplement of $K$ in $M$. Hence $M = L + K$ and $L \cap K \subseteq \delta(M)$. Then $f(M) = f(L) + f(K)$ and $f(L \cap K) \subseteq f(\delta(M)) \subseteq \delta(f(M))$ by Lemma 2.1. As ker($f$) $\subseteq L$, we have $f(L \cap K) = f(L) \cap f(K)$ and so $f(L)$ is a weak generalized $\delta$-supplement of $f(K)$ in $f(M)$. □

**Lemma 3.10.** Let $M$ be a module, $U$ a cofinite submodule of $M$ and $N \leq M$ a weak generalized $\delta$-supplemented module. If $N + U$ has a weak generalized $\delta$-supplement in $M$, then $U$ also has.

Proof. Let $X$ be a weak generalized $\delta$-supplement of $N + U$ in $M$. We have $\frac{N + X + U}{X + U} \cong \frac{N}{N \cap (X + U)} \cong \frac{M}{M/U}$. Note that $\frac{M}{M/U}$ is finitely generated. So $N \cap (X + U)$ has a weak generalized $\delta$-supplement $Y$ in $N$; i.e. $Y + [N \cap (X + U)] = N$ and $y \cap N \cap (X + U) = Y \cap (X + U) \subseteq \delta(N) \subseteq \delta(M)$. Now we have

$$M = U + X + N = U + X + Y + [N \cap (X + U)] = U + X + Y$$

and also

$$U \cap (X + Y) \subseteq X \cap (Y + U) + Y \cap (X + U) \subseteq X \cap (N + U) + Y \cap (X + U) \subseteq \delta(M)$$

That is $X + Y$ is a weak generalized $\delta$-supplement of $U$ in $M$. □

**Proposition 3.11.** Any sum of CWG-$\delta$-$S$ modules is again CWG-$\delta$-$S$.

Proof. Let $\{M_i\}_{i \in I}$ be set of CWG-$\delta$-$S$ modules and $M = \sum_{i \in I} M_i$. Suppose that $N$ is a cofinite submodule of $M$ and $\frac{M}{N}$ is finitely generated by $\{x_1 + N, x_2 + N, ..., x_k + N\}$. Thus $M = Rx_1 + Rx_2 + ... + Rx_k + N$. For every $i \in \{1, 2, ..., k\}$ we have $x_i \in \sum_{j \in F_i} M_j$ for some finite set $F_i \subseteq I$. Therefore $Rx_1 + Rx_2 + ... + Rx_k \leq \sum_{j \in F} M_j$ where $F = \bigcup_{i=1}^{k} F_i$. Suppose that $F = \{i_1, i_2, ..., i_r\}$. Then $M = N + \sum_{s=1}^{r} M_{i_s}$. Since $M = M_{i_r} + (N + \sum_{s=1}^{r-1} M_{i_s}$ has a trivial weak generalized $\delta$-supplement 0 and $M_{i_r}$ is a CWG-$\delta$-$S$ module, $N + \sum_{s=1}^{r-1} M_{i_s}$ has a weak generalized $\delta$-supplement in $M$ by Lemma 3.10. Similarly $N + \sum_{s=1}^{r-2} M_{i_s}$ has a weak generalized $\delta$-supplement in $M$ and so on. After we have used Lemma 3.10 $r$ times, we will obtain that $N$ has a weak generalized $\delta$-supplement in $M$, as required. □

**Corollary 3.12.** If $M$ is a CWG-$\delta$-$S$ module, then every $M$-generated module is again CWG-$\delta$-$S$.

Proof. Follows immediately from Proposition 3.3 and Proposition 3.11. □
Lemma 3.13. Let $M$ be a module and $X$ a cofinite (maximal) submodule of $M$. If $Y$ is a weak generalized $\delta$-supplement of $X$ in $M$, then $X$ has a finitely generated (cyclic) weak generalized $\delta$-supplement in $M$ contained in $Y$.

Proof. We have $\frac{X+Y}{X} \cong \frac{X+Y}{X} = \frac{M}{X}$ is finitely generated. Suppose that $\frac{X+Y}{X} = \langle x_1 + X \cap Y, x_2 + X \cap Y, \ldots, x_n + X \cap Y \rangle$. If $Z = \langle x_1, x_2, \ldots, x_n \rangle$, then $Z \leq Y$ and $Z + X \cap Y = Y$. Now $Z + X = Z + X \cap Y + X = Y + X = M$ and also $Z \cap X \leq Y \cap X \leq \delta(M)$. Therefore $Z$ is a finitely generated weak generalized $\delta$-supplement of $X$ contained in $Y$.

Now if $X$ is maximal, then $\frac{X}{X+Y}$ is simple and especially cyclic and by the similar way there is a cyclic submodule $W$ of $Y$ which is a weak generalized $\delta$-supplement of $X$ in $M$. □

Recall that ([1], Proposition 10.4]) if $M$ is a finitely generated module, then $\text{Rad}(M) \ll M$. Similarly we have the following lemma.

Lemma 3.14. If $M$ is a finitely generated module, then $\delta(M) \ll \delta M$.

In the next proposition we proceed with a weak condition to derive a strong property for such submodules. Then from this proposition we present some corollaries.

Proposition 3.15. Let $M$ be a module and $X$ a cofinite submodule of $M$. Moreover suppose that $Y$ is a weak generalized $\delta$-supplement of $X$ in $M$ and every finitely generated submodule $K$ of $Y$ satisfies in condition that $\delta(K) = K \cap \delta(M)$. Then $X$ has a finitely generated $\delta$-supplement in $M$.

Proof. We have $X + Y = M$ and $X \cap Y \subseteq \delta(M)$. Since $\frac{M}{X}$ is finitely generated, by Lemma 3.13 $X$ has a finitely generated weak generalized $\delta$-supplement $K$ in $M$ contained in $Y$. So $M = X + K$ and $X \cap K \subseteq \delta(M)$. Now $X \cap K \leq K \cap \delta(M) = \delta(K)$. By Lemma 3.14, $\delta(K) \ll \delta K$ and hence $K$ is a $\delta$-supplement of $X$ in $M$. □

The next corollary follows immediately from Proposition 3.15.

Corollary 3.16. Let $M$ be a module and $X$ a cofinite submodule of $M$. Moreover suppose that $Y$ is a weak generalized $\delta$-supplement of $X$ in $M$ and every finitely generated submodule $K$ of $Y$ is a direct summand of $M$. Then $X$ has a finitely generated generalized $\delta$-supplement in $M$.

Theorem 3.17. Let $M$ be a module such that every finitely generated submodule $K$ of $M$ satisfies in condition $K \cap \delta(M) = \delta(K)$. Then the following statements are equivalent:
(1) $M$ is cofinitely $\delta$-supplemented;
(2) $M$ is $CG-\delta$-S;
(3) $M$ is cofinitely weak $\delta$-supplemented;
(4) $M$ is $CWG-\delta$-S.

Proof. $1 \implies 2 \implies 4$ and $1 \implies 3 \implies 4$ are clear.

So it is enough to prove $4 \implies 1$. Suppose that $X$ is a cofinite submodule of $M$. Since $M$ is $CWG-\delta$-S, $X$ has a weak generalized $\delta$-supplement in $M$. Now by Proposition 3.15, $X$ has a $\delta$-supplement in $M$ and this completes the proof. $\square$

Corollary 3.18. Suppose that $M$ is a module such that every finitely generated submodule of $M$ is a direct summand. Then the following statements are equivalent

(1) $M$ is cofinitely $\delta$-supplemented;
(2) $M$ is $CG-\delta$-S;
(3) $M$ is cofinitely weak $\delta$-supplemented;
(4) $M$ is $CWG-\delta$-S.

Proof. If $K$ is a direct summand of $M$, then $K \cap \delta(M) = \delta(K)$. Now apply Theorem 3.17. $\square$

Corollary 3.19. Suppose that $M$ is a finitely generated module such that for every finitely generated submodule $K$ of $M$ we have $K \cap \delta(M) = \delta(K)$. Then the following statements are equivalent

(1) $M$ is $\delta$-supplemented;
(2) $M$ is generalized $\delta$-supplemented;
(3) $M$ is weak $\delta$-supplemented;
(4) $M$ is weak generalized $\delta$-supplemented.

Furthermore in this case every finitely generated submodule of $M$ is a $\delta$-supplement.

Proof. The first part follows from Theorem 3.17. To see the second part, suppose that $K$ is a finitely generated submodule of $M$. Then $K$ has a weak $\delta$-supplement $L$ in $M$. Therefore $L + K = M$ and $L \cap K \subseteq K \cap \delta(M) = \delta(K) \triangleleft K$; i.e. $K$ is a $\delta$-supplement of $L$ in $M$. $\square$

Example 3.20. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Then for every submodule $N = m\mathbb{Z}$ of $M$ we have $0 = \delta(N) = N \cap \delta(M)$. Also $M$ is not $\delta$-supplemented, so $M$ is not generalized $\delta$-supplemented and especially $M$ is not weak generalized $\delta$-supplemented by Corollary 3.19.

Theorem 3.21. For a module $M$ the following are equivalent
(1) $M$ is a CWG-$\delta$-$S$ module;
(2) $\frac{M}{\delta(M)}$ is a CWG-$\delta$-$S$ module;
(3) Every cofinite submodule of $M$ is a direct summand.

Proof. 1 $\implies$ 2 follows from Proposition 3.3.
2 $\implies$ 3 is clear since $\delta(\frac{M}{\delta(M)}) = 0$.
To see 3 $\implies$ 1 suppose that $K$ is a cofinite submodule of $M$. Then $\frac{K+\delta(M)}{\delta(M)}$ is a cofinite submodule of $\frac{M}{\delta(M)}$. Hence there exists a submodule $\frac{L}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that
$$\frac{K+\delta(M)}{\delta(M)} + \frac{L}{\delta(M)} = \frac{M}{\delta(M)}$$
and $\frac{K+\delta(M)}{\delta(M)} \cap \frac{L}{\delta(M)} = 0$
Therefore $K + L = M$ and $K \cap L \subseteq \delta(M)$ as desired. □

Corollary 3.22. Let $M$ be a CWG-$\delta$-$S$ module. Then

(1) Every maximal submodule of $\frac{M}{\delta(M)}$ is a direct summand.
(2) Every maximal submodule of $\frac{M}{\delta(M)}$ is a weak generalized $\delta$-supplement.
(3) Every maximal submodule of $M$ is weak generalized $\delta$-supplement.

Proof. It is clear that conditions 1, 2 and 3 are equivalent by Theorem 3.21. Now if $M$ is CWG-$\delta$-$S$, then 3 holds, since every maximal submodule is cofinite. □

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